The approximation of Dirichlet series with infinite order in the right half plane

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Abstract

We study the error in approximating Dirichlet series of infinite order which converges in the right half plane by a Dirichlet polynomial. We introduce the concept of γ_U order of a function and obtain some necessary and sufficient conditions on the growth of Dirichlet series with finite γ_U-order.

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1 Introduction, and some definitions

Let the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-s\lambda_n} \tag{1.1}$$

where

$$0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n < \ldots, \quad \lambda_n \to \infty \quad \text{as} \quad n \to \infty. \tag{1.2}$$

$s = \sigma + it$ (σ, t are real variables), a_n are non-zero complex numbers,

$$\limsup_{n\to\infty} (\lambda_{n+1} - \lambda_n) = b < +\infty \tag{1.3}$$

and

$$\limsup_{n\to\infty} \frac{\log^+ |a_n|}{\lambda_n} = 0. \tag{1.4}$$

In this case, the abscissas of convergence and absolute convergence is zero and hence $f(s)$ is an analytic function in the right half plane

$$H = \{s = \sigma + it : \sigma > 0, t \in \mathbb{R}\}.$$

Let A be the class of all functions $f(s)$ satisfy (1.2)-(1.4) and analytic in $\mathbb{R}(s) > 0$, and let $\overline{A}_\alpha$ be the class of all functions $f(s)$ satisfy (1.2)-(1.3) and analytic in $\mathbb{R}(s) \geq \alpha$, where $-\infty < \alpha < +\infty$. Thus, if $0 < \alpha < \infty$ and $f(s) \in A$ then $f(s) \in \overline{A}_\alpha$, if $-\infty < \alpha < 0$ and $f(s) \in \overline{A}_\alpha$ then $f(s) \in A$. Let $\prod_k$ be the class of all exponential polynomials of degree at most k, that is

$$\prod_n = \left\{ \sum_{i=1}^{n} b_i \exp(-s\lambda_i); (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n \right\}.$$

For $f(s) \in A$, let

$$M(\sigma, F) = \sup_{-\infty < t < +\infty} |f(\sigma + it)|$$

and

$$\mu(\sigma, F) = \max_{n \in \mathbb{N}} \{|a_n| e^{-\lambda_n \sigma}\}.$$

These are respectively called maximum modulus and maximum term of $f(s)$ for $Re(s) = \sigma > 0$ respectively.

Definition 1.1. [20] *Let $f(s) \in A$, the order of $f(s)$ can be defined by*

$$\rho = \limsup_{\sigma \to 0^+} \frac{\log \log^+ M(\sigma, f)}{-\log \sigma},$$

where

$$\log^+ x = \left\{ \begin{array}{ccc} \log x & , & x \geq 1 \\ 0 & , & x \leq 1 \end{array} \right\}$$

For $\rho = 0$, $0 < \rho < +\infty$ and $\rho = \infty$, $f(s)$ is of zero order, finite order and infinite order respectively. Considerable attention has been paid to the growth and the value distribution of analytic functions defined by Dirichlet series: [1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 21] for some results. Let $\mathfrak{I}$ be the class of all functions $\gamma(x)$ satisfy the following conditions:
(i) $\gamma(x)$ is defined on $[a, \infty)$, $a > 0$ is positive, strictly increasing, differentiable and tends to $+\infty$ as $x \to \infty$.
(ii) $x\gamma'(x) = o(1)$ as $x \to \infty$.

Definition 1.2. [4] *If the Dirichlet series $f(s)$ of infinite order satisfies*

$$\limsup_{\sigma \to 0^+} \frac{\gamma(\log^+ M(\sigma, f))}{-\log \sigma} = \overline{\rho},$$

where $\gamma(x) \in \mathfrak{I}$, then $\overline{\rho}$ is called the γ-order of $f(s)$.

By introducing the concept of γ-order the author *[20]* studied the growth of functions of infinite order which are represented by a class of Dirichlet series convergence in the half-plane, and obtained the following theorem:

Theorem 1.1. ([20]) *If $f(s) \in A$ be of γ order and ρ_γ be finite, then*

$$\limsup_{n \to \infty} \frac{\gamma(\lambda_n)}{\log_{\lambda_n} - \log^+ \log |a_n|} = \rho_\gamma = \limsup_{\sigma \to 0^+} \frac{\gamma(\log M(\sigma, f))}{-\log(\sigma)}.$$

Remark 1.1. *If p-order [5], is regarded as a special case of γ-order of Dirichlet series. For example the function $d(x) = \log_p x$, $p \geq 2$, $p \in \mathbb{N}_+$ satisfies the conditions (i) and (ii) where p is a positive integer and $\log_1 x = \log x$, and $\log_p x = \log(\log_{p-1} x)$.*

Remark 1.2. *Moreover γ-order is more precise than p-order to some extent. In fact, for $p(\geq 2)$ is a positive integer, we can find the function $\gamma(x) \in \Im$ and a positive real function $M(x)$ satisfying $\rho_p(M) = \infty$, $\rho_{p+1}(M) = 0$ and $\rho_\gamma(M) = t$, for example let*

$$M(x) = \exp_{p+1}\left\{(t.\log x)^{\frac{1}{h}}\right\}, \quad \gamma(x) = (\log_{p+1} x)^h$$

where t is a finite positive real constant and $0 < h < 1$ then

$$\rho_\gamma(M) = \limsup_{x\to\infty} \frac{\gamma(\log M(x))}{\log x} = t(0 < t < \infty).$$

$$\rho_p(M) = \limsup_{x\to\infty} \frac{\log_p(\log M(x))}{\log x} = \infty,$$

and

$$\rho_{p+1}(M) = \limsup_{x\to\infty} \frac{\log_{p+1}(\log M(x))}{\log x} = 0.$$

Remark 1.3. *If $\overline{\rho} = \infty$ in Definition 2, then $f(s)$ is a Dirichlet series of infinite γ-order.*

Now a question arises, If ρ_γ is infinite then what may happen in Theorem 1.1. The purpose of this paper is to deal with this problem. For this we are using the help of type function U(x) in [5] to enlarge the growth of the denominator $-\log(\sigma)$ and we are trying to obtain the main results as follows:

Theorem 1.2. *Let $f(s) \in A$ be of infinite γ-order then*

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log^+ m(\sigma, f))}{\log U\left(\frac{1}{\sigma}\right)} = T \iff \limsup_{\sigma\to 0^+} \frac{\gamma(\log^+ M(\sigma, f))}{\log U\left(\frac{1}{\sigma}\right)} = T.$$

where $0 < T < \infty$, $U(x) = x^{\rho(x)}$ satisfies the following conditions:
(i) $\rho(x)$ is monotone and $\lim_{x\to\infty} \rho(x) = \infty$.
(ii) $\lim_{x\to\infty} \frac{\log U(x')}{\log U(x)} = 1$, where $x' = x + \frac{x}{\log U(x)}$.

Proof. From the Lemma 2.1 and the Lemma 2.2 in the next Section 2, we can prove the conclusion of the Theorem 1.2 easily. □

Definition 1.3. *If Dirichlet series $f(s)$ of infinite has infinite γ-order and satisfies*

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log^+ M(\sigma, f))}{\log U\left(\frac{1}{\sigma}\right)} = T,$$

then T is called the γ_U-order of Dirichlet series $f(s)$.

Theorem 1.3. *Let $f(s) \in A$ be of infinite γ-order, then*

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log^+ M(\sigma, f))}{\log U\left(\frac{1}{\sigma}\right)} = T \iff \limsup_{n\to\infty} \frac{\gamma(\log^+ |a_n|)}{\log U\left(\frac{\lambda_n}{\log^+ |a_n|}\right)} = T,$$

where $0 < \underline{T} < \infty$
For $f(s) \in A_\alpha$ *and* $-\infty < \alpha < +\infty$, *let* $E_n(f, \alpha)$ *be the error in approximating the function f(s) by exponential polynomials of degree n in uniform norm then*

$$E_n(f,\alpha) = \inf_{p\in\prod_n} \|f - p\|_\alpha, n = 1, 2, 3, ...$$

where

$$\|f - p\|_\alpha = \max_{-\infty\to+\infty} |f(\alpha + it) - p(\alpha + it)|.$$

The author [18] *investigated the relationship between the error* $E_n(f, \alpha)$ *and the growth order of* $f(s)$, *and obtained some equivalence relationship between* $E_n(f, \alpha)$ *and the regular growth of* $f(s)$ *with finite order as follows:*

Theorem 1.4. [20] *Let* $f(s) \in A$ *be of finite order* ρ *then for any real number* $(0 < \rho < \infty)$

$$\lim_{\sigma\to 0^+} \frac{\log^+ M_u(\sigma, f)}{U_1\left(\frac{1}{\sigma}\right)} = 1 \iff \limsup_{n\to\infty} \frac{\log^+ [E_n(f,\alpha)\exp(\alpha\lambda_{n+1})]}{BU_1\left(\frac{\lambda_{n+1}}{\log^+[E_n(f,\alpha)\exp(\alpha\lambda_{n+1})]}\right)} = 1,$$

and there exists an increasing positive integer $\{n_\nu\}$ *satisfying*

$$\lim_{\nu\to\infty} \frac{\log^+ [E_{n_\nu}(f,\alpha)\exp(\alpha\lambda_{n_\nu+1})]}{BU_1\left(\frac{\lambda_{n_\nu+1}}{log^+[E_{n_\nu}(f,\alpha)\exp(\alpha\lambda_{n_\nu+1})]}\right)} = 1, \lim_{\nu\to\infty} \frac{\lambda_{n_\nu+1}}{\lambda_{n_\nu}} = 1,$$

where $B = \frac{(1+\rho)^{1+\rho}}{\rho^\rho}$ *and* $U_1(r) = r^{\rho(r)}$, $\rho(r)$ *satisfying the following conditions:*
(i) there exist a real number r_o, $\rho(r)$ *is non-negative continuous, monotone on* $[r_o, +\infty)$ *and tends to* ρ *as* $r \to \infty$.
(ii) $\lim_{r\to\infty} \rho'(r) r \log r = 0$.
(iii)$U_1(Kr) = [K^\rho + o(1)]U_1(r)(r \to \infty)$ *for every positive integer K, and* $U_1(r)$ *is an increasing function on* $r \geq r'_0 > r_0$.

Remark. This type function $U_1(r)$ is different from the type function $U(x)$ in Theorem 1.4 Recently the authors [19] further investigate the relations between the error $E_n(f, \alpha)$ and the growth order of $f(s)$, when $f(s)$ has infinite order by introducing the concept of γ-order.

Theorem 1.5. [22] *Let* $f(s) \in A$ *be of finite order* ρ_γ *then for any real number* $(0 < \alpha < +\infty)$

$$\limsup_{n\to\infty} \frac{\gamma(\lambda_n)}{\log \lambda_n - \log^+ \log (E_n(f,\alpha)e^{\alpha\lambda_n})} = \rho_\gamma.$$

In this paper we will investigate the problem on $\rho_\gamma = \infty$, *in the Theorem* 1.5 *by using type function* $U(x)$ *in* [11] *to enlarge the growth of the denominator* $-log\sigma$ *and obtain the following theorem.*

Theorem 1.6. *If Dirichlet series* $f(s) \in A$ *with fixed real number* $0 < \alpha < +\infty$, *we have*

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log^+ M(\sigma, f))}{\log U\left(\frac{1}{\sigma}\right)} = T \iff \limsup_{n\to\infty} \phi_n(f, \alpha, \lambda_n) = T,$$

where

$$\phi_n(f,\alpha,\lambda_n) = \frac{\gamma\left(\log^+ [E_{n-1}(f,\alpha)\exp(\alpha\lambda_n)]\right)}{\log U\left(\frac{\lambda_n}{\log^+[E_{n-1}(f,\alpha)\exp(\alpha\lambda_n)]}\right)}.$$

2 Lemmas

Lemma 2.1. [22] *Let $\gamma(x) \in \Im$ and $\psi(x)$ be the function such that*

$$\rho = \limsup_{x\to\infty} \frac{\log \psi(x)}{\log x}(0 \le \rho < \infty),$$

and if the real function $M(x)$ satisfies

$$\limsup_{x\to\infty} \frac{\gamma(\log M(x))}{\log x} = \nu(> 0),$$

then

$$\limsup_{x\to\infty} \frac{\gamma(\psi(x)\log M(x))}{\log x} = \nu$$

Proof. We consider two cases as follows:
Case I. If $\psi(x)$ is not a constant.
From the assumption of Lemma 2.1, we can get that $\psi(x) \to \infty$ as $x \to \infty$ and for sufficiently large x we have $\psi(x) > 1$. If $\gamma(x) \in \Im$ and $\lim_{x\to\infty} M(x) = \infty$ then from the Cauchy mean value theorem there exist ξ ($\log M(x) < \xi < \gamma(x)\log M(x)$) satisfying

$$\frac{\gamma(\psi(x)\log M(x)) - \gamma(\log M(x))}{\log \psi(x)\log M(x) - \log\log M(x)} = \frac{\gamma'(\xi)}{(\log \xi)'} = \xi\gamma'(\xi).$$

that is

$$\gamma(\psi(x)\log M(x)) = \gamma(\log M(x)) + \log \psi(x)\xi\gamma'(\xi), \tag{2.1}$$

Since, $x\gamma'(x) = o(1)$ as $x \to \infty$ and

$$\rho = \limsup_{x\to\infty} \frac{\log \psi(x)}{\log x}, (0 \le \rho < \infty)$$

by (2.1) we can get the conclusion of Lemma 2.1.
Case II. If $\psi(x)$ is constant.
By using the some argument as in Case I, we can prove that the conclusion of Lemma 2.1 is true. Thus this complete the proof of Lemma 2.1. The following Lemma 2.1 is very crucial in the study of the growth of analytic functions represented by Dirichlet series which shows that the relation between $M(\sigma, f)$ and $m(\sigma, f)$ of such functions.

□

Lemma 2.2. [21] *If Dirichlet series* (1.1) *satisfying* (1.2) − (1.3) *then for any given $\epsilon \in (0\ 1)$ and for $\sigma(> 0)$ sufficiently reaching zero, we have*

$$m(\sigma, f) \le M(\sigma, f) \le K(\epsilon)\left(\frac{1}{\sigma}\right)m((1-\epsilon), f).$$

where $K(\epsilon)$ is a positive constant depending on ϵ and $f(s)$.

3 The proof of Theorem 3

Proof. We first prove sufficient part ($\Longleftarrow$) of the theorem
Suppose

$$\limsup_{n\to\infty} \frac{\gamma(\log^+ |a_n|)}{\log U\left(\frac{\lambda_n}{\log^+ |a_n|}\right)} = T,$$

then for any positive real number $\tau > 0$ for sufficiently large n we have

$$\log^+ |a_n| < \beta\left((T+\tau)\log U\left(\frac{\lambda_n}{\log^+ |a_n|}\right)\right)$$

where $\beta(x)$ is the inverse of $\gamma(x)$
Let $V(x)$ is the inverse function of $U(x)$ then

$$\frac{\gamma(\log^+ |a_n|)}{T+\tau} < \log U\left(\frac{\lambda_n}{\log^+ |a_n|}\right).$$

that is

$$\log^+ |a_n| < \frac{\lambda_n}{V\left(\exp\left(\frac{\gamma(\log^+ |a_n|)}{T+\tau}\right)\right)} \qquad \Rightarrow \log^+ |a_n| < \lambda_n\left[V\left(\exp\left(\frac{\gamma(\log^+ |a_n|)}{T+\tau}\right)\right)\right]^{-1}.$$

Thus

$$\log^+ |a_n| e^{-\lambda_n \sigma} < \lambda_n\left[\left(V\left(\exp\left(\frac{\gamma(\log^+ |a_n|)}{T+\tau}\right)\right)\right)^{-1} - \sigma\right]. \tag{3.1}$$

For any fixed and sufficiently small $\sigma > 0$
Set

$$J = \beta\left[(T+\tau)\log U\left(\frac{1}{\sigma} + \frac{1}{\sigma\log U(\frac{1}{\sigma})}\right)\right]. \tag{3.2}$$

that is

$$\frac{1}{\sigma} + \frac{1}{\sigma\log U(\frac{1}{\sigma})} = V\left(\exp\left(\frac{\gamma(J)}{T+\tau}\right)\right). \tag{3.3}$$

If $\log A_n^* \leq J$ then for sufficiently $\sigma(>0)$ let $V\left(\exp\left(\frac{\gamma(I)}{T+\tau}\right)\right) \geq 1$ then from (3.2) and the properties of $U(x)$

$$\begin{aligned}
\log^+(|a_n| e^{-\lambda_n \sigma}) &\leq I\left[\left(V\left(\exp\left(\frac{\gamma(\log^+ |a_n|)}{T+\tau}\right)\right)\right)^{-1} - \sigma\right] \\
&\leq J = \beta\left((T+\tau)\log U\left(\frac{1}{\sigma} + \frac{1}{\sigma\log U(\frac{1}{\sigma})}\right)\right) \\
&\leq \beta\left((T+\tau)\log\left((1+o(1))U\left(\frac{1}{\sigma}\right)\right)\right).
\end{aligned} \tag{3.4}$$

If $\log^+ |a_n| > J$ then from (3.1) and (3.2), we have

$$\log(|a_n|e^{-\lambda_n\sigma}) \leq \lambda_n \left(\left(V\left(\exp\left(\frac{\gamma(\log^+ |a_n|)}{T+\tau}\right)\right)\right)^{-1} - \sigma\right)$$

$$\leq \lambda_n \left(\left(V\left(\exp\left(\frac{\gamma(J)}{T+\tau}\right)\right)\right)^{-1} - \sigma\right)$$

$$< 0. \tag{3.5}$$

From (3.3) and (3.4)

$$\log\ m(\sigma, f) \leq J\left((T+\tau)\log\left((1+o(1))U\left(\frac{1}{\sigma}\right)\right)\right)$$

$$\leq J\left((T+\tau)\log U\left(\frac{1}{\sigma}\right)\right). \tag{3.6}$$

Since τ is arbitrary, from (3.6) and Lemma 2.2

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log M(\sigma, f))}{\log U(\frac{1}{\sigma})} \leq \limsup_{n\to\infty} \frac{\gamma(\log^+ |a_n|)}{\log U(\frac{\lambda_n}{\log^+ |a_n|})} = T.$$

Now let if possible

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log M(\sigma, f))}{\log U(\frac{1}{\sigma})} < T. \tag{3.7}$$

Then there exist a real number $\epsilon (0 < \epsilon < \frac{T}{2})$ for any positive number n and sufficiently small $\sigma > 0$, from Lemma 2.2, we have

$$\log^+(|a_n|e^{-\lambda_n\sigma}) \leq \log M(\sigma, f) \leq \beta\left((T-2\epsilon)\log U\left(\frac{1}{\sigma}\right)\right). \tag{3.8}$$

From (3.7), there exist a subsequence $\{n_\nu\}$ we have

$$\gamma(\log^+ |a_{n_\nu}|) > (T-2\epsilon)\log U\left(\frac{\lambda_{n_\nu}}{\log^+ |a_{n_\nu}|}\right). \tag{3.9}$$

Choose a sequence $\{\sigma_\nu\}$ satisfying

$$\beta\left((T-2\epsilon)\log U\left(\frac{1}{\sigma_\nu}\right)\right) = \frac{\log^+ |a_{n_\nu}|}{1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ |a_{n_\nu}|}\right)}. \tag{3.10}$$

From (3.8)

$$\log^+ |a_{n_\nu}| - \lambda_{n_\nu}\sigma_\nu \ \leq\ \beta\left((T-2\epsilon)\log U\left(\frac{1}{\sigma_\nu}\right)\right) = \frac{\log^+ |a_{n_\nu}|}{1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ |a_{n_\nu}|}\right)},$$

that is

$$\frac{1}{\sigma_\nu} \ \leq\ \frac{\lambda_{n_\nu}}{\log^+ |a_{n_\nu}|}\left(1 + \frac{1}{\log U\left(\frac{\lambda_{n_\nu}}{\log |a_{n_\nu}|}\right)}\right).$$

Thus

$$U\left(\frac{1}{\sigma_\nu}\right) \le U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\left(1+\frac{1}{\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)}\right)\right).$$

$$\le U^{(1+o(1))}\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right). \tag{3.11}$$

From (3.10) and (3.11), we have

$$\begin{aligned}\log^+|a_{n_\nu}| &= \beta\left((T-2\epsilon)\log U\left(\frac{1}{\sigma_\nu}\right)\right)\left(1+\log U\left(\frac{\lambda_{n\nu}}{\log^+|a_{n_\nu}|}\right)\right)\\ &= \beta\left((T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)\right)\left(1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)\right).\end{aligned}$$

Thus from the Cauchy mean value theorem and there exist a real number ξ between x_1 and x_2, where

$$x_1 = \beta\left((T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)\right)$$

and

$$x_2 = \beta\left((T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)\right)\left(1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)\right).$$

such that

$$\begin{aligned}&\gamma(\log^+|a_{n_\nu}|)\\ &= \gamma\left\{\left\{1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)\right\}\beta\left\{(T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)\right\}\right\}\\ &= \gamma\left(\beta\left((T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)\right)\right)+\log\left(1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)\right)\xi\gamma'(\xi).\end{aligned}$$

Since

$$\lim_{\nu\to\infty}\frac{\log\left(1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)\right)}{\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)} = 0.$$

Then for sufficiently large ν

$$\gamma(\log^+|a_{n_\nu}|) = (T-2\epsilon)(1+o(1))\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right)+K_1\xi\gamma'(\xi)\log U\left(\frac{\lambda_{n_\nu}}{\log^+|a_{n_\nu}|}\right), \tag{3.12}$$

where K_1 is a constant.
From (3.8) and (3.12), we get a contradiction.
Thus

$$\limsup_{\sigma\to 0^+}\frac{\gamma(\log M(\sigma,f))}{\log U\left(\frac{1}{\sigma}\right)} = T.$$

Hence, the sufficiency of Theorem 3 is completed.

□

Similarly we can prove the necessary part of Theorem 3 by using similar argument as in the proof of sufficiency of Theorem 3. Thus, the proof of Theorem 3 is completed.

4 The proof of Theorem 6

Proof. We prove the conclusion of the Theorem 6 by using the properties of two functions $\gamma(x)$ and $U(x)$, this method is different from the previous method to some extent.
We first prove sufficient part ($\Longleftarrow$) of the Theorem 6
Suppose that

$$\lim_{n\to\infty}\phi_n(f,\alpha,\lambda_n)=\limsup_{n\to\infty}\frac{\gamma(\log^+(E_{(n-1)}(f,\alpha)e^{\alpha\lambda_n})}{\log U\left(\frac{\lambda_n}{\log^+(E_{n-1}(f,\alpha)e^{\alpha\lambda_n})}\right)}=T. \tag{4.1}$$

Let $A_n=E_{(n-1)}(f,\alpha)e^{\alpha\lambda_n}, n=1,2,\ldots,$ then for any positive real number $\epsilon>0$, for sufficiently large n

$$\log^+ A_n<\beta\left\{(T+\epsilon)\log U\left(\frac{\lambda_n}{\log^+ A_n}\right)\right\}.$$

where $\beta(x)$ is the inverse function of $\gamma(x)$. Let $V(x)$ and $U(x)$ be two reciprocally inverse functions, then we have

$$V\left(\exp\left\{\frac{\gamma(\log^+ A_n)}{T+\epsilon}\right\}\right)<\frac{\lambda_n}{\log^+ A_n},$$

and hence

$$\log^+ A_n\le\lambda_n\left(V\left(\exp\left\{\frac{\gamma(\log^+ A_n)}{T+\epsilon}\right\}\right)\right)^{-1}.$$

Thus we have

$$\log^+ A_ne^{-\lambda_n\sigma}\le\lambda_n\left[\left(V\left(\exp\left(\frac{\gamma(\log^+ A_n)}{T+\epsilon}\right)\right)\right)^{-1}-\sigma\right]. \tag{4.2}$$

For any fixed and sufficiently small $\sigma>0$, set

$$I=\beta\left[(T+\epsilon)\log U\left(\frac{1}{\sigma}+\frac{1}{\sigma\log U(\frac{1}{\sigma})}\right)\right],$$

that is,

$$\frac{1}{\sigma}+\frac{1}{\sigma\log U(\frac{1}{\sigma})}=V\left(\exp\left(\frac{\gamma(I)}{T+\epsilon}\right)\right) \tag{4.3}$$

If $\log^+ A_n\le I$, then for sufficiently large n, let

$$V\left(\exp\left(\frac{\gamma(\log^+ A_n)}{T+\epsilon}\right)\right)\ge 1,$$

for $\sigma > 0$, and follows from (4.2) and (4.3), and the definition of $U(x)$

$$\log^+(A_n e^{-\lambda_n \sigma}) \leq \lambda_n \left[\left(V \left(\exp\left(\frac{\gamma(\log^+ A_n)}{T+\epsilon} \right) \right) \right)^{-1} - \sigma \right]$$

$$\leq I = \beta\left((T+\epsilon) \log U \left(\frac{1}{\sigma} + \frac{1}{\sigma \log U(\frac{1}{\sigma})} \right) \right)$$

$$\leq \beta\left((T+\epsilon) \log \left((1+o(1)) U\left(\frac{1}{\sigma} \right) \right) \right). \tag{4.4}$$

If $\log^+ A_n > I$ then from (4.2) and (4.3)

$$\log(A_n e^{-\lambda_n \sigma}) \leq \lambda_n \left(\left(V \left(\exp\left(\frac{\gamma(\log^+ |a_n|)}{T+\epsilon} \right) \right) \right)^{-1} - \sigma \right)$$

$$\leq \lambda_n \left(\left(V \left(\exp\left(\frac{\gamma(J)}{T+\epsilon} \right) \right) \right)^{-1} - \sigma \right)$$

$$< 0. \tag{4.5}$$

For sufficiently large n from (4.4) and (4.5)

$$\log(A_n e^{-\lambda_n \sigma}) \leq \beta \left\{ (T+\epsilon) \log \left((1+o(1)) U\left(\frac{1}{\sigma} \right) \right) \right\}.$$

From the definition of $E_n(f, \alpha)$, there exists $p(s) \in \prod_{n-1}$ and a real constant $K > 1$ such that

$$\|f - p\|_\alpha \leq K E_{n-1}(f, \alpha). \tag{4.6}$$

Since $f(s) \in \overline{A}_\alpha$ and from [10], for any real number t_0, $v (\neq 0)$,

$$\lim_{R\to+\infty} \frac{1}{R} \int_{t_0}^{R} e^{vit} dt = 0, a_n e^{-\alpha\lambda_n} = \lim_{R\to+\infty} \frac{1}{R} \int_{t_0}^{R} f(\alpha+it) e^{\lambda_n it} dt. \tag{4.7}$$

From (4.7) for any real number $x \neq 0$

$$\lim_{R\to+\infty} \frac{1}{R} \int_{t_0}^{R} e^{x(\alpha+it)} dt = 0. \tag{4.8}$$

Thus from (4.7) and (4.8), for any $p_1(s) \in \prod_{n-1}$

$$a_n e^{-\alpha\lambda_n} = \lim_{R\to+\infty} \frac{1}{R} \int_{t_0}^{R} [f(\alpha+it) - p_1(\alpha+it)] e^{\lambda_n it} dt.$$

that is

$$|a_n| e^{-\alpha\lambda_n} \leq \|f - p_1\|_\alpha. \tag{4.9}$$

From (4.6) and (4.9), we get

$$|a_n| e^{-\alpha\lambda_n} \leq K E_{n-1}(f, \alpha) \tag{4.10}$$

Since $A_n = E_{n-1}(f,\alpha)e^{\alpha\lambda_n}$ and ϵ is arbitrary from (4.10) by the Lemma 2.1 and the Theorem 1.2 we get

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log M(\sigma, f))}{\log U(\frac{1}{\sigma})} = T.$$

Suppose that

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log^+ M(\sigma, f))}{\log U\left(\frac{1}{\sigma}\right)} = \zeta < T.$$

Thus, there exist any real number $\delta(0 < \delta < \frac{\zeta}{2})$, and for any sufficiently small $\sigma > 0$ from Lemma 2.2 we have

$$\log^+(M(\sigma, f)) \leq \beta\left((\zeta - 2\delta)\log U\left(\frac{1}{\sigma}\right)\right). \tag{4.11}$$

For any sufficiently small $\sigma > 0$, and $0 < \alpha < \sigma < \infty$

$$E_{n-1}(f,\alpha) \leq \|f - p_{n-1}\| \leq \sum_{k=n}^{\infty} |a_k| e^{-\alpha\lambda_k} \leq M(\sigma, f)\sum_{k=n}^{\infty} e^{-\lambda_n(\alpha-\sigma)}, \tag{4.12}$$

where $p_{n-1}(s) = \sum_{k=1}^{n-1} a_k e^{s\lambda_k}$ From (1.3), we take $0 < b' < b$ satisfies $\lambda_{n+1} - \lambda_n \geq b'$ for a subsequence of $\{n\}$.Thus for sufficiently small $\sigma > 0$ such that $\sigma \leq \frac{\alpha}{2}$ from (4.12)

$$\begin{aligned}
E_{n-1}(f,\alpha) &\leq M(\sigma, f)e^{-\lambda_n(\alpha-\sigma)}\sum_{k=n}^{\infty} e^{-(\lambda_k-\lambda_n)(\alpha-\sigma)} \\
&\leq M(\sigma, f)e^{-\lambda_n(\alpha-\sigma)}e^{\frac{\alpha}{2}nb'}\sum_{k=n}^{\infty} e^{-\frac{\alpha}{2}kb'} \\
&\leq M(\sigma, f)e^{-\lambda_n(\alpha-\sigma)}\left(1 - e^{\frac{\alpha}{2}b'}\right)^{-1} \\
&\leq M(\sigma, f)e^{-\lambda_n(\alpha-\sigma)}K_2^{-1}
\end{aligned}$$

or

$$M(\sigma, f) \geq K_2 E_{n-1}(f,\alpha)e^{\lambda_n(\alpha-\sigma)}$$

or

$$M(\sigma, f) \geq K_2 A_n e^{\lambda_n\sigma} \tag{4.13}$$

where $K_2 = \left(1 - e^{\frac{\alpha}{2}b'}\right)$ from (4.11) and (4.13) nothing the properties of the function

$$\begin{aligned}
\log^+ A_n e^{\lambda_n\sigma} &\leq \log^+(M(\sigma, f)) \\
&\leq \beta\left((\zeta - 2\delta)\log U\left(\frac{1}{\sigma}\right)\right).
\end{aligned} \tag{4.14}$$

From (4.1) there exist a subsequence $\{\lambda_{n_\nu}\}$ for sufficiently large ν we have

$$\gamma(\log^+ A_{n_\nu}) > (T - \delta)\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right). \tag{4.15}$$

Take a sequence $\{\sigma_\nu\}$ satisfying

$$\beta\left((\zeta-2\delta)\log U\left(\frac{1}{\sigma}\right)\right)=\frac{\log^+ A_{n_\nu}}{1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)}. \tag{4.16}$$

From equation (4.14) and (4.16), we get

$$\log^+ A_{n_\nu}-\lambda_{n_\nu}\le\beta\left((\zeta-2\delta)\log U\left(\frac{1}{\sigma_\nu}\right)\right)\le\frac{\log^+ A_{n_\nu}}{1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)}. \tag{4.17}$$

that is

$$\frac{1}{\sigma_\nu}\le\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\left(1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)\right).$$

Thus

$$\begin{aligned}U\left(\frac{1}{\sigma_\nu}\right)&\le U\left\{\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\left(1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)\right)\right\}\\&\le U^{(1+o(1))}\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)\end{aligned} \tag{4.18}$$

From equation (4.17) and (4.18)

$$\log^+ A_{n_\nu}=\beta\left((\zeta-2\delta)\log U\left(\frac{1}{\sigma_\nu}\right)\right)\left(1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)\right)$$

Thus from the Cauchy mean value theorem, there exist a real number ζ between

$$\beta\left((\zeta-2\delta)\log U\left(\frac{1}{\sigma_\nu}\right)\right)\quad\text{and}\quad\beta\left((\zeta-2\delta)\log U\left(\frac{1}{\sigma_\nu}\right)\right)\left(1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)\right)$$

such that

$$\begin{aligned}\gamma(\log^+ A_{n_\nu})&=\gamma\left(\beta\left((\zeta-2\delta)\log U\left(\frac{1}{\sigma_\nu}\right)\right)\left(1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)\right)\right)\\&=\gamma\left\{\beta\left\{(\zeta-2\delta)(1+o(1))\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)\right\}\right\}\\&\quad+\log\left\{1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)\right\}\xi\gamma'(\xi).\end{aligned}$$

Since

$$\lim_{\nu\to\infty}\frac{\log\left\{1+\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)\right\}}{\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right)}=0.$$

Then for sufficiently large ν

$$\gamma(\log^+ A_{n_\nu})=(\zeta-2\delta)\log U\left(\frac{1}{\sigma_\nu}\right)+K_2\xi\gamma'(\xi)\log U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}}\right). \tag{4.19}$$

where K_2 is a constant. From (4.16) and (4.19) and $0 < \delta < \frac{\zeta}{2}$ we get contradiction. Thus we can get

$$\limsup_{\sigma\to 0^+} \frac{\gamma(\log^+ M(\sigma, f))}{\log U(\frac{1}{\sigma})} = T.$$

Hence the sufficient part of Theorem 1.6 is completed. We can prove the necessary part of Theorem 1.6 by using the similar argument as in the proof of the sufficient part of the Theorem 1.6. Thus the proof of the Theorem 1.6 is completed. □

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