New Class of Weakly Berwald Space with a Special (α,β)-metric

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Abstract

As a generalization of Berwald spaces, we have the ideas of Douglas spaces and Landsberg spaces. S. Bacso defined a weakly-Berwald space as another generalization of Berwald spaces. Matsumoto developed the concepts of (α,β)-metric in 1972, which is a Finsler metric generated from a Riemannian metric α and a differential 1-form β. In this paper, we investigated an important class of (α,β)-metrics of the form $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^2}{\beta}$, which is recognized as a special form of (α,β)-metric on an n-dimensional manifold, and we obtain the criteria for such metrics to be weakly-Berwald metrics. A Finsler space with a special (α,β)-metric is a weakly Berwald space if and only if B^m_m is a 1-form. We have shown that under certain geometric and algebraic circumstances, it transforms into a weakly Berwald space.

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1 Introduction

In 1972 [7], Matsumoto introduced the concept of a (α,β)-metric on a Finsler space $F^n = (M^n, F)$ and it has been studied by numerous authors [2], [8], [11]. The study of several well-known metrics, such as the Randers metric and the Kropina metric, has significantly contributed to the expansion of Finsler geometry and its applications to relativity theory. A Finsler metric $F(x,y)$ is known as (α,β)-metric, if F is a positively homogeneous function of α and β of degree one, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and 1-form $\beta = b_i(x)y^i$ on M^n.

Let $F^n = (M^n, F)$ be an n-dimensional Finsler space where M^n be an n-dimensional differential manifold and fundamental function F. Let the fundamental tensor $g_{ij} = \dot{\partial}_i\dot{\partial}_j\frac{F^2}{2}$, where $\dot{\partial}_i$ represents $\frac{\partial}{\partial y^i}$ and we define G_i as follows

$$G_i = \frac{1}{4}\left(y^r(\partial_r\dot{\partial}_iF^2) - \partial_iF^2\right),$$

and $G^i = g^{ij}G_j$ here ∂_i means $\frac{\partial}{\partial x^i}$ and g^{ij} is inverse of g_{ij} fundamental tensor. The co-efficients of (G^i_{jk}, G^i_j) of the Berwald connection $B\Gamma$ are determined as $G^i_{jk} = \dot{\partial}_kG^i_j$ and $G^i_j = \dot{\partial}_jG^i$. A Berwald space is a Finsler space that satisfy the criterion $G^h_{ijk} = 0$, which means that the Berwald connection's co-efficients G^h_{ij} are functions of the position (x^i) alone. Thus the equation $y_r G^r_{ijk} = 0$ holds, so $2G^i = G^i_{rs} y^r y^s$ are homogeneous polynomials of degree two in (y^i), so $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials of degree three in (y^i). Then, as two extensions of Berwald spaces, we can study the concepts of Landsberg spaces and Douglas spaces. The third extension of Berwald spaces is the concept of weakly-Berwald spaces. As a result, if a Finsler space satisfies the criterion $G_{ij} = 0$, it is referred to as a weakly-Berwald space.

Berwald space is a Finsler space, if G^i_{jk} are the functions of position alone, that is, Berwald connection $B\Gamma$ is linear. If the (hv)-Ricci curvature tensor $G_{jk} = 0$, a Finsler space is said to be a weakly Berwald space. The spray functions G^i of a Finsler space with an (α,β)-metric are given by $2G^i = \gamma^i_{00} + 2B^i$, where γ^i_{jk} represents the Christoffel symbols in the associated Riemannian space (M^n, α). Then we have $G^i_{jk} = B^i_{jk} + \gamma^i_{jk}$ and $G^i_j = B^i_j + \gamma^i_{0j}$, where $\dot{\partial}_k B^i_j = B^i_{jk}$ and $\dot{\partial}_j B^i = B^i_j$. Then a Finsler space with an (α,β)-metric is a weakly Berwald space if and only if $B^m_m = \dfrac{\partial B^m}{\partial y^m}$ is a one-form.

In [1], S. Bacso proposed the concept of weakly-Berwald space as another extension of Berwald spaces as well as a necessary condition for the existence of a weakly Berwald Finsler space of Kropina type. IL-Yong Lee have investigated weakly Berwald spaces with special (α,β)-metric in [6]. In 2004, R. Yoshikawa [13] developed the conditions for generalised Kropina and Matsumoto spaces to be weakly-Berwald and Berwald spaces, respectively. In [12], Akbar Tayebi obtained a new class of weakly Berwald Finsler metric. Gauree Shanker has obtained the conditions for Finsler space with a second approximate Matsumoto metric to be weakly Berwald space in [10]. In [9], Narasimhamurthy has proved that under some conditions, a Finsler space with special (α,β)-metric becomes a weakly-Berwald space. Pradeep Kumar et al. [4], have proved the Finsler space with special (α,β)-metric to be Berwald space. Recently, Khoshdani [3] has discussed the characteristics of weakly Berwald space for fourth-root (α,β)-metric.

In this paper, we extend the study on weakly Berwald spaces with a special metric. We proposed a special (α,β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\alpha^2}{\beta}$, where μ_1, μ_2 and μ_3 are constants in [5]. We can clearly observe that there may occur special cases for certain given conditions as follows: For constants $\mu_1 = 1$, $\mu_2 = 1$ and $\mu_3 = 0$, the special metric becomes $F = \alpha + \beta$ and is known as Randers metric. Similarly, for constants $\mu_1 = 0$, $\mu_2 = 0$ and $\mu_3 = 1$, then special metric becomes $F = \dfrac{\alpha^2}{\beta}$ and is known as Kropina metric.

Firstly, we gave a brief introduction to Berwald and Weakly-Berwald space in section one. We discussed the basic notations and conditions for a Finsler space F^n with an (α,β)-metric to be a weakly Berwald space in section two. Finally, we have obtained the conditions for Finsler space with a special form of (α,β)-metric, F to be Weakly Berwald space.

2 Weakly-Berwald space with respect to (α,β)-metric

The conditions for a Finsler space with a (α,β)-metric to be a weakly-Berwald space is discussed in this section.
Let $F^n = (M^n, F)$ be a Finsler space defined on n-dimensional differential manifold equipped with (α,β)-metric $F(\alpha,\beta)$, where Riemannian metric $\alpha^2 = a_{ij}(x)y^i y^j$ and one-form $\beta = b_i(x)y^i$. The symbol (;) in this paper stands for h-covariant derivation in the space (M,α) with regard to the Riemannian connection, while γ^i_{jk} stands for Christoffel symbols in the space (M,α). The notations are as follows [1]

$$\begin{aligned}
&i. \quad b^i = a^{ir}b_r, \quad b^2 = a^{rs}b_r b_s,\\
&ii. \quad 2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i},\\
&iii. \quad r^i_j = a^{ir}r_{rj}, \quad s^i_j = a^{ir}s_{rj}, \quad r_i = b_r r^r_i, \quad s_i = b_r s^r_i.
\end{aligned}$$

Now we consider the function $G^i(x,y)$ of F^n with an (α,β)-metric. According to [7], they are written in the form

$$
\begin{aligned}
2G^m &= 2B^m + \gamma^m_{00},\\
B^m &= \frac{\alpha F_\beta}{F_\alpha} s^m_0 + \frac{E^*}{\alpha} y^m - \frac{\alpha F_{\alpha\alpha}}{F_\alpha}\left(\frac{1}{\alpha}y^m - \frac{\alpha}{\beta}b^m\right)C^*,
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{aligned}
C^* &= \frac{\alpha\beta\left(r_{00}F_\alpha - 2\alpha s_0 F_\beta\right)}{2\left(\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha}\right)},\\
\gamma^2 &= b^2\alpha^2 - \beta^2, \quad E^* = \left(\frac{\beta F_\beta}{F}\right)C^*,
\end{aligned}
\tag{2.2}
$$

and

$$
F_\alpha = \frac{\partial F}{\partial \alpha}, \quad F_\beta = \frac{\partial F}{\partial \beta}, \quad F_{\alpha\alpha} = \frac{\partial^2 F}{\partial \alpha^2}, \quad F_{\alpha\beta} = \frac{\partial^2 F}{\partial\alpha\partial\beta}, \quad F_{\alpha\alpha\alpha} = \frac{\partial^3 F}{\partial \alpha^3}.
\tag{2.3}
$$

Since a homogeneous polynomials $\gamma^i_{00} = \gamma^i_{jk}(x)y^jy^k$ in $\left(y^i\right)$ of degree two, on the other hand, it is well-known that a Finsler space with an $(\alpha,\beta)-$ metric is a Berwald space, if and only if a homogeneous polynomials B^m in $\left(y^i\right)$ of degree two and Berwald connection $B\Gamma$ is linear.
Differentiating equation (2.1) by y^n and contracting m and n in the obtained equation, we get

$$
\begin{aligned}
B^m_m = &\left\{\dot\partial_m\left(\frac{\beta F_\beta}{\alpha F}\right)y^m + \frac{n\beta F_\beta}{\alpha F} - \dot\partial_m\left(\frac{\alpha F_{\alpha\alpha}}{F_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right)\right\}C^*\\
&- \frac{\alpha F_{\alpha\alpha}}{F_\alpha}\left\{\dot\partial_m\left(\frac{1}{\alpha}\right)y^m + \frac{1}{\alpha}\delta^m_m - \dot\partial_m\left(\frac{\alpha}{\beta}\right)b^m\right\}C^* + \dot\partial_m\left(\frac{\alpha F_\beta}{F_\alpha}\right)s^m_0\\
&+ \left(\frac{\beta F_\alpha F_\beta - \alpha F F_{\alpha\alpha}}{\alpha F F_\alpha}\right)\left(\dot\partial_m C^*\right)y^m + \left(\frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha}\right)\left(\dot\partial_m C^*\right)b^m.
\end{aligned}
\tag{2.4}
$$

Since $F = F(\alpha,\beta)$ is a positively homogeneous function of α and β of degree one, we have

$$
\begin{aligned}
F_\alpha\alpha + F_\beta\beta = F, \quad F_{\alpha\alpha}\alpha + F_{\alpha\beta}\beta = 0,\\
F_{\beta\alpha}\alpha + F_{\beta\beta}\beta = 0, \quad F_{\alpha\alpha\alpha}\alpha + F_{\alpha\alpha\beta}\beta = -F_{\alpha\alpha}.
\end{aligned}
\tag{2.5}
$$

Using the above and the homogeneity of $\left(y^i\right)$, we obtain the following

$$
\dot\partial_m\left(\frac{\beta F_\beta}{\alpha F}\right)y^m = -\frac{\beta F_\beta}{\alpha F},
\tag{2.6}
$$

$$
\dot\partial_m\left(\frac{\alpha F_{\alpha\alpha}}{F_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right) = \frac{\gamma^2}{(\beta F_\alpha)^2}\left\{F_\alpha F_{\alpha\alpha} + \alpha F_\alpha F_{\alpha\alpha\alpha} - \alpha\left(F_{\alpha\alpha}\right)^2\right\},
\tag{2.7}
$$

$$
\left\{\dot\partial_m\left(\frac{1}{\alpha}\right)y^m + \frac{1}{\alpha}\delta^m_m - \dot\partial_m\left(\frac{\alpha}{\beta}\right)b^m\right\} = \frac{1}{\alpha\beta^2}\left\{\gamma^2 + (n-1)\beta^2\right\},
\tag{2.8}
$$

$$(2.9)\qquad \left(\dot{\partial}_m C^*\right) y^m = 2C^*,$$

$$(2.10)\qquad \begin{aligned}\left(\dot{\partial}_m C^*\right) b^m =& \frac{1}{2\alpha\beta\Omega^2}\left[\Omega\{\beta\left(\gamma^2+2\beta^2\right)W + 2\alpha^2\beta^2 F_\alpha r_0 - \alpha\beta\gamma^2 F_{\alpha\alpha} r_{00} - 2\alpha(\beta^3 F_\beta\right.\\ &\left.+\alpha^2\gamma^2 F_{\alpha\alpha})s_0\} - \alpha^2\beta W\left\{2b^2\beta^2 F_\alpha - \gamma^4 F_{\alpha\alpha\alpha} - b^2\alpha\gamma^2 F_{\alpha\alpha}\right\}\right],\end{aligned}$$

$$(2.11)\qquad \dot{\partial}_m\left(\frac{\alpha F_\beta}{F_\alpha}\right)s_0^m = \frac{\alpha^2 F F_{\alpha\alpha} s_0}{(\beta F_\alpha)^2},$$

where

$$(2.12)\qquad \begin{aligned} &W = \left(r_{00}F_\alpha - 2\alpha s_0 F_\beta\right),\\ &\Omega = \left(\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha}\right), \quad \text{provided that } \Omega \neq 0\Big)\\ &Y_i = a_{ir}y^r, \quad s_{00}=0, \quad b^r s_r = 0, \quad a^{ij}s_{ij}=0.\end{aligned}$$

Substituting (2.2) to (2.3) and (2.5) to (2.10) into (2.4), we get

$$(2.13)\qquad B^m_m = \frac{1}{2\alpha F\left(\beta F_\alpha\right)^2\Omega^2}\left\{2\Omega^2 A C^* + 2\alpha F\Omega^2 B s_0 + \alpha^2 F F_\alpha F_{\alpha\alpha}\left(C r_{00} + D s_0 + E r_0\right)\right\},$$

where

$$\begin{aligned} A =&(t+1)\beta^2 F_\alpha\left(\beta F_\alpha F_\beta - \alpha F F_{\alpha\alpha}\right) + \alpha\gamma^2 F\left\{\alpha\left(F_{\alpha\alpha}\right)^2 - 2F_\alpha F_{\alpha\alpha} - \alpha F_\alpha F_{\alpha\alpha\alpha}\right\},\\ B =&\alpha^2 F F_{\alpha\alpha},\\ C =&\beta\gamma^2\left\{-\beta^2\left(F_\alpha\right)^2 + 2b^2\alpha^3 F_\alpha F_{\alpha\alpha} - \alpha^2\gamma^2\left(F_{\alpha\alpha}\right)^2 + \alpha^2\gamma^2 F_\alpha F_{\alpha\alpha\alpha}\right\},\\ D =&2\alpha\left\{\beta^3\left(\gamma^2-\beta^2\right)F_\alpha F_\beta - \alpha^2\beta^2\gamma^2 F_\alpha F_{\alpha\alpha} - 2\alpha\beta\gamma^2\left(\gamma^2+2\beta^2\right)F_\beta F_{\alpha\alpha}\right.\\ &\left.-\alpha^3\gamma^4\left(F_{\alpha\alpha}\right)^2 - \alpha^2\beta\gamma^4 F_\beta F_{\alpha\alpha\alpha}\right\},\\ E =&2\alpha^2\beta^2 F_\alpha\Omega.\end{aligned}$$

After summarizing the above, we have

Theorem 2.1. *A Finsler space F^n with an special (α,β)-metric is a weakly-Berwald space if $G^m_m = B^m_m + \gamma^m_{0m}$ is a homogeneous polynomial in (y^m) of degree one, where B^m_m is given by* (2.12) *and* (2.13)*, provided that $\Omega \neq 0$.*

Lemma 2.1. *[2] If $\alpha^2 \equiv 0(\bmod \beta)$, that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension n is equal to 2 and b^2 vanishes. In this case we have 1 -form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

3 Finsler space with a special (α,β)-metric

In this section, we have investigate the Finsler space with the special (α,β)-metric to be a weakly Berwald space. Let us consider $F^n = (M^n, F)$ be a Finsler space with a special (α,β)-metric

$$(3.1)\qquad F(\alpha,\beta) = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^2}{\beta},$$

where μ_1, μ_2 and μ_3 are constants. The conditions for F^n with the metric (3.1) being a weakly Berwald space have now been determined. For F^n with metric (3.1), we have

$$F_\alpha = \mu_1 + 2\mu_3\frac{\alpha}{\beta}, \quad F_\beta = \mu_2 - \mu_3\frac{\alpha^2}{\beta^2}, \quad F_{\alpha\alpha} = \frac{2\mu_3}{\beta}, \quad F_{\alpha\alpha\alpha} = 0. \tag{3.2}$$

Substituting (3.2) into (2.1), we have

$$\begin{aligned} B^m =& \frac{\left(\mu_2\beta^2 - \mu_3\alpha^2\right)\{r_{00}\left(\mu_1\beta^2 + 2\mu_3\alpha\beta\right) - 2\alpha s_0\left(\mu_2\beta^2 - \mu_3\alpha^2\right)\}}{2\left(\mu_3\alpha^2 + \mu_1\alpha\beta + \mu_2\beta^2\right)\left[\mu_1\beta^3 + 2b^2\mu_3\alpha^3\right]}y^m \\ &- \frac{2\mu_3\{r_{00}\beta\left(\mu_1\beta + 2\mu_3\alpha\right) - 2\alpha s_0\left(\mu_2\beta^2 - \mu_3\alpha^2\right)\}}{\left(\mu_1\beta + 2\mu_3\alpha\right)\left[\mu_1\beta^3 + 2b^2\mu_3\alpha\right]}y^m + \frac{\alpha\left(\mu_2\beta^2 - \mu_3\alpha^2\right)}{\beta\left(\mu_1\beta + 2\mu_3\alpha\right)}s_0^m \\ &+ \frac{2\mu_3\alpha^3\left[r_{00}\beta\left(\mu_1\beta + 2\mu_3\alpha\right) - 2\alpha s_0\left(\mu_2\beta^2 - \mu_3\alpha^2\right)\right]}{\beta\left(\mu_1\beta + 2\mu_3\alpha\right)\left[\mu_1\beta^3 + 2b^2\mu_3\alpha^3\right]}b^m. \end{aligned} \tag{3.3}$$

and again substituting (3.2) into (2.2), (2.4) and (2.13) in respective quantities, we get

$$\begin{aligned} A =& \frac{4\mu_3\gamma^2\alpha}{\beta^3}\left(\mu_1\beta + \mu_3\alpha\right)\left(\mu_3\alpha^2 + \mu_1\alpha\beta + \mu_2\beta^2\right) - \frac{(t+1)}{\beta}\left(\mu_1\beta + 2\mu_3\alpha\right) \\ &\left(4\mu_3^2\alpha^3 + 3\mu_1\mu_2\alpha^2\beta - \mu_1\mu_2\beta^3\right), \\ B =& \frac{2\mu_3\alpha^2\left(\mu_1\alpha\beta + \mu_2\beta^2 + \mu_3\alpha^2\right)}{\beta^2}, \\ C =& \frac{-\gamma^2}{\beta}\left[-4\mu_1\mu_3b^2\alpha^3\beta - 4\mu_3^2b^2\alpha^4 + \mu_1^2\beta^4 + 4\mu_1\mu_3\alpha\beta^3\right], \\ D =& \left[4\mu_3^2b^2\alpha^6 - 8\mu_2\mu_3b^4\alpha^6 - 6\mu_1\mu_3b^2\alpha^8\beta + 4\mu_2\mu_3b^2\alpha^4\beta^2 + 2\mu_1\mu_2b^2\alpha^3\beta^3\right. \\ &\left. + 8\mu_1\mu_3\alpha^3\beta^3 - 4\mu_1\mu_2\alpha\beta^5\right], \\ E =& 2\alpha^2\left(\mu_1\beta^3 + 2\mu_3b^2\alpha^3\right)\left(\mu_1\beta + 2\mu_3\alpha\right), \\ \Omega =& \frac{1}{\beta}\left(\mu_1\beta^3 + 2\mu_3b^2\alpha^3\right), \\ W =& \frac{1}{\beta^2}\left[r_{00}\beta\left(\mu_1\beta + 2\mu_3\alpha\right) - 2\alpha s_0\left(\mu_2\beta^2 - \mu_3\alpha^2\right)\right], \\ C^* =& \frac{\alpha\left[r_{00}\beta\left(\mu_1\beta + 2\mu_3\alpha\right) - 2\alpha s_0\left(\mu_2\beta^2 - \mu_3\alpha^2\right)\right]}{2\left(\mu_1\beta^3 + 2\mu_3b^2\alpha^3\right)}, \\ E^* =& \frac{\alpha\left(\mu_2\beta^2 - \mu_3\alpha^2\right)\left[r_{00}\beta\left(\mu_1\beta + 2\mu_3\alpha\right) - 2\alpha s_0\left(\mu_2\beta^2 - \mu_3\alpha^2\right)\right]}{2\left(\mu_1\alpha\beta + \mu_2\beta^2 + \mu_3\alpha^2\right)\left(\mu_1\beta^3 + 2\mu_3b^2\alpha^3\right)}. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (2.12), we get

$$\frac{B^m_m}{\beta^3}\left[2\alpha\left(\mu_1\beta^3+2\mu_3b^2\alpha^3\right)^2(\mu_1\beta+2\mu_3\alpha)^2\left(\mu_1\alpha\beta+\mu_2\beta^2+\mu_3\alpha^2\right)\right]-\frac{1}{\beta^5}[-\alpha$$
$$\left(\mu_1\beta^3+2\mu_3b^2\alpha^3\right)\left(2\mu_2s_0\alpha^3-2\mu_2s_0\alpha\beta^2+2\mu_3r_{00}\alpha\beta+\mu_2r_{00}\beta^2\right)\{4\mu_3b^2\alpha^6-$$
$$\mu_1^2\mu_2\beta^6+4\mu_3^3\alpha^4\beta^2+12\mu_1\mu_3^2\alpha^3\beta^3-\mu_1^2\mu_3\alpha^2\beta^4-4\mu_2\mu_3^2\alpha^2\beta^4+8\mu_3t\alpha^4\beta^2-\mu_1^2$$
$$\mu_2t\beta^6+4\mu_1^2\mu_3b^2\alpha^4\beta^2+4\mu_2\mu_3^2b^2\alpha^4\beta^2+\mu_1\mu_3^2b^2\alpha^5\beta+10t\mu_1\mu_3^2\alpha^3\beta^3+3t\mu_1^2\mu_3\alpha^2\beta^4$$
$$-6\mu_1\mu_2\mu_3\alpha\beta^5+4\mu_1\mu_2\mu_3\alpha^3\alpha^3-2t\mu_1\mu_2\mu_3\alpha\beta^5\}]$$
$$-\frac{2s_0}{\beta^5}\left[2\mu_3\alpha^3\left(2\mu_3b^2\alpha^3+\mu_1\beta^3\right)^2\left(\mu_1\alpha\beta+\mu_2\beta^2+\mu_3\alpha^2\right)^2\right]-\frac{r_{00}}{\beta^4}\left[2\mu_3\alpha^2\left(\beta^2-b^2\alpha^2\right)\right.$$
$$(\mu_1\beta+2\mu_3\alpha)\left(\mu_1\alpha\beta+\mu_2\beta^2+\mu_3\alpha^2\right)\left(-4\mu_1\mu_3b^2\alpha^3\beta-4\mu_3^2b^2\alpha^4+4\mu_1\mu_3\alpha\beta^3\right)\Big]-\frac{s_0}{\beta^3}$$
$$\left[4\mu_3\alpha^2(\mu_1\beta+2\mu_3\alpha)\left(\mu_1\alpha\beta+\mu_2\beta^2+\mu_3\alpha^2\right)(-4\mu_3b^4\alpha^5+2\mu_3^2b^2\alpha^5-3\mu_1\mu_3b^2\alpha^4\beta+\right.$$
$$2\mu_2\mu_3b^2\alpha^3\beta^2+\mu_1\mu_2\alpha^2\beta^3+4\mu_1\mu_3\alpha^2\beta^3-2\mu_1\mu_2\beta^5)\Big]-\frac{r_0}{\beta^3}\left[4\mu_3\alpha^4\left(\mu_1\beta^3+2\mu_3b^2\alpha^3\right)\right.$$
$$(\mu_1\beta+2\mu_3\alpha)^2\left(\mu_1\alpha\beta+\mu_2\beta^2+\mu_3\alpha^2\right)\Big]=0.$$

Above equation can be re-written as

$$2B^m_m\left[2\mu_1^4\mu_2\alpha\beta^{12}+a_1\alpha^2\beta^{11}+a_2\alpha^3\beta^{10}+a_3\alpha^4\beta^9+a_4\alpha^5\beta^8+a_5\alpha^6\beta^7+a_6\alpha^7\beta^6+a_7\alpha^8\beta^5\right.$$
$$\left.+a_8\alpha^9\beta^4+64\mu_1\mu_3^4b^2\alpha^{10}\beta^3+32\mu_3^5b^4\alpha^{11}\beta^2\right]-r_{00}\left[(1+t)\mu_1^4\mu_2\alpha\beta^{11}+10\mu_1^3\mu_2\mu_3\alpha^2\beta^{10}-\right.$$
$$a_9\alpha^3\beta^9+a_{10}\alpha^4\beta^8+a_{11}\alpha^5\beta^7+a_{12}\alpha^6\beta^6+a_{13}\alpha^7\beta^5+a_{14}\alpha^8\beta^4-32(t+1)\mu_3^5b^2\alpha^9\beta^3+16$$
$$(3.5)\qquad \left.b^4\mu_3^5\alpha^{11}\beta\right]-2s_0\left[-2(1+t)\mu_1^3\mu_2^2\alpha^2\beta^{11}+a_{15}\alpha^3\beta^{10}+8\mu_1^3\mu_2\mu_3\alpha^4\beta^9+a_{16}\alpha^5\beta^8+a_{17}\alpha^6\beta^7\right.$$
$$\left.+a_{18}\alpha^7\beta^6+a_{19}\alpha^8\beta^5+a_{20}\alpha^9\beta^4+a_{21}\alpha^{10}\beta^3+a_{22}\alpha^{11}\beta^2-16\mu_1\mu_3^4b^4\alpha^{12}\beta\right]-r_0\left[4\mu_1^3\mu_2\mu_3\right.$$
$$\alpha^4\beta^9+a_{23}\alpha^5\beta^8+a_{24}\alpha^6\beta^7+a_{25}\alpha^7\beta^6+a_{26}\alpha^8\beta^5+a_{27}\alpha^9\beta^4+64\mu_1\mu_3^4b^2\alpha^{10}\beta^3+32\mu_3^5b^2$$
$$\left.\alpha^{11}\beta^2\right]=0,$$

where

$$a_1=2\mu_1^4\mu_3+8\mu_1^3\mu_2\mu_3^2,\qquad a_2=10\mu_1^4\mu_3+8\mu_1^2\mu_2\mu_3^2,$$
$$a_3=64\mu_1\mu_3^4b^2-16\mu_1^3\mu_2^2+8\mu_1^3\mu_2\mu_3,\qquad a_4=32\mu_1^2\mu_2\mu_3^2b^2+8\mu_1^4\mu_3b^2+8\mu_1^2\mu_3^3,$$
$$a_5=40\mu_1^3\mu_3^2b^2+32\mu_1\mu_2\mu_3^3b^2,\qquad a_6=64\mu_1^2\mu_3^3b^2+8\mu_1^2\mu_2\mu_3^2b^4,$$
$$a_7=32\mu_1\mu_3^4b^2+8\mu_1^3\mu_3^2b^2+32\mu_1\mu_2\mu_3^3,\qquad a_8=40\mu_1^2\mu_2^3b^4+32\mu_2\mu_3^4b^2,$$
$$a_9=3(1-t)\mu_1^4\mu_3+(28+4t)\mu_1^2\mu_2\mu_3^2,$$
$$a_{10}=(14-16t)\mu_1^2\mu_3^3-6\mu_1^4\mu_3b^2+(t-2)\mu_1^3\mu_2\mu_3,$$
$$a_{11}=(20-28t)\mu_1^2\mu_3^3-6\mu_1^4\mu_3b^2+8(t-2)\mu_1^2\mu_2\mu_3^2b^2,$$
$$a_{12}=(8-16t)\mu_1\mu_3^4-6(6+t)\mu_1^3\mu_3^2b^2+8(t-2)\mu_1\mu_2\mu_3^3b^2,$$
$$a_{13}=-(80+32t)\mu_1^2\mu_3^3b^2-8\mu_1^2\mu_2\mu_3^2b^4,$$
$$a_{14}=-(56t+80)\mu_1^2\mu_3^3\mu_3^2b^2+8\mu_1^3\mu_3^2(1-b^2),$$
$$a_{15}=-14\mu_1^2\mu_3^3b^2-2(7+2t)\mu_1^2\mu_2^2\mu_3,$$
$$a_{16}=8(2+3t)\mu_1^2\mu_2\mu_3^2+2(3-2t)\mu_1^2\mu_2^2\mu_3b^2,$$
$$a_{17}=2(7-6t)\mu_1^3\mu_3^2+8(3-2t)\mu_1\mu_2\mu_3^3,$$

$$a_{18} = 2(11-10t)\mu_1^2\mu_3^3 - 8\mu_2^2\mu_3^3b^2 + 4(7+4t)\mu_1^2\mu_2\mu_3^2b^2,$$
$$a_{19} = 8(1+2t)\mu_1\mu_3^4 - 6\mu_1^3\mu_3^2b^2 + 24\mu_1\mu_2\mu_3^2b^4 + 8(7+6t)\mu_1\mu_2\mu_3^2,$$
$$a_{20} = 16(2t+3)\mu_2\mu_3^4b^2 + (10b^2-26)\mu_1^2\mu_3^3b^2 + 8\mu_2^2\mu_3^3b^4 + 8\mu_1^2\mu_2\mu_3^2b^4,$$
$$a_{21} = 8\mu_1\mu_2\mu_3^3b^4 - 8(1+5t)\mu_1\mu_3^4b^2,$$
$$a_{22} = 4\mu_1^4\mu_3 + 16\mu_1^2\mu_2\mu_3^2, \qquad a_{23} = 8(1-4t)\mu_3^5b^2 - 8\mu_1^2\mu_3^3b^4,$$
$$a_{24} = 20\mu_1^3\mu_3^2 + 16\mu_1\mu_2\mu_3^3, \qquad a_{25} = 32\mu_1^2\mu_3^3 + 8\mu_1^2\mu_2\mu_3^2b^2,$$
$$a_{26} = 16\mu_1\mu_3^4 + 8\mu_1^3\mu_3^2b^2 + 32\mu_1\mu_2\mu_3^3b^2, \qquad a_{27} = 32\mu_2\mu_3^3b^2 + 40\mu_1^2\mu_3^3b^2.$$

Now we assume that F^n is a weakly Berwald space, then B^m_m is $hp(1)$. Since α is irrational in (y^i), the equation (3.5) is divided into two equations as follows

$$\beta K_1 B^m_m + L_1 r_{00} + \alpha^2 M_1 s_0 + \alpha^2 N_1 r_0 = 0, \tag{3.6}$$

$$\beta^3 K_2 B^m_m + L_2 r_{00} + \alpha^2\beta^2 M_2 s_0 + \alpha^2\beta^2 N_2 r_0 = 0, \tag{3.7}$$

where

$$K_1 = 2\mu_1^4\mu_2\beta^{10} + a_2\alpha^2\beta^8 + a_4\alpha^4\beta^6 + a_6\alpha^6\beta^4 + 32\mu_3^5b^4\alpha^{11}\beta^2,$$
$$K_2 = a_1\beta^8 + a_3\alpha^2\beta^6 + a_5\alpha^4\beta^4 + a_7\alpha^6\beta^2 + 64\mu_1\mu_3^4b^2\alpha^8,$$
$$L_1 = -\{(1+t)\mu_1^4\mu_2\beta^{10} + a_9\alpha^2\beta^8 + a_{11}\alpha^4\beta^6 + a_{13}\alpha^6\beta^4 - 32(t+1)\mu_3^5b^2\alpha^8\beta^2 + 16b^4\mu_3^5\alpha^{10}\},$$
$$L_2 = -\{10\mu_1^3\mu_2\mu_3\beta^{10} + a_{10}\alpha^2\beta^8 + a_{12}\alpha^4\beta^6 + a_{14}\alpha^6\beta^4 + 16b^4\mu_3^5\alpha^{11}\beta\},$$
$$M_1 = -2\{2(1+t)\mu_1^3\mu_2^2\beta^{10} + 8\mu_1^3\mu_2\mu_3\alpha^2\beta^8 + a_{17}\alpha^4\beta^6 + a_{19}\alpha^6\beta^4 + a_{21}\alpha^8\beta^2 - 16\mu_1\mu_3^4b^4\alpha^{10}\},$$
$$M_2 = -2\{a_{15}\beta^8 + a_{16}\alpha^2\beta^6 + a_{18}\alpha^4\beta^4 + a_{20}\alpha^6\beta^2 + a_{22}\alpha^8\},$$
$$N_1 = -2\{4\mu_1^3\mu_2\mu_3\alpha^4\beta^8 + a_{24}\alpha^4\beta^6 + a_{26}\alpha^6\beta^4 + 64\mu_1\mu_3^4b^2\alpha^8\beta^2\},$$
$$N_2 = -2\{a_{23}\alpha^2\beta^6 + a_{25}\alpha^4\beta^4 + a_{27}\alpha^6\beta^2 + 32\mu_3^5b^2\alpha^8\}.$$

Eliminating B^m_m from equations (3.6) and (3.7), we get

$$Fr_{00} + \alpha^2\beta^3 G s_0 + \alpha^2\beta^3 H r_0 = 0, \tag{3.8}$$

where

$$F = \beta^2K_2L_1 - K_1L_2,$$
$$G = K_2M_1 - K_1M_2,$$
$$H = K_2N_1 - K_1N_2.$$

And (3.8) implies

$$\left(\frac{F}{\alpha^2\beta^3}\right) r_{00} + Gs_0 + Hr_0 = 0. \tag{3.9}$$

Since only the term $\epsilon_1\alpha^{18}$ of Gs_0 in (3.9) does not contain β, we must have $hp(18)V_{18}$ such that

$$\alpha^{18}s_0 = \beta V_{18}, \tag{3.10}$$

here $\epsilon_1 = -64b^4\left(\mu_3^5a_{22} - 32\mu_1^2\mu_3^8b^2\right)$.

Initially consider that $\alpha^2 \not\equiv 0 \pmod{\beta}$ and $b^2 \neq 0$. Equation (3.10) shows the existence of a function $q(x)$ satisfy $V_{18} = q\alpha^{18}$ and hence $s_0 = q\beta$, equation (3.9) reduces to

$$\left(\frac{F}{\alpha^2\beta^3}\right) r_{00} + Gq\beta + Hr_0 = 0,$$

implies

$$Fr_{00} + Gq\alpha^2\beta^4 + \alpha^2\beta^3 Hr_0 = 0.$$

Only the term $544\mu_1^{10}b^8\alpha^{18}r_{00}$ of the above relation does not contain β. Thus there exist $hp(19)U_{19}$ satisfying $544\mu_1^{10}b^8\alpha^{18}r_{00} = \beta U_{19}$. It is a contradiction, which implies that $q = 0$. Hence we obtain $s_0 = 0$, $s_j = 0$. Then equation (3.8) becomes

$$Fr_{00} + \alpha^2\beta^3 Hr_0 = 0. \tag{3.11}$$

Only the term $\left(2\mu_1^7\mu_2^2\mu_3 - (t-1)\mu_1^4\mu_2 a_1\right)\beta^{20}r_{00}$ of (3.11) seemingly does not contain α^2 and hence we must have $hp(20)V_{20}$ such that $\beta^{20}r_{00} = \alpha^2 V_{20}$. From $\alpha^2 \not\equiv 0 \pmod{\beta}$ there exist a function $g(x)$ such that

$$r_{00} = \alpha^2 g(x); \quad r_{ij} = a_{ij}g(x). \tag{3.12}$$

Transvecting above by $b^i y^j$, we have

$$r_0 = \beta g(x); \quad r_j = b_j g(x). \tag{3.13}$$

Plugging (3.12) and (3.13) into (3.11), we get

$$g(x)\left(F + \beta^3 H\right) = 0. \tag{3.14}$$

Assume that $g(x) \neq 0$, from equation (3.14) we have

$$F + \beta^3 H = 0.$$

The term $544\mu_1^{10}b^8\alpha^{18}r_{00}$ of above does not contain β. Then there exist $hp(17)V_{17}$ satisfying $544\mu_1^{10}b^8\alpha^{18} = \beta V_{17}$, here V_{17} this implies $V_{17} = 0$, provided that $b^2 \neq 0$. Hence $g(x) = 0$ must hold and we get

$$r_{00} = 0, \quad r_{ij} = 0 \quad and \quad r_0 = 0; \quad r_j = 0.$$

Conversely, substituting $r_{00} = 0$, $s_0 = 0$, *and* $r_0 = 0$ into equation (3.5), we have $B^m_m = 0$. That is, the Finsler space with (3.1) is a Weakly Berwald space.

Consequently, we assume that the Finsler space with (3.1) is a Berwald space. As a result of the preceding discussion, we have r_{00}, $s_0 = 0$ *and* $r_0 = 0$, indicating that the space is weakly Berwald space. When we plug the above into (3.3), we get $B^m_m = 0$, noting that the Finsler space with (3.1) is a Berwald space. Hence s_{ij} is holds good.

Now consider $\alpha^2 \equiv 0 (\bmod\ \beta)$, Lemma(2.1) shows that $t = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$. From these conditions (3.8) is rewritten in the form

$$F' r_{00} + \beta\delta G' s_0 = 0, \tag{3.15}$$

Where

$$F^{'} =\beta^{10}\left(10\mu_1^7\mu_2^2\mu_3 - (1+t)\mu_1^4\mu_2 a_1\right) - \beta^9\delta\left(a_1a_9 - 2a_{10}\mu_1^4\mu_2 - 10a_2\mu_1^3\mu_2\mu_3 + (1+t)\mu_1^4\mu_2a_3\right) + \beta^8\delta^2\left(2a_{12}\mu_1^4\mu_2 + a_2a_{10} + 10a_4\mu_1^3\mu_2\mu_3 - a_3a_9 - a_1a_{11} - (1+t)\mu_1^4\mu_2a_5\right) + \beta^7\delta^3\left(2\mu_1^4\mu_2a_{14} + a_2a_{12} + 10\mu_1^3\mu_2\mu_3a_6 + a_4a_{10} - a_1a_{13} - a_3a_{11} - a_5a_9 - (1+t)\mu_1^4\mu_2a_7\right) + \beta^6\delta^4\left(32(1+t)\mu_3^5b^2a_1 - a_3a_{13} - a_5a_{11} - a_7a_9 + a_2a_{14} + a_6a_{10} + a_4a_{12} - 64(1+t)\mu_1^5\mu_2\mu_3^4\right) + \beta^5\delta^5\left(32(1+t)\mu_3^5b^2a_3 - a_5a_{13} - a_7a_{11} + a_4a_{14} + a_6a_{12} - 64\mu_1\mu_3^4b^2a_9 + 320\mu_1^4\mu_2\mu_3^5b^4\right) + \beta^6\delta^4\left(-16\mu_3^5b^4a_3 + 32(1+t)\mu_3^5b^2 - a_7a_{13} - 64\mu_1\mu_3^4b^2a_{11} + 16\mu_3^5b^4a_2 + 32\mu_3^5b^4a_{10} + a_6a_{14}\right) + \beta^7\delta^3\left(32\mu_3^5b^4a_{12} + 16\mu_3^5b^4a_4 - 64\mu_1\mu_3^4b^2 + 32(1+t)\mu_3^5b^2a_7 - 16\mu_3^5b^4a_5\right) + \beta^8\delta^2\left(2048(1+t)\mu_1\mu_3^9b^4 - 16\mu_3^5b^4a_7 + 16\mu_3^5b^4a_6 + 32\mu_3^5b^4a_{14}\right) + 1024\mu_1\mu_3^9b^4\delta^9\beta + 544\mu_3^{10}b^8\delta^{10},$$

$$G^{'} =\beta^9\left(4\mu_1^4\mu_2 - 4(1+t)\mu_1^3\mu_2^2a_1\right) + 2\delta\beta^8\left(2\mu_1^4\mu_2a_{16} + a_2a_{15} - 8t\mu_1^3\mu_2\mu_3 - 2(1+t)\mu_1^3\mu_2^2\right) + 2\delta^2\beta^7\left(-a_1a_{17} - 8t\mu_1^3\mu_2\mu_3a_3 - 2(1+t)\mu_1^3\mu_2^2a_5 + 2\mu_1^4\mu_2a_{18} + a_2a_{16} + a_4a_{15}\right) + 2\delta^3\beta^6\left(2\mu_1^4\mu_2a_{20} + a_2a_{18} + a_4a_{16} + a_6a_{15} - a_1a_{19} - a_3a_{17} - 8\mu_1^3\mu_2\mu_3ta_5 - 2(1+t)\mu_1^3\mu_2^2a_7\right) + 2\delta^4\beta^5\left(2\mu_1^4\mu_2a_{22} + a_2a_{20} + a_4a_{18} + a_6a_{16} - a_1a_{21} - a_3a_{19} - a_5a_{17} - 8\mu_1^3\mu_2\mu_3ta_7 - 128(1+t)\mu_1^4\mu_2^2\mu_3^4b^2\right) + 2\beta^4\delta^5\left(a_2a_{22} + a_4a_{20} + a_6a_{18} - a_5a_{19} + 32\mu_3^5b^4a_{15} + 32\mu_1\mu_3^4a_1 - a_3a_{21} - a_7a_{17} + 16\mu_1\mu_3^4b^4a_1 - 512\mu_1^4\mu_2\mu_3^5b^2\right) + 2\beta^3\delta^6\left(a_4a_{22} + a_6a_{20} + 32\mu_3^5b^2a_{16} - a_5a_{21} - a_5a_{19} + 16\mu_1\mu_3^4b^4a_3 - 64\mu_1\mu_3^4b^2a_{17}\right) + 2\beta^2\delta^7\left(a_6a_{22} + 32\mu_3^5b^4a_{18} + 16\mu_1\mu_3^4b^4a_5 - a_7a_{21} - 64\mu_1\mu_3^4b^2a_{19}\right) + 2\beta\delta^8\left(16\mu_1\mu_3^4b^4a_7 - 64\mu_1\mu_3^4a_{21} + 32\mu_3^5b^4a_{20}\right) + \delta^9\left(64\mu_3^5b^4a_{22} - 1024\mu_1^2\mu_3^8b^6\right).$$

Since only the term $\left(10\mu_1^7\mu_2^2\mu_3 - (1+t)\mu_1^4\mu_2a_1\right)\beta^{10}r_{00}$ of $F^{'}r_{00} + \beta\delta G^{'}s_0$ in (3.15) seemingly does not contain δ. we must have $hp(1)V_1$ such that $r_{00} = \delta V_1$. We have $s_0 = 0$, $s_j = 0$, now (3.15) becomes

$$F^{'}r_{00} = 0, \tag{3.16}$$

Which implies

$$r_{00} = 0, \quad r_{ij} = 0 \quad and \quad r_0 = 0; \quad r_j = 0.$$

Consequently from r_{00}, $r_0 = 0$ *and* $s_0 = 0$, we have $B^m_m = 0$. Thus the space with (3.1) is weakly Berwald space. Hence we state the following

Theorem 3.1. *Let F be a Finsler space with* (α,β)*-metric* (3.1) *is weakly Berwald space if and only if the following properties satisfies*

i. $\alpha^2 \not\equiv 0(\text{mod }\beta)$ *implies* $r_{ij} = 0$ *and* $s_j = 0$,

ii. $\alpha^2 \equiv 0(\text{mod }\beta)$ *implies* $t = 2$, $b^2 = 0$ *and* $r_{ij} = 0$, $s_j = 0$ *are satisfied, where* $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$.

4 Conclusion

In this paper, we look at a Finsler space where the (hv)-Ricci tensor G_{ij} vanishes but the (hv)-curvature tensor G^h_{ijk} does not always equal zero. The primary goal of this research is to study an special (α,β)-metric of a weakly Berwald Finsler space and to show a required condition for the existence of a weakly Berwald Finsler space of the (α,β)-metric (3.1).

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