Relative $(p,q,t)L$-th order and $(p,q,t)L$-th type based some growth analysis of wronskian

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Abstract

In the paper we establish some new results depending on the comparative growth properties of composite transcendental entire and meromorphic functions using relative $(p,q,t)L$-th order, $(p,q,t)L$-th type and Wronskian generated by one of the factors.

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1 Introduction

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [6, 9, 14, 15]. We also use the standard notations and definitions of the theory of entire functions which are available in [13] and therefore we do not explain those in details. Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z|=r$ as $M_f(r)=\max_{|z|=r}|f(z)|$. If f is non-constant then it has the following property:

Property (A) [2] : A non-constant entire function f is said to have the Property (A) if for any $\sigma>1$ and for all sufficiently large values of r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [2].

When f is meromorphic, the Nevanlinna's characteristic function $T_f(r)$ of f plays the same role as $M_f(r)$.

The integrated counting function $N_f(r,a)(\overline{N}_f(r,a))$ of a-points (distinct a-points) of f is defined as

$$N_f(r,a)=\int_0^r \frac{n_f(t,a)-n_f(0,a)}{t}dt+n_f(0,a)\log r$$

$$\Big(\overline{N}_f(r,a)=\int_0^r \frac{\overline{n}_f(t,a)-\overline{n}_f(r,a)}{t}dt+\overline{n}_f(0,a)\log r\Big),$$

where we denote by $n_f(t,a)(\overline{n}_f(t,a))$ the number of a-points (distinct a-points) of f in $|z|\leq t$ and an ∞ -point is a pole of f . In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and

$\overline{N}_f(r)$ respectively. The function $N_f(r,a)$ is called the enumerative function. On the other hand, the function $m_f(r) \equiv m_f(r,\infty)$ known as the proximity function is defined as

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x = \max(\log x, 0)$ for all $x \geqslant 0$ and an ∞ -point is a pole of f.

Analogously, $m_{\frac{1}{f-a}}(r) \equiv m_f(r,a)$ is defined when a is not an ∞-point of f.

Thus the Nevanlinna's characteristic function $T_f(r)$ corresponding to f is defined as

$$T_f(r) = N_f(r) + m_f(r).$$

When f is entire, $T_f(r)$ coincides with $m_f(r)$ as $N_f(r) = 0$.

However, for a meromorphic function f, the Wronskian determinant $W(f) = W(a_1,a_2,....a_k,f)$ is defined as

$$W(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1^{'} & a_2^{'} & . & . & . & a_k^{'} & f^{'} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}$$

where $a_1,a_2,....a_k$ are linearly independent meromorphic functions and small with respect to f (i.e., $T_{a_i}(r) = S(r,f)$ for $i = 1,2,3...k$, for details about $S(r,f)$ one may see [6], $p.22$). From the Nevanlinna's second fundamental theorem, it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum\limits_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ ($cf.$ [6], $p.43$) where $\delta(a; f) = 1 - \limsup\limits_{r\to\infty} \frac{N(r,a;f)}{T_f(r)} = \liminf\limits_{r\to\infty} \frac{m(r,a;f)}{T_f(r)}$. If in particular $\sum\limits_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

Moreover, if f is a non-constant entire function, then $T_f(r)$ is a strictly increasing and continuous function of r. Also its inverse $T_f^{-1} : (T_f(0),\infty) \to (0,\infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$. Also the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of the Nevanlinna's characteristic functions of the meromorphic functions f and g.

However let us consider that $x \in [0,\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers. We define $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Further we assume that throughout the present paper a,l,p,q,m and n always denote positive integers and $t \in \mathbb{N} \cup \{-1,0\}$. Now considering this, we just recall that Shen et al. [12] defined the (m,n)-φ order and (m,n)-φ lower order of entire function f which are as follows:

Definition 1.1. [12] Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function and $m \geq n$. The (m,n)-φ order $\rho^{(m,n)}(f,\varphi)$ and (m,n)-φ lower order $\lambda^{(m,n)}(f,\varphi)$ of entire function f are defined as:

$$\rho^{(m,n)}(f,\varphi) = \overline{\lim_{r\to\infty}} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)} \text{ and } \lambda^{(m,n)}(f,\varphi) = \underline{\lim_{r\to\infty}} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)}.$$

If f is a meromorphic function, then

$$\rho^{(m,n)}(f,\varphi) = \overline{\lim_{r\to\infty}} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)} \text{ and } \lambda^{(m,n)}(f,\varphi) = \underline{\lim_{r\to\infty}} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)}.$$

Further for any non-decreasing unbounded function $\varphi : [0,+\infty) \to (0,+\infty)$, if we assume $\lim_{r\to+\infty} \frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)} = 1$ for all $\alpha > 0$, then for any entire function f, using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.$ [6]$\}$, one can easily verify that (see [12])

$$\rho^{(m,n)}(f,\varphi) = \varlimsup_{r\to\infty} \frac{\log^{[m]} M_f(r)}{\log^{[n]}\varphi(r)} = \varlimsup_{r\to\infty} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]}\varphi(r)}$$

$$\left(\lambda^{(m,n)}(f,\varphi) = \varliminf_{r\to\infty} \frac{\log^{[m]} M_f(r)}{\log^{[n]}\varphi(r)} = \varliminf_{r\to\infty} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]}\varphi(r)}\right)$$

when $m > 1$.

If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r$, then the above definition reduces to the following definition:

Definition 1.2. The (p,q)-th order and (p,q)-th lower order of an entire function f are defined as:

$$\rho^{(p,q)}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \varliminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

If f is a meromorphic function, then

$$\rho^{(p,q)}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \varliminf_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

Definition 1.2 avoids the restriction $p \geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [7].

However the above definition is very useful for measuring the growth of entire and meromorphic functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire or meromorphic function f. For details about generalized order one may see [11]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire or meromorphic function f.

In this connection we just recall the following definition of index-pair where we will give a minor modification to the original definition (see e.g. [7]):

Definition 1.3. An entire function f is said to have index-pair (p,q) if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots. \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots. \end{cases}$$

Analogously one can easily verify that Definition 1.3 of index-pair can also be applicable to a meromorphic function f.

However, the function f is said to be of regular (p,q) growth when (p,q)-th order and (p,q)-th lower order of f are the same. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

For entire functions, Somasundaram and Thamizharasi [10] introduced the notions of the growth indicators L-order and L-lower order where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a' i.e., $\lim_{r\to\infty} \frac{L(ar)}{L(r)} = 1$ where $L \equiv L(r)$ is a positive continuous function increasing slowly. The more generalized concept of L-order and L-lower order for entire function are L^*-order and L^*-lower order. Their definitions are as follows:

Definition 1.4. [10] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log[re^{L(r)}]} \text{ and } \lambda_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log[re^{L(r)}]}.$$

When f is meromorphic one can easily verify that

$$\rho_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log[re^{L(r)}]} \text{ and } \lambda_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log[re^{L(r)}]}.$$

If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r \cdot \exp^{[t+1]} L(r)$, then Definition 1.1 turn into the definitions of $(p,q,t)L$-th order and $(p,q,t)L$-th lower order of an entire function f which are as follows:

$$\rho_f^L(p,q,t) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}$$
$$\text{and } \lambda_f^L(p,q,t) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

If f is a meromorphic function, then

$$\rho_f^L(p,q,t) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}$$
$$\text{and } \lambda_f^L(p,q,t) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

In order to compare the relative growth of two entire functions having same non zero finite $(p,q,t)L$-th order, one may introduce the definition of $(p,q,t)L$-th type (respectively $(p,q,t)L$-th lower type) of entire functions having finite positive finite $(p,q,t)L$-th order in the following manner:

Definition 1.5. [4] Let f be an entire function with non-zero finite $(p,q,t)L$-th order $\rho_f^L(p,q,t)$. The $(p,q,t)L$-th type denoted by $\sigma_f^L(p,q,t)$ and $(p,q,t)L$-th lower type denoted by $\overline{\sigma}_f^L(p,q,t)$ are respectively defined as follows:

$$\sigma_f^L(p,q,t) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_f^L(p,q,t)}}$$

and

$$\overline{\sigma}_f^L(p,q,t) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_f^L(p,q,t)}}.$$

Analogously in order to determine the relative growth of two entire functions having same non zero finite $(p,q,t)L$-th lower order one may introduce the definition of $(p,q,t)L$-th weak type of entire functions having finite positive $(p,q,t)L$-th lower order in the following way:

Definition 1.6. [4] The $(p,q,t)L$-th weak type denoted by $\tau_f^L(p,q,t)$ of an entire function f is defined as follows:

$$\tau_f^L(p,q,t) = \underline{\lim}_{r\to+\infty} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda_f^L(p,q,t)}},\ 0 < \lambda_f^L(p,q,t) < \infty.$$

Also one may define the growth indicator $\overline{\tau}_f^L(p,q,t)$ of an entire function f in the following manner :

$$\overline{\tau}_f^L(p,q,t) = \overline{\lim}_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda_f^L(p,q,t)}},\ 0 < \lambda_f^L(p,q,t) < \infty.$$

Mainly the growth investigation of entire or meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire or meromorphic function with respect to a new entire function, the notions of relative growth indicators [2, 8] will come. Extending this notion,one may introduce the definitions of relative $(p,q,t)L$-th order and relative $(p,q,t)L$ -th lower order of a meromorphic function f with respect to another entire function g in the following way:

Definition 1.7. [4] Let f be a meromorphic function and g be an entire function. Then relative $(p,q,t)L$-th order denoted as $\rho_g^{(p,q,t)L}(f)$ and relative $(p,q,t)L$-th lower order denoted as $\lambda_g^{(p,q,t)L}(f)$ of a meromorphic function f with respect to an entire function g are defined by

$$\rho_g^{(p,q,t)L}(f) = \overline{\lim}_{r\to\infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r + \exp^{[t]} L(r)}$$

$$\text{and } \lambda_g^{(p,q,t)L}(f) = \underline{\lim}_{r\to\infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. Actually in the paper we establish some new results depending on the comparative growth properties of composite transcendental entire and meromorphic functions using relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of meromorphic function with respect to an entire function and that of Wronskian generated by one of the factors.

2 Main Results

In this section first we present some lemmas which will be needed in the sequel.

Lemma 2.1. [1] Let f be a meromorphic function and g be an entire function, then for all sufficiently large values of r,

$$T_{f\circ g}(r) \leqslant \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Lemma 2.2. [5] Let f be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then

$$\beta T_f(r) < T_f(\alpha r^\delta).$$

Lemma 2.3. [3] Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ and g be a transcendental entire function having the maximum deficiency sum with regular (m,p) growth where $m > 1$. Then the relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of $W(f)$ with respect to $W(g)$ are same as those of f with respect to g, i.e.,

$$\rho_{W(g)}^{(p,q,t)L}(W(f)) = \rho_g^{(p,q,t)L}(f) \text{ and } \lambda_{W(g)}^{(p,q,t)L}(W(f)) = \lambda_g^{(p,q,t)L}(f).$$

Now we present the main results of the paper.

Theorem 2.1. Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth. Also let f be a meromorphic function and h be an entire function such that $\rho_h^{(p,q,t)L}(f)<\infty$, $\rho_k^{(l,n,t)L}(g)>0$ and $\rho_g^L(m,n,t)<\infty$ where $m>q$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p+m-q]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]}T_{W(k)}^{-1}(T_{W(g)}(r))+\exp^{[t]}L(M_g(r))}\leq\frac{\rho_g^L(m,n,t)}{\rho_k^{(l,n,t)L}(g)},$$

when $\exp^{[t]}L(M_g(r))=o\{\log^{[l]}T_{W(k)}^{-1}(T_{W(g)}(r))\}$ as $r\to\infty$.

Proof. Let us suppose that $\eta>2$ and $\delta\to 1+$ in Lemma 2.2. As $T_h^{-1}(r)$ is an increasing function of r, from Lemma 2.1, Lemma 2.2 and the inequality $T_g(r)\leq\log^+M_g(r)$ {cf. [6]} for all sufficiently large values of r, it follows that

$$T_h^{-1}(T_{f\circ g}(r))\leqslant T_h^{-1}[\{1+o(1)\}T_f(M_g(r))],$$

$$i.e.,\ T_h^{-1}(T_{f\circ g}(r))\leqslant\eta\cdot[T_h^{-1}(T_f(M_g(r)))]^{\delta},$$

$$i.e.,\ \log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leqslant\log^{[p]}T_h^{-1}(T_f(M_g(r)))+O(1),\tag{2.1}$$

$$i.e.,\ \log^{[p]}T_h^{-1}(T_{f\circ g}(r))$$
$$\leq\ (\rho_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q]}M_g(r)+\exp^{[t]}L(M_g(r))+O(1)],$$

$$i.e.,\log^{[p]}T_h^{-1}(T_{f\circ g}(r))$$
$$\leq\ (\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]}M_g(r)\Big[1+\frac{\exp^{[t]}L(M_g(r))+O(1)}{\log^{[q]}M_g(r)}\Big],$$

$$i.e.,\log^{[p+1]}T_h^{-1}(T_{f\circ g}(r))\leq\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)+\log^{[q+1]}M_g(r)$$
$$+\log\Big[1+\frac{\exp^{[t]}L(M_g(r))+O(1)}{\log^{[q]}M_g(r)}\Big].$$

Taking $\log\Big(1+\frac{\exp^{[t]}L(M_g(r))+O(1)}{\log^{[q]}M_g(r)}\Big)\sim\frac{\exp^{[t]}L(M_g(r))+O(1)}{\log^{[q]}M_g(r)}$, we get for all sufficiently large values of r,

$$\log^{[p+1]}T_h^{-1}(T_{f\circ g}(r))\leq\log^{[q+1]}M_g(r)+\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)$$
$$+\frac{\exp^{[t]}L(M_g(r))+O(1)}{\log^{[q]}M_g(r)},$$

$$i.e.,\ \log^{[p+1]}T_h^{-1}(T_{f\circ g}(r))$$
$$\leq\log^{[q+1]}M_g(r)\Big[1+\frac{\exp^{[t]}L(M_g(r))+O(1)+\log^{[q]}M_g(r)\cdot\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\log^{[q]}M_g(r)\cdot\log^{[q+1]}M_g(r)}\Big],$$

$$i.e., \ \log^{[p+2]} T_h^{-1}(T_{f\circ g}(r)) \leq \log^{[q+2]} M_g(r) \log\Big[1 + \frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f) + \varepsilon)}{\log^{[q]} M_g(r) \cdot \log^{[q+1]} M_g(r)}\Big].$$

Again using $\log(1 + x) \sim x$ for $x = \frac{\exp^{[t]} L(M_g(r))+O(1)+\log^{[q]} M_g(r)\cdot\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+1} \log^{[k]} M_g(r)}$, we get from above for all sufficiently large positive numbers of r ,

$$\log^{[p+2]} T_h^{-1}(T_{f\circ g}(r)) \leq \log^{[q+2]} M_g(r) + \frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f) + \varepsilon)}{\prod_{k=q}^{q+1} \log^{[k]} M_g(r)}.$$

Continuing this process, we get

$$\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r)) \leq \log^{[q+m-q]} M_g(r) + \frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f) + \varepsilon)}{\prod_{k=q}^{q+m-q-1} \log^{[k]} M_g(r)},$$

$$i.e., \log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r)) \leq \log^{[m]} M_g(r) + \frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f) + \varepsilon)}{\prod_{k=q}^{m-1} \log^{[k]} M_g(r)},$$

$$i.e., \ \log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r)) \leq (\rho_g^L(m,n,t) + \varepsilon)[\log^{[n]} r + \exp^{[t]} L(r)]$$

$$+\frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f) + \varepsilon)}{\prod_{k=q}^{m-1} \log^{[k]} M_g(r)}. \tag{2.2}$$

Again in view of Lemma 2.3, we have for a sequence of values of r tending to infinity that

$$\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) \geq (\rho_{W(k)}^{(l,n,t)L}(W(g)) - \varepsilon)[\log^{[n]} r + \exp^{[t]} L(r)],$$

$$i.e., \ \log^{[n]} r + \exp^{[t]} L(r) \leq \frac{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))}{(\rho_k^{(l,n,t)L}(g) - \varepsilon)}. \tag{2.3}$$

Hence from (2.2) and (2.3), it follows for a sequence of values of r tending to infinity that

$$\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r)) \leq \Big(\frac{\rho_g^L(m,n,t) + \varepsilon}{\rho_k^{(l,n,t)L}(g) - \varepsilon}\Big) \cdot \log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))$$

$$+\frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f) + \varepsilon)}{\prod_{k=q}^{m-1} \log^{[k]} M_g(r)},$$

$$i.e,\ \frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))}$$

$$\leq \Big(\frac{\rho_g^L(m,n,t)+\varepsilon}{\rho_k^{(l,n,t)L}(g)-\varepsilon}\Big)\cdot \frac{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))}$$

$$+ \frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r)\cdot \log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{[\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))]\cdot \prod_{k=q}^{m-1} \log^{[k]} M_g(r)},$$

$$i.e,\ \frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))}$$

$$\leq \frac{\frac{\rho_g^L(m,n,t)+\varepsilon}{\rho_k^{(l,n,t)L}(g)-\varepsilon}}{1+\frac{\exp^{[t]} L(M_g(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))}} + \frac{1+\frac{O(1)+\log^{[q]} M_g(r)\cdot\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\exp^{[t]} L(M_g(r))}}{[1+\frac{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))}{\exp^{[t]} L(M_g(r))}]\cdot \prod_{k=q}^{m-1}\log^{[k]} M_g(r)}. \tag{2.4}$$

Since $\exp^{[t]} L(M_g(r)) = o\{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))\}$ as $r\to\infty$ and $\varepsilon(>0)$ is arbitrary we obtain from (2.4) that

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))} \leq \frac{\rho_g^L(m,n,t)}{\rho_k^{(l,n,t)L}(g)}.$$

Thus the theorem is established.

In the line of Theorem 2.1, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 2.2. Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth. Also let f be a meromorphic function and h be an entire function such that $\rho_h^{(p,q,t)L}(f)<\infty$, $\lambda_k^{(l,n,t)L}(g)>0$ and $\lambda_g^L(m,n,t)<\infty$ where $m>q$. If h satisfies the Property (A), then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))} \leq \frac{\lambda_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)},$$

when $\exp^{[t]} L(M_g(r)) = o\{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))\}$ as $r\to\infty$.

Now we state the following two theorems without their proofs as those can be carried out in the line of Theorem 2.1 and Theorem 2.2:

Theorem 2.3. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\rho_h^{(p,q,t)L}(f)<\infty$ and $\rho_g^L(m,n,t)<\infty$ where $m>n=q$. If h satisfies the Property (A), then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))} \leq \frac{\rho_g^L(m,n,t)}{\rho_h^{(p,q,t)L}(f)},$$

when $\exp^{[t]} L(M_g(r)) = o\{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(r))\}$ as $r\to\infty$.

Theorem 2.4. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\lambda_h^{(p,q,t)L}(f)\leq\rho_h^{(p,q,t)L}(f)<\infty$ and $\lambda_g^L(m,n,t)<\infty$ where $m>n=q$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p+m-q]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]}T_{W(h)}^{-1}(T_{W(f)}(r))+\exp^{[p-1]}L(M_g(r))}\leq\frac{\lambda_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)},$$

when $\exp^{[t]}L(M_g(r))=o\{\log^{[p]}T_{W(h)}^{-1}(T_{W(f)}(r))\}$ as $r\to\infty$.

Theorem 2.5. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\lambda_h^{(p,q,t)L}(f)<\infty$ or $0<\rho_h^{(p,q,t)L}(f)<\infty$ and $\sigma_g^L(m,n,t)<\infty$ where $m-1\leq q$, $a>1$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]}T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}))}\leq\ \sigma_g^L(m,n,t),$$

when $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$.

Proof. Let us consider $0<\lambda_h^{(p,q,t)L}(f)<\infty$. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from (2.1) for a sequence of values of r tending to infinity that

$$\log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leq(\lambda_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q]}M_g(r)+\exp^{[t]}L(M_g(r))]+O(1),$$

$$\begin{aligned}&i.e.,\ \log^{[p]}T_h^{-1}(T_{f\circ g}(r))\\ \leq\ &(\lambda_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[m-1]}M_g(r)+\exp^{[t]}L(M_g(r))]+O(1),\end{aligned}$$

$$i.e.,\ \log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leq(\lambda_h^{(p,q,t)L}(f)+\varepsilon)\cdot$$

$$[(\sigma_g^L(m,n,t)+\varepsilon)[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}+\exp^{[t]}L(M_g(r))]+O(1)\ . \tag{2.5}$$

Also, we obtain in view of Lemma 2.3 for all sufficiently large values of r that

$$\begin{aligned}&\log^{[p]}T_{W(h)}^{-1}T_{W(f)}(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)})\\ &\geq(\lambda_{W(h)}^{(p,q,t)L}(W(f))-\varepsilon)[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}\\ &+(\lambda_{W(h)}^{(p,q,t)L}(W(f))-\varepsilon)\exp^{[t]}[L(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)})],\end{aligned}$$

$$\begin{aligned}&i.e.,\ \log^{[p]}T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}))\\ &\qquad\geq(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}.\end{aligned}$$

Now from (2.5) and above it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]}T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}))}$$

$$\leq (\lambda_h^{(p,q,t)L}(f)+\varepsilon)[(\sigma_g^L(m,n,t)+\varepsilon)$$
$$\cdot\frac{[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}+\exp^{[t]} L(M_g(r))]}{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}}$$
$$+\frac{O(1)}{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}},$$

$$i.e.,\ \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}))}$$
$$\leq \frac{(\lambda_h^{(p,q,t)L}(f)+\varepsilon)\cdot\left[(\sigma_g^L(m,n,t)+\varepsilon)+\frac{\exp^{[t]} L(M_g(r))}{[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}}\right]}{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)}$$

$$+\frac{O(1)}{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}}. \tag{2.6}$$

As $\alpha<\rho_g^L(m,n,t)$ and $\exp^{[t]} L(M_g(r))=o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$, we obtain that

$$\lim_{r\to\infty}\frac{\exp^{[t]} L(M_g(r))}{[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}}=0\,. \tag{2.7}$$

Since $\varepsilon(>0)$ is arbitrary, it follows from (2.6) and (2.7) that

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}))}\leq\ \sigma_g^L(m,n,t)\,.$$

Similarly if we consider $0<\rho_h^{(p,q,t)L}(f)<\infty$, then using the same technique one can easily verify that

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}))}\leq\ \sigma_g^L(m,n,t)\,.$$

Thus the theorem is established.

In the line of Theorem 2.5, the following theorems can be carried out and therefore their proofs are omitted:

Theorem 2.6. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\lambda_h^{(p,q,t)L}(f)<\infty$ or $0<\rho_h^{(p,q,t)L}(f)<\infty$ and $\overline{\tau}_g^L(m,n,t)<\infty$ where $m-1\leq q$, $a>1$. If h satisfies the Property (A), then

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\lambda_g^L(m,n,t)}))}\leq\ \overline{\tau}_g^L(m,n,t),$$

when $\exp^{[t]} L(M_g(r))=o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$.

Theorem 2.7. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\lambda_h^{(p,q,t)L}(f)\leq\rho_h^{(p,q,t)L}(f)<\infty$ and $\overline{\sigma}_g^L(m,n,t)<\infty$ where $m-1\leq q$, $a>1$. If h satisfies the Property (A), then

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]}T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}))}\leq\frac{\rho_h^{(p,q,t)L}(f)\cdot\overline{\sigma}_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)},$$

when $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$.

Theorem 2.8. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\lambda_h^{(p,q,t)L}(f)\leq\rho_h^{(p,q,t)L}(f)<\infty$ and $\tau_g^L(m,n,t)<\infty$ where $m-1\leq q$, $a>1$. If h satisfies the Property (A), then

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]}T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\lambda_g^L(m,n,t)}))}\leq\frac{\rho_h^{(p,q,t)L}(f)\cdot\tau_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)},$$

when $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$.

Now we state the following three theorems without their proofs as those can be carried out in the line of Theorem 2.5.

Theorem 2.9. Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth where $m>1$. Also let f be a meromorphic function and h be an entire function such that $\lambda_k^{(l,n,t)L}(g)>0$, $\lambda_h^{(p,q,t)L}(f)<\infty$ and $\sigma_g^L(m,n,t)<\infty$ where $m-1\leq q$. If h satisfies the Property (A), then

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]}T_{W(k)}^{-1}(T_{W(g)}(\exp^{[n]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}))}\leq\frac{\lambda_h^{(p,q,t)L}(f)\cdot\sigma_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)},$$

when $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$.

Theorem 2.10. Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth where $m>1$. Also let f be a meromorphic function and h be an entire function such that $\rho_k^{(l,n,t)L}(g)>0$, $\rho_h^{(p,q,t)L}(f)<\infty$ and $\sigma_g^L(m,n,t)<\infty$ where $m-1\leq q$. If h satisfies the Property (A), then

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]}T_{W(k)}^{-1}(T_{W(g)}(\exp^{[n]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}))}$$
$$\leq \frac{\rho_h^{(p,q,t)L}(f)\cdot\sigma_g^L(m,n,t)}{\rho_k^{(l,n,t)L}(g)},$$

when $\exp^{[t]}L(M_g(r)) = o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$.

Theorem 2.11. Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth where $m>1$. Also let f be a meromorphic function and h be an entire function such that $\lambda_k^{(l,n,t)L}(g)>0$, $\rho_h^{(p,q,t)L}(f)<\infty$ and $\overline{\sigma}_g^L(m,n,t)<\infty$ where $m-1\leq q$. If h satisfies the Property (A), then

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]}T_{W(k)}^{-1}(T_{W(g)}(\exp^{[n]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}))}$$
$$\leq \frac{\rho_h^{(p,q,t)L}(f)\cdot\overline{\sigma}_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)},$$

when $\exp^{[t]}L(M_g(r)) = o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$.

We omit the proof of Theorem 2.11 as it can be easily established in the line of Theorem 2.7.

Further we state the following theorem which is based on $(m,n,t)L$-th weak type:

Theorem 2.12. Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth where $m>1$. Also let f be a meromorphic function and h be an entire function such that $\lambda_k^{(l,n,t)L}(g)>0$, $\rho_h^{(p,q,t)L}(f)<\infty$ and $\tau_g^L(m,n,t)<\infty$ where $m-1\leq q$. If h satisfies the Property (A), then

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]}T_{W(k)}^{-1}(T_{W(g)}(\exp^{[n]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\lambda_g^L(m,n,t)}))}$$
$$\leq \frac{\rho_h^{(p,q,t)L}(f)\cdot\tau_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)},$$

when $\exp^{[t]}L(M_g(r)) = o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$.

Proof of the above theorem can be carried out in the line of Theorem 2.11 and therefore its proof is omitted.

Using the concept of the growth indicator $\overline{\tau}_g^L(m,n,t)$ of an entire function g, we may state the subsequent three theorems without their proofs since those can be carried out in the line of Theorem 2.12.

Theorem 2.13. Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth where $m>1$. Also let f be a meromorphic function and h be an entire function such that $\lambda_k^{(l,n,t)L}(g)>0$, $\lambda_h^{(p,q,t)L}(f)<\infty$ and $\overline{\tau}_g^L(m,n,t)<\infty$ where $m-1\leq q$. If h satisfies the Property

(A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(\exp^{[n]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\lambda_g^L(m,n,t)}))}$$
$$\leq \quad \frac{\lambda_h^{(p,q,t)L}(f)\cdot\overline{\tau}_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)},$$

when $\exp^{[t]} L(M_g(r)) = o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$.

Theorem 2.14. Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth where $m>1$. Also let f be a meromorphic function and h be an entire function such that $\rho_k^{(l,n,t)L}(g)>0$, $\rho_h^{(p,q,t)L}(f)<\infty$ and $\overline{\tau}_g^L(m,n,t)<\infty$ where $m-1\leq q$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(\exp^{[n]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\lambda_g^L(m,n,t)}))}$$
$$\leq \quad \frac{\rho_h^{(p,q,t)L}(f)\cdot\overline{\tau}_g^L(m,n,t)}{\rho_k^{(l,n,t)L}(g)},$$

when $\exp^{[t]} L(M_g(r)) = o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$.

Theorem 2.15. Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth where $m>1$. Also let f be a meromorphic function and h be an entire function such that $\lambda_k^{(l,n,t)L}(g)>0$, $\rho_h^{(p,q,t)L}(f)<\infty$ and $\overline{\tau}_g^L(m,n,t)<\infty$ where $m-1\leq q$. If h satisfies the Property (A), then

$$\varlimsup_{r\to\infty}\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(\exp^{[n]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\lambda_g^L(m,n,t)}))}$$
$$\leq \quad \frac{\rho_h^{(p,q,t)L}(f)\cdot\overline{\tau}_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)},$$

when $\exp^{[t]} L(M_g(r)) = o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$.

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