Coefficient estimates for subclasses of analytic and biunivalent functions based on subordination

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Abstract

We introduce the unified biunivalent classes $M_{\mu}^{a,c}[\gamma,\phi]$, $\Im_{\mu}^{a,c}[\beta,\phi]$, $\Re_{\mu}^{a,c}[\alpha,\phi]$ based on subordination and obtained the coefficients estimates for Taylor Maclaurin Coefficients $|a_2|$ and $|a_3|$. Further we pointed out several new or known consequences of our result.
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1 Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of $\mathcal{A}$, which consists of functions of the form (1.1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in U.
[7] Some of the important and well-investigated subclass of the univalent function class S includes the class $S^*(\alpha), (0 \le \alpha < 1)$ of star like functions of order α in U and the class $K(\alpha), (0 \le \alpha < 1)$ of convex function of order α

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (z \in U)$$

and

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha. \qquad (z \in U).$$

[16] For the two functions f and g are analytic in U, we say that the function $f(z)$ is subordinate $g(z)$ in U and written as

$$f(z) \prec g(z), (z \in U)$$

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If there exist a Schwartz function $w(z)$, analytic in U with $w(0) = 0, |w(z)| < 1\,(z \in U)$ such that $f(z) = g(w(z)), (z \in U)$. [16] In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0), f(U) \subset g(U)$.
Let ϕ be an analytic function with positive real part in U such that $\phi(0) = 1$, $\phi^{'}(0) > 0$ and $\phi(U)$ is symmetric with respect to the real axis. Hence we have,

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \qquad (B_1 > 0). \tag{1.2}$$

In [10] Ma and Minda introduced the unified classes,

$$\mathcal{S}^*(\phi) = \left(f : f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \prec \phi(z); z \in U \right)$$

$$\mathcal{K}^*(\phi) = \left(f : f \in \mathcal{A}, 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z); z \in U \right)$$

For the choice

$$\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha)z + 2(1 - \alpha)z^2 + ... \qquad (0 \le \alpha < 1)$$

here $B_1 = B_2 = 2(1 - \alpha)$
or

$$\phi(z) = \left(\frac{1 + z}{1 - z}\right)^{\beta} == 1 + 2\beta z + 2\beta^2 z^2 + .. \qquad (0 < \beta \le 1).$$

Here $B_1 = 2\beta, B_2 = 2\beta^2$. The classes $\mathcal{S}^*(\phi)$ and $\mathcal{K}^*(\phi)$ consist of functions known as the Starlike (resp. Convex) functions of order α or strongly starlike (respectively.Convex) functions of order β respectively.

Definition 1.1. *For $\mu > 0$ and $a, c \in \mathbb{C}$ are such that $\Re(c - a) \ge 0$, Raina and Sharma [14] [see also[1],[2]] defined the integral operator $\mathcal{J}_{\mu}^{a,c} : \mathcal{A} \to \mathcal{A}$ as following,*
i) For $\Re(c - a) \ge 0$ and $\Re(a) > -\mu$ by

$$\mathcal{J}_{\mu}^{a,c} f(z) = \frac{\Gamma(c + \mu)}{\Gamma(a + \mu)\Gamma(c - a)} \int_0^1 (1 - t)^{c-a-1} t^{a-1} f(zt^{\mu}) dt \tag{1.3}$$

ii) for a=c by

$$\mathcal{J}_{\mu}^{a,a} f(z) = f(z). \tag{1.4}$$

Where Γ stands for Euler's Gamma function (which is valid for all complex number except the non-positive integer) for $f(z)$ defined by (1.1)*, it is easily from*(1.2)*and* (1.3) *that*

$$\mathcal{J}_{\mu}^{a,c} f(z) = z + \frac{\Gamma(c + \mu)}{\Gamma(a + \mu)} \sum_{k=2}^{\infty} \frac{\Gamma(a + k\mu)}{\Gamma(c + k\mu)} a_k z^k, (\mu > 0, \Re(c) \ge \Re(a) > -\mu). \tag{1.5}$$

It is well known that every function $f \in S$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z, \quad (z \in U)$ and $f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f); \;\; r_0(f) \ge \frac{1}{4}\right).$
where,

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... \;. \tag{1.6}$$

[7] A function f in D is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions in U given by (1.1).
In [9] Lewin first investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$.
Netanyahu [11] on the other hand showed that $\max_{f\in\Sigma} \ |a_2| = \dfrac{4}{3}$. The coefficient estimate problem for each of the Taylor-Maclaurin coefficients $|a_n|$ ($n \geq 3$; $n \in \mathbb{N}$) is still an open problem. Followed by Brannan and Taha [5] and many other researchers [3], [6], [8], [13], [15] introduced certain subclasses of biunivalent functions class Σ.
A function f is bistarlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\phi)$ and $\mathcal{K}^*_\Sigma(\phi)$, where ϕ already defined in (1.2)
In order to prove our main results we shall need the following lemma.

Lemma 1.1. *[12]. If $h(z) \in P$ the class of functions analytic in U with $\Re\,(h(z)) > 0$ then $|c_n| \leq 2$ for $n \in \mathbb{N}$ where $h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ for $z \in U$.*

2 Main Results

Definition 2.1. *A function f is given by (1.1) is said to be in the class $M^{a,c}_\mu[\gamma, \phi]$ if the following conditions are satisfied:*

$$f \in \Sigma \quad , \quad \left[\frac{z(J^{a,c}_\mu f(z))'}{J^{a,c}_\mu f(z)}\right]^\gamma \left[1 + \frac{z(J^{a,c}_\mu f(z))''}{(J^{a,c}_\mu f(z))'}\right]^{1-\gamma} \prec \phi(z) \qquad (z \in U). \tag{2.1}$$

and

$$\left[\frac{w(J^{a,c}_\mu g(w))'}{J^{a,c}_\mu g(w)}\right]^\gamma \left[1 + \frac{w(J^{a,c}_\mu g(w))''}{(J^{a,c}_\mu g(w))'}\right]^{1-\gamma} \prec \phi(w) \qquad (w \in U). \tag{2.2}$$

Where $\gamma \geq 0$ and the function g and ϕ defined by (1.6)and (1.2) respectively. We first state and prove the following result.

Theorem 2.1. *Let f given by (1.1) be in the function class $M^{a,c}_\mu[\gamma, \phi]$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left[(6-4\gamma)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)} + (\frac{\gamma^2}{2} + \frac{5\gamma}{2} - 4)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right] B_1^2 + (B_1 - B_2)(2-\gamma)^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}} \tag{2.3}$$

and

$$a_3| \leq \frac{B_1}{(6-4\gamma)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}} + \frac{{B_1}^2}{(2-\gamma)^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2} \tag{2.4}$$

Proof. Since $f \in M^{a,c}_\mu[\gamma, \phi]$, there exist two analytic functions $u, v : U \to U$ with $u(0) = v(0) = 0$ such that

$$\left[\frac{z(J^{a,c}_\mu f(z))'}{J^{a,c}_\mu f(z)}\right]^\gamma \left[1 + \frac{z(J^{a,c}_\mu f(z))''}{(J^{a,c}_\mu f(z))'}\right]^{1-\gamma} = \phi(u(z)) \tag{2.5}$$

and

$$\left[\frac{w(J^{a,c}_\mu g(w))'}{J^{a,c}_\mu g(w)}\right]^\gamma \left[1 + \frac{w(J^{a,c}_\mu g(w))''}{(J^{a,c}_\mu g(w))'}\right]^{1-\gamma} = \phi(v(w)) \tag{2.6}$$

where $z, w \in U$. Define the functions s and t as

$$s(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \tag{2.7}$$

$$t(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1 w + d_2 w^2 + d_3 w^3 + ... \tag{2.8}$$

clearly s and t are analytic function in U and $s(0) = t(0) = 1$. Since $u, v : U \to U$, the functions s and t have positive real part in U. Hence by lemma 1.1,

$$|c_n| \le 2, |d_n| \le 2, (n \in N). \tag{2.9}$$

Solving for $u(z)$ and $v(w)$, we get:

$$u(z) = \frac{s(z)-1}{s(z)+1} = \frac{1}{2}\left[c_1 z + (c_2 - \frac{c_1^2}{2})z^2 + ...\right](z \in U)$$

and

$$v(w) = \frac{t(w)-1}{t(w)+1} = \frac{1}{2}\left[d_1 w + (d_2 - \frac{d_1^2}{2})w^2 + ...\right](w \in U)$$

Using these expression in (1.2), we obtain

$$\phi(u(z)) = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2 C_1^2\right]z^2 + ... \tag{2.10}$$

and

$$\phi(v(w)) = 1 + \frac{1}{2}B_1 d_1 w + \left[\frac{1}{2}B_1(d_2 - \frac{d_1^2}{2}) + \frac{1}{4}B_2 d_1^2\right]w^2 + ... \tag{2.11}$$

Expanding the L.H.S. of (2.5)and (2.6) and then equating the coefficients of z, z^2, w, w^2 we get

$$(2-\gamma)\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}a_2 = \frac{1}{2}B_1 c_1 \tag{2.12}$$

$$(6-4\gamma)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}a_3 + (\frac{\gamma^2}{2} + \frac{5\gamma}{2} - 4)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2 a_2^2 = \frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2 C_1^2 \tag{2.13}$$

$$-(2-\gamma)\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}a_2 = \frac{1}{2}B_1 d_1 \tag{2.14}$$

$$(6-4\gamma)(2a_2^2 - a_3)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)} + (\frac{\gamma^2}{2} + \frac{5\gamma}{2} - 4)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2 a_2^2 = \frac{1}{2}B_1(d_2 - \frac{d_1^2}{2}) + \frac{1}{4}B_2 d_1^2 \tag{2.15}$$

From (2.12) and (2.14) we get

$$c_1 = -d_1 \tag{2.16}$$

and

$$8(2-\gamma)^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2 a_2^2 = B_1^2(c_1^2+d_1^2). \tag{2.17}$$

Adding (2.13) and (2.15) we get

$$4\left[2(6-4\gamma)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+2(\frac{\gamma^2}{2}+\frac{5\gamma}{2}-4)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]a_2^2=2B_1(c_2+d_2)+(B_2-B_1)(c_1^2+d_1^2) \tag{2.18}$$

This on using(2.17) we get

$$a_2^2=\frac{B_1^3(c_2+d_2)}{4\left[(6-4\gamma)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+(\frac{\gamma^2}{2}+\frac{5\gamma}{2}-4)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]B_1^2+8(B_1-B_2)(2-\gamma)^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}. \tag{2.19}$$

Applying Lemma 1.1 in (2.19) and using (2.9) we get the desired estimates on $|a_2|$as

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\left[(6-4\gamma)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+(\frac{\gamma^2}{2}+\frac{5\gamma}{2}-4)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]B_1^2+(B_1-B_2)(2-\gamma)^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}}. \tag{2.20}$$

Next, to find the estimates on $|a_3|$ Subtracting (2.15) from (2.13) we get

$$2(6-4\gamma)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}(a_3-a_2^2)=\frac{2B_1(c_2-d_2)+(B_2-B_1)(c_1^2-d_1^2)}{4}$$

Which on using (2.16), gives:

$$a_3 = a_2^2+\frac{B_1(c_2-d_2)}{4(6-4\gamma)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}}. \tag{2.21}$$

Using (2.17) in (2.21), we get:

$$a_3 = \frac{B_1^2(c_1^2+d_1^2)}{8(2-\gamma)^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}+\frac{B_1(c_2-d_2)}{4(6-4\gamma)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}}. \tag{2.22}$$

Applying Lemma 1.1 in (2.22) and using (2.9) we get the desired estimates on $|a_3|$ as

$$|a_3|\leq\frac{B_1}{(6-4\gamma)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}}+\frac{B_1^2}{(2-\gamma)^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}. \tag{2.23}$$

This completes the proof of Theorem. □

Definition 2.2. *A function f is given by* (1.1) *is said to be in the class $\mathfrak{S}_\mu^{a,c}[\beta,\phi]$ if the following conditions are satisfied:*

$$f\in\Sigma \quad , \quad \left[\frac{z(J_\mu^{a,c}f(z))'}{J_\mu^{a,c}f(z)}\right]^\beta \prec \phi(z) \qquad (z\in U). \tag{2.24}$$

and

$$\left[\frac{w(J_\mu^{a,c}g(w))'}{J_\mu^{a,c}g(w)}\right]^\beta \prec \phi(w) \qquad (w\in U). \tag{2.25}$$

Where $\beta>0$ and the function g and ϕ defined by (1.6)*and* (1.2) *respectively. We first state and prove the following result.*

Theorem 2.2. *Let f given by* (1.1) *be in the function class $\mathfrak{S}_{\mu}^{a,c}[\beta,\phi]$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left[2\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)} + (\frac{\beta^2-3\beta}{2})\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]B_1^2 + (B_1-B_2)\beta^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}} \tag{2.26}$$

and

$$|a_3| \leq \frac{B_1}{2\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}} + \frac{B_1^2}{\beta^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2} \tag{2.27}$$

Proof. Since $\mathfrak{S}_{\mu}^{a,c}[\beta,\phi]$, there exist two analytic functions $u, v : U \to U$ with $u(0) = v(0) = 0$ such that

$$\left[\frac{z(J_{\mu}^{a,c}f(z))'}{J_{\mu}^{a,c}f(z)}\right]^{\beta} = \phi(u(z)) \tag{2.28}$$

and

$$\left[\frac{w(J_{\mu}^{a,c}g(w))'}{J_{\mu}^{a,c}g(w)}\right]^{\beta} = \phi(v(w)) \tag{2.29}$$

where $z, w \in U$. Define the functions s and t as in Theorem 1 and then proceed similarly upto (2.11). Expanding the L.H.S. of (2.24) and (2.25), we obtain:

$$\begin{aligned}\left[\frac{z(J_{\mu}^{a,c}f(z))'}{J_{\mu}^{a,c}f(z)}\right]^{\gamma} =& 1 + \beta\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}a_2z + \\ &\left[2\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}a_3 + \frac{\beta(\beta-3)}{2}\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]z^2 + \ldots.\end{aligned} \tag{2.30}$$

$$\begin{aligned}\left[\frac{w(J_{\mu}^{a,c}g(w))'}{J_{\mu}^{a,c}g(w)}\right]^{\beta} =& 1 - \beta\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}a_2w + \\ &\left[2\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}(2a_2^2-a_3) + \frac{\beta(\beta-3)}{2}\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]w^2 + \ldots.\end{aligned} \tag{2.31}$$

Now, using(2.10) (2.11) (2.30) (2.31) in (2.28) and (2.29) and then equating the coefficients of z, z^2, w, w^2, we get

$$\beta\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}a_2 = \frac{1}{2}B_1c_1 \tag{2.32}$$

$$2\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}a_3 + \frac{\beta(\beta-3)}{2}\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2 a_2^2 = \frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2C_1^2 \tag{2.33}$$

$$-\beta\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}a_2 = \frac{1}{2}B_1d_1 \tag{2.34}$$

$$2\beta(2a_2^2-a_3)\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+\frac{\beta(\beta-3)}{2}\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2 a_2^2=\frac{1}{2}B_1(d_2-\frac{d_1^2}{2})+\frac{1}{4}B_2d_1^2 \tag{2.35}$$

From (2.32)and (2.34) we get

$$c_1=-d_1 \tag{2.36}$$

and

$$8\beta^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2 a_2^2=B_1^2(c_1^2+d_1^2). \tag{2.37}$$

Adding (2.33)and(2.35) we get

$$4\left[4\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+(\beta^2-3\beta)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]a_2^2=2B_1(c_2+d_2)+(B_2-B_1)(c_1^2+d_1^2) \tag{2.38}$$

This on using(2.37) in(2.38), we get:

$$a_2^2=\frac{B_1^3(c_2+d_2)}{\left[8\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+4(\frac{(\beta^2-3\beta)}{2})\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]B_1^2+4(B_1-B_2)\beta^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}. \tag{2.39}$$

Applying Lemma 1.1 in(2.39) and using (2.9)we get the desired estimates on $|a_2|$as

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\left[2\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+(\frac{\beta^2-3\beta}{2})\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]B_1^2+(B_1-B_2)\beta^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}}. \tag{2.40}$$

Next, to find the estimates on $|a_3|$, subtracting (2.35)from(2.33) and then using (2.36) we get:

$$a_3=a_2^2+\frac{B_1(c_2-d_2)}{8\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}}. \tag{2.41}$$

Using (2.37) in (2.41) we get:

$$a_3=\frac{B_1^2(c_1^2+d_1^2)}{8\beta^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}+\frac{B_1(c_2-d_2)}{8\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}}. \tag{2.42}$$

Applying Lemma 1.1 in (2.42) and using (2.9) we get the desired estimates on $|a_3|$ as

$$|a_3|\leq\frac{B_1}{2\beta\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}}+\frac{B_1^2}{\beta^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}. \tag{2.43}$$

This completes the proof of Theorem. □

Definition 2.3. *A function f is given by* (1.1) *is said to be in the class $\Re_\mu^{a,c}[\alpha,\phi]$ if the following conditions are satisfied:*

$$f\in\Sigma \quad , \quad \left[1+\frac{z(J_\mu^{a,c}f(z))''}{(J_\mu^{a,c}f(z))'}\right]^\alpha\prec\phi(z) \qquad (z\in U). \tag{2.44}$$

and

$$\left[1+\frac{w(J_{\mu}^{a,c}g(w))''}{(J_{\mu}^{a,c}g(w))'}\right]^{\alpha} \prec \phi(w) \qquad (w \in U). \tag{2.45}$$

Where $\alpha > 0$ and the function g and ϕ defined by (1.6)*and* (1.2) *respectively.*
We first state and prove the following result.

Theorem 2.3. *Let f given by* (1.1) *be in the function class $\Re_{\mu}^{a,c}[\alpha,\phi]$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2\left[3\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+(\alpha^2-3\alpha)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]B_1^2+4(B_1-B_2)\alpha^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}} \tag{2.46}$$

and

$$|a_3| \leq \frac{B_1}{6\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}}+\frac{B_1^2}{4\alpha^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2} \tag{2.47}$$

Proof. Since $\Re_{\mu}^{a,c}[\alpha,\phi]$, there exist two analytic functions $u, v : U \to U$ with $u(0) = v(0) = 0$ such that

$$\left[1+\frac{z(J_{\mu}^{a,c}f(z))''}{(J_{\mu}^{a,c}f(z))'}\right]^{\alpha} = \phi(u(z)) \tag{2.48}$$

and

$$\left[1+\frac{w(J_{\mu}^{a,c}g(w))''}{(J_{\mu}^{a,c}g(w))'}\right]^{\alpha} = \phi(v(w)) \tag{2.49}$$

where $z, w \in U$. Define the functions s and t as in Theorem 1 and then proceed similarly upto (2.11). Expanding (2.48) and (2.49), we obtain:

(2.50)
$$\left[1+\frac{z(J_{\mu}^{a,c}f(z))''}{(J_{\mu}^{a,c}f(z))'}\right]^{\alpha} = 1+2\alpha\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}a_2z+ \left[6\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}a_3+(2\alpha^2-6\alpha)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]z^2+....$$

$$\left[1+\frac{w(J_{\mu}^{a,c}g(w))''}{(J_{\mu}^{a,c}g(w))'}\right]^{\alpha} = 1-2\alpha\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}a_2w+ \left[6\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}(2a_2^2-a_3)+...\right]w^2+.... \tag{2.51}$$

Now, using(2.10)(2.11)(2.50)(2.51) in (2.48) and (2.49) and then equating the coefficients of z, z^2, w, w^2, we get

$$2\alpha\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}a_2 = \frac{1}{2}B_1c_1 \tag{2.52}$$

$$\left[6\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}a_3+(2\alpha^2-6\alpha)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2 a_2^2\right]=\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2C_1^2 \tag{2.53}$$

$$-2\alpha\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}a_2=\frac{1}{2}B_1d_1 \tag{2.54}$$

$$\left[6\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}(2a_2^2-a_3)+(2\alpha^2-6\alpha)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2 a_2^2\right]=\frac{1}{2}B_1(d_2-\frac{d_1^2}{2})+\frac{1}{4}B_2d_1^2. \tag{2.55}$$

From (2.52) and (2.54) we get

$$c_1=-d_1 \tag{2.56}$$

and

$$32\alpha^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2 a_2^2=B_1^2(c_1^2+d_1^2). \tag{2.57}$$

Adding (2.53) and (2.55) we get

$$4\left[12\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+4(\alpha^2-3\alpha)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]a_2^2=2B_1(c_2+d_2)+(B_2-B_1)(c_1^2+d_1^2) \tag{2.58}$$

This on using (2.57) in(2.58),we get:

$$a_2^2=\frac{B_1^3(c_2+d_2)}{16\left[3\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+(\alpha^2-3\alpha)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]B_1^2+32(B_1-B_2)\alpha^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}. \tag{2.59}$$

Applying Lemma 1.1 in(2.58) and using (2.9) we get the desired estimates on $|a_2|$as

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{2\left[3\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}+(\alpha^2-3\alpha)\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2\right]B_1^2+4(B_1-B_2)\alpha^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}}. \tag{2.60}$$

Next, to find the estimates on $|a_3|$, subtracting (2.55)from(2.53) and then using (2.56) we get:

$$a_3=a_2^2+\frac{2B_1(c_2-d_2)}{48\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}}. \tag{2.61}$$

Using (2.37) in (2.41) we get:

$$a_3=\frac{B_1^2(c_1^2+d_1^2)}{32\alpha^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}+\frac{2B_1(c_2-d_2)}{48\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}}. \tag{2.62}$$

Applying Lemma 1.1 in (2.62) and and using (2.9) we get the desired estimates on $|a_3|$ as

$$|a_3|\le\frac{B_1}{6\alpha\frac{\Gamma(c+\mu)\Gamma(a+3\mu)}{\Gamma(a+\mu)\Gamma(c+3\mu)}}+\frac{B_1^2}{4\alpha^2\left(\frac{\Gamma(c+\mu)\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)}\right)^2}. \tag{2.63}$$

This completes the proof of Theorem. □

Remark 2.1. *Taking $a=c$, $\gamma=1$ in the class $M_\mu^{a,c}[\gamma,\phi]$and $\beta=1$ in the class $\Im_\mu^{a,c}[\beta,\phi]$. We have $M_\mu^{a,c}[\gamma,\phi]=\mathcal{S}_\Sigma^*(\phi)$ and $\Im_\mu^{a,c}[\beta,\phi]=\mathcal{S}_\Sigma^*(\phi)$.*

Remark 2.2. *Taking $a=c$, $\gamma=0$ in the class $M_\mu^{a,c}[\gamma,\phi]$and $\alpha=1$ in the class $\Re_\mu^{a,c}[\alpha,\phi]$. We have $M_\mu^{a,c}[\gamma,\phi]=\mathcal{K}_\Sigma^*(\phi)$ and $\Re_\mu^{a,c}[\alpha,\phi]=\mathcal{K}_\Sigma^*(\phi)$.*

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