Existence of fixed point for rational type contraction in $\mathcal{F}$-metric space

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Abstract

In this manuscript, we prove some fixed point result for rational type contraction in the context of $\mathcal{F}$-metric spaces. Our result generalize and extend some already proved result in the literature. We also give an example to support our result.

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1 Introduction and Preliminaries

Banach contraction principle is one of the most pivotal result in the study of nonlinear analysis. Due to its applicability this principle has been generalized in different directions by various researchers. There are many authors in the literature who have extended and generalized it. In recent years, various extensions of metric space have been introduced(see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [13], [14], [15], [17], [18], [19]). Very recently, In 2018 Jleli and Samet [12] introduced the notion of $\mathcal{F}$-metric space in the following way.

Let $\mathcal{F}$ be the set of function f : $(0,\infty) \longrightarrow$ R satisfying the following conditions:

($\mathcal{F}_1$) f is non-decreasing, i.e., $0 < s < t \Longrightarrow f(s) \leq f(t)$.

($\mathcal{F}_2$) for every sequence $\{t_n\} \subseteq R^+$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} f(t_n) = -\infty$

Definition 1.1. *[12] Suppose that U is a nonempyt set, and* $\rho : U \times U \longrightarrow [0,\infty)$ *is a given mapping. Suppose that there exists* $(f,\alpha) \in \mathcal{F} \times [0,\infty)$ *such that*

(ρ_1) $(\mu,\nu) \in U \times U$, $\rho(\mu,\nu) = 0 \Longleftrightarrow \mu = \nu$

(ρ_2) $\rho(\mu,\nu) = \rho(\nu,\mu)$, *for all* $(\mu,\nu) \in U \times U$

(ρ_3) *for every* $(\mu,\nu) \in U \times U$, *for every* $N \in \mathbb{N}$, $N \geq 2$, *and for every* $(u_i)_{i=1}^N \subset U$ *with* $(u_1,u_N) = (\mu,\nu)$, *we have*

$$\rho(\mu,\nu) > 0 \implies f(\rho(\mu,\nu)) \leq f\left(\sum_{i=1}^{N-1} \rho(\mu_i,\mu_{i+1})\right) + \alpha.$$

Then ρ is said to be an $\mathcal{F}$-metric on U, and the pair (U,ρ) is said to be an $\mathcal{F}$-metric space.

Jleli and Samet [12] showed that any metric space is an $\mathcal{F}$-metric but the converse is not true in general, which validate the concept of $\mathcal{F}$-metric space is more general than the standard metric space concept.

[12] Let $U = \mathrm{R}$ and $\rho : U \times U \longrightarrow [0, \infty)$ be defined as follow:

$$\rho(\mu, \nu) = \begin{cases} (\mu - \nu)^2 & , (\mu, \nu) \in [0, 3] \times [0, 3], \\ | \mu - \nu | & otherwise, \end{cases}$$

and let f(t) = ln(t) for all t > 0 and $\alpha = \ln(3)$. Then ρ is an $\mathcal{F}$-metric on U. But, ρ is not a metric on U. Since, $\rho(1, 3) = 4 \geq \rho(1, 2) + \rho(2, 3) = 2$.

[12] Let $U = \mathrm{R}$ and $\rho : U \times U \longrightarrow [0, \infty)$ be defined as follow:

$$\rho(\mu, \nu) = \begin{cases} e^{|\mu-\nu|}, & if\ \mu \neq \nu, \\ 0, & if\ \mu = \nu, \end{cases}$$

Then again ρ is an $\mathcal{F}$-metric on U. But it is not a metric on U.

Definition 1.2. *[12] Let (U,ρ) be an $\mathcal{F}$-metric space and $\{\mu_n\}$ be a sequence in U.*

A sequence $\{\mu_n\}$ is called $\mathcal{F}$-convergent to $\mu \in U$, if $\lim_{n\to\infty} \rho(\mu_n, \mu) = 0$.

A sequence $\{\mu_n\}$ is $\mathcal{F}$-Cauchy, if and only if $\lim_{n,m\to\infty} \rho(\mu_n, \mu_m) = 0$.

An $\mathcal{F}$-metric space (U,ρ) is said to be $\mathcal{F}$-complete, if every $\mathcal{F}$-Cauchy sequence in U is $\mathcal{F}$-convergent to some element in U.

In 2018, Olatinwo and Ishole [16] proved the following fixed point result in complete metric space.

Theorem 1.1. *Let (U,ρ) be a complete metric space and Q : U ⟶ U be a self mapping such that there exist $\alpha, p, q, r, \omega, \eta \in R^+$ and $\beta \in [0, 1)$ satisfying*

$$\rho(Q\mu, Q\nu) \leq \alpha \frac{[p + \rho(\mu, Q\mu)][\rho(\nu, Q\nu)]^r[\rho(\nu, Q\mu)]^q}{\rho(\mu, \nu) + \omega\rho(\mu, Q\nu) + \eta\rho(\nu, Q\mu)} + \beta\rho(\mu, \nu)$$

for all $(\mu, \nu) \in U \times U$ and $\rho(\mu, \nu) + \omega\rho(\mu, Q\nu) + \eta\rho(\nu, Q\mu) > 0$. Then Q has a unique fixed point in U.

The purpose of this paper is to establish Olatinwo and Ishole rational type contraction in the context of $\mathcal{F}$-metric spaces and established some new fixed point result. We also suppport our main result by providing a non-trivial example.

2 Main Results

First we define Olatinwo and Ishole [16] contraction in the context of $\mathcal{F}$-metric space and then establish a new fixed point theorem.

Definition 2.1. *Suppose that (U,ρ) is an $\mathcal{F}$-metric space. A self mapping Q on U is said to be Olatinwo and Ishole contraction, if there exist $\alpha, p, q, r, \omega, \eta \in R^+$ and $\beta \in [0, 1)$ such that*

$$\rho(Q\mu, Q\nu) \leq \alpha \frac{[p+\rho(\mu, Q\mu)][\rho(\nu, Q\nu)]^r[\rho(\nu, Q\mu)]^q}{\rho(\mu,\nu)+\omega\rho(\mu, Q\nu)+\eta\rho(\nu, Q\mu)} + \beta\rho(\mu,\nu)$$

for all $(\mu,\nu) \in U \times U$ and $\rho(\mu,\nu)+\omega\rho(\mu, Q\nu)+\eta\rho(\nu, Q\mu) > 0$.

Theorem 2.1. *Suppose that (U,ρ) is an $\mathcal{F}$-metric space, and let $Q : U \to U$ is an Olatinwo and Ishole contraction. If (U,ρ) is $\mathcal{F}$-complete, then Q has a unique fixed point $\mu^* \in X$. Moreover, for any $\mu_0 \in X$, the sequence $\{\mu_n\} \subset X$ defined by*

(2.1) $$\mu_{n+1} = Q(\mu_n),\ n \in \mathbb{N}$$

is $\mathcal{F}$-convergent to μ^*.

Proof. Let $\mu_0 \in U$ be an arbitrary element. Let the sequence $\{\mu_n\} \subset U$ be defined by (2.1). As $Q : U \to U$ is Olatinwo and Ishole contraction, we have

$$\begin{aligned}
\rho(\mu_n,\mu_{n+1}) &= \rho(Q(\mu_{n-1}), Q(\mu_n)) \\
&\leq \alpha \frac{[p+\rho(\mu_{n-1}, Q\mu_{n-1})][\rho(\mu_n, Q\mu_n)]^r[\rho(\mu_n, Q\mu_{n-1})]^q}{\rho(\mu_{n-1},\mu_n)+\omega\rho(\mu_{n-1}, Q\mu_n)+\eta\rho(\mu_n, Q\mu_{n-1})} + \beta\rho(\mu_{n-1},\mu_n) \\
&\leq \alpha \frac{[p+\rho(\mu_{n-1},\mu_n)][\rho(\mu_n,\mu_{n+1})]^r[\rho(\mu_n,\mu_n)]^q}{\rho(\mu_{n-1},\mu_n)+\omega\rho(\mu_{n-1},\mu_{n+1})+\eta\rho(\mu_n,\mu_n)} + \beta\rho(\mu_{n-1},\mu_n) \\
&\leq \beta^n\rho(\mu_0,\mu_1),\quad n \in N
\end{aligned}$$

This implies that

$$\sum_{i=n}^{m-1} \rho(\mu_i,\mu_{i+1}) \leq \frac{\beta^n}{1-\beta}\rho(\mu_0,\mu_1),\ m > n$$

Since,

$$\lim_{n\to\infty} \frac{\beta^n}{1-\beta}\rho(\mu_0,\mu_1) = 0,$$

there exist some $N \in \mathbb{N}$ such that

$$0 < \frac{\beta^n}{1-\beta}\rho(\mu_0,\mu_1) < \delta,\quad n \geq N$$

Next, let $(f,\alpha) \in \mathcal{F} \times [0,\infty)$ be such that (ρ_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$ there exist $\delta > 0$ such that

(2.2) $$0 < t < \delta \implies f(t) < f(\epsilon) - \alpha$$

Hence, by (2.2) and $(\mathcal{F}_1)$, we have

(2.3) $$f\left(\sum_{i=n}^{m-1} \rho(\mu_i,\mu_{i+1})\right) \leq f\left(\frac{\beta^n}{1-\beta}\rho(\mu_0,\mu_1)\right) < f(\epsilon) - \alpha$$

for m > n $\geq N$. Using (ρ_3) and (2.3), we obtain $\rho(\mu_n,\mu_m) > 0$, m > n $\geq N$ implies

$$f(\rho(\mu_n,\mu_m)) \leq f\left(\sum_{i=n}^{m-1} \rho(\mu_i,\mu_{i+1})\right) + \alpha < f(\epsilon).$$

Thus, we get $\rho(\mu_n, \mu_m) < \epsilon$, m > n $\geq N$. This show that the sequence $\{\mu_n\}$ is $\mathcal{F}$ - Cauchy. Since, the pair (U,ρ) is $\mathcal{F}$-complete, there exist $\mu^* \in U$ such that the sequence $\{\mu_n\}$ is $\mathcal{F}$-convergent to μ^*, that is,

$$\lim_{n\to\infty} \rho(\mu_n, \mu^*) = 0. \tag{2.4}$$

We shall show that μ^* is a fixed point of Q. We declare by contradiction that $\rho(Q(\mu^*), \mu^*) > 0$. By ρ_3, we get

$$f(\rho(Q(\mu^*), \mu^*)) \leq f(\rho(Q(\mu^*), Q(\mu_n)) + \rho(Q\mu_n, \mu^*)) + \alpha,\ n \in N.$$

As $Q : U \longrightarrow U$ is Olatinwo and Ishole contraction, we get

$$\begin{aligned} f(\rho(Q(\mu^*), \mu^*)) &\leq f\left(\alpha \frac{[p + \rho(\mu^*, Q\mu^*)][\rho(\mu_n, Q\mu_n)]^r [\rho(\mu_n, Q\mu^*)]^q}{\rho(\mu^*, \mu_n) + \omega\rho(\mu^*, Q\mu_n) + \eta\rho(\mu_n, Q\mu^*)}\right) \\ &+ f\left(\beta\rho(\mu^*, \mu_n) + \rho(\mu_{n+1}, \mu^*)\right) + \alpha \end{aligned}$$

for n $\in N$.

Now, using $(\mathcal{F}_2)$ and (2.4), we have

$$\lim_{n\to\infty} f\left(\alpha \frac{[p + \rho(\mu^*, Q\mu^*)][\rho(\mu_n, Q\mu_n)]^r [\rho(\mu_n, Q\mu^*)]^q}{\rho(\mu^*, \mu_n) + \omega\rho(\mu^*, Q\mu_n) + \eta\rho(\mu_n, Q\mu^*)} + \beta\rho(\mu^*, \mu_n) + \rho(\mu_{n+1}, \mu^*)\right)$$
$$+ \alpha = -\infty$$

which is a contradiction. Hence, we have $\rho(Q(\mu^*), \mu^*) = 0$, that is, $Q(\mu^*) = \mu^*$.

Therefore, μ^* is a fixed point of Q.

Now, we shall prove that μ^* is the unique fixed point of Q.

Let $\mu^*, \nu^* \in U \times U$ are two fixed points of Q with $\mu^* \neq \nu^*$ that is

$$Q(\mu^*) = \mu^*, \qquad Q(\nu^*) = \nu^*.$$

As $Q : U \longrightarrow U$ is Olatinwo and Ishole contraction, so

$$\begin{aligned} \rho(\mu^*, \nu^*) &= \rho(Q\mu^*, Q\nu^*) \\ &\leq \alpha \frac{[p + \rho(\mu^*, Q\mu^*)][\rho(\nu^*, Q\nu^*)]^r [\rho(\nu^*, Q\mu^*)]^q}{\rho(\mu^*, \nu^*) + \omega\rho(\mu^*, Q\nu^*) + \eta\rho(\nu^*, Q\mu^*)} + \beta\rho(\mu^*, \nu^*) \\ &\leq \alpha \frac{[p + \rho(\mu^*, \mu^*)][\rho(\nu^*, \nu^*)]^r [\rho(\nu^*, \mu^*)]^q}{\rho(\mu^*, \nu^*) + \omega\rho(\mu^*, \nu^*) + \eta\rho(\nu^*, \mu^*)} + \beta\rho(\mu^*, \nu^*) \\ &< \beta\rho(\mu^*, \nu^*) \end{aligned}$$

which is a contradiction. Therefore, $\mu^* = \nu^*$.

Hence, μ^* is the unique fixed point of Q. □

Now, we shall give an example to validate our main result.

Let $U = [0, 1]$ with the metric $\rho : U \times U \longrightarrow [0, \infty)$ defined as $\rho(\mu, \nu) = |\ \mu - \nu\ |$. Taking f(t) = lnt and $\alpha = 0$, then (U, ρ) is an $\mathcal{F}$-metric space which is also $\mathcal{F}$-complete.

Assume $Q\mu = \frac{\mu}{4}$ and $\beta = \frac{1}{4}$, p=0, r=q=ω=η=α=1, we have

$$\rho(Q\mu, Q\nu) = \left| \frac{\mu}{4} - \frac{\nu}{4} \right| \leq \frac{|\mu - \frac{\mu}{4}||\nu - \frac{\nu}{4}||\nu - \frac{\mu}{4}|}{|\mu - \nu| + |\mu - \frac{\nu}{4}| + |\nu - \frac{\mu}{4}|} + \frac{1}{4}|\mu - \nu|$$

and since, $\dfrac{|\mu - \frac{\mu}{4}||\nu - \frac{\nu}{4}||\nu - \frac{\mu}{4}|}{|\mu - \nu| + |\mu - \frac{\nu}{4}| + |\nu - \frac{\mu}{4}|} \geq 0$

Therefore, Q satisfies all the conditions of Theorem 2.1 and Q has a unique fixed point $\mu = 0 \in [0, 1]$.

If $\alpha = 0$. Then Theorem 2.1 reduces to the Banach contraction principle in an $\mathcal{F}$-metric space.

Corollary 2.1. *Assume (U, ρ) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $Q : U \longrightarrow U$ be a self mapping satisfying*

$$\rho(Q\mu, Q\nu) \leq \beta\rho(\mu, \nu)$$

for all $\mu, \nu \in U$, where $0 \leq \beta < 1$. Then Q has a unique fixed point.

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