Properties of Adjoints of Generalized Slant Toeplitz Operators

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Abstract
Let T be the unit circle $\{z \in \mathbb{C} :| z |= 1\}$ and $\varphi = \sum_n a_n e^{in\theta}$ be a bounded measurable function on T. The k^{th} order slant Toeplitz operator (for each integer $k \geq 2$) U_φ on the space $L^2(T)$ is defined as $\langle U_\varphi e^m, e^n \rangle = a_{kn-m}$ for all n,m in $\mathbb{Z}, \mathbb{Z}$ being the set of integers and $\{e_n(z) = z^n, z \in T\}$ is an orthonormal basis of $L^2(T)$. In this paper we discuss the point spectrum of U_φ^* and we show that the point spectrum of U_φ^* is contained in a circle of positive radius. Furthermore we throw some light on the relation between U_φ^* and the isometries .

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1 Introduction

The study of Toeplitz operators began almost in the beginning of 20th century by O. Toeplitz. A lot of work on Toeplitz operators has been done by different mathematicians in the world. Toeplitz operators became a subject of investigations for the researchers. Motivated by these, M. C. Ho [5,6,7,8] in the year 1996 introduced a new class of operators called slant Toeplitz operators on the space $L^2(T)$. Further, these notions have been generalized [1] to k^{th} order slant Toeplitz operators , for $k \geq 2$ and many mathmaticians including [9,10] have studied properties of these operators simultaneously. Let $\varphi(z) = \sum_{i=-\infty}^{\infty} a_i z^i$ or $\varphi = \sum_n a_n e^{in\theta}$ be a bounded measurable function on the unit circle T, where $a_i = \langle \varphi, z^i \rangle$ is the i^{th} Fourier coefficient of φ and $\{z^i : z \in \mathbb{Z}\}$ is the standard basis of the space $L^2(T)$. For $k \geq 2$,the k^{th} order slant Toeplitz operator U_φ[1] is defined as follows:

$$U_\varphi(z^t) = \sum_{i=-\infty}^{\infty} a_{ki-t} z^i.$$

That is, $U_\varphi = W_k M_\varphi$ where M_φ is multiplication operator induced by φ and W_k is an operator on $L^2(T)$ defined by

$$W_k(z^i) = \begin{cases} z^{i/k} & \text{if i is divisible by k} \\ 0 & \text{otherwise .} \end{cases}$$

We can see that the adjoint U_φ^* of k^{th} order slant Toeplitz operator U_φ, is given by

$$U_\varphi^*(z^j) = (W_k W_\varphi)^* z^j = \sum_{t=-\infty}^{\infty} \bar{a}_{kj-t} z^t.$$

Let $[T, \mathcal{A}, \mu]$ be a probability space and let $\tau_k : T \to T$ be a measure preserving continuous map. Also let $T : L^p \to L^p (1 \leq p < \infty)$ be defined as $Tf = f \circ \tau_k$, for any f in $L^p(T)$, where $\tau_k : T \to T$ for each $k \geq 2$ is defined as $\tau(e^{i\theta}) = e^{ki\theta}$.Define

$$S_n f = \frac{1}{n} \sum_{t=0}^{n-1} T^t f$$

for all n.Then for each $k \geq 2$,τ_k is ergodic [13] if and only if the T-invariant function are constants.Also

$$(U_\varphi)^*(f(z)) = M_{\overline{\varphi}} W_k^* f = \overline{\varphi}(f \circ \tau_k).$$

That is, U_φ^* is a weighted composition operator on T as for all f in L^2

$$(U_\varphi)^* f = \overline{\varphi}(f \circ \tau_k).$$

2 Spectrum of U_φ^*

In [1] it is proved that if φ is invertible in L^∞ then the spectrum of U_φ contains a closed disc. Then in [3] it is shown that the spectrum of the operator U_φ, if φ is continuous, is a closed disc and furthermore if φ does not vanish on T, then the interior of its spectrum consists of eigen values of infinite multiplicities. This shows that the point spectrum of U_φ is large but it is not true for U_φ^* as proved in the following theorem .

Theorem 2.1. *Let φ be an L^∞-function on the unit circle such that* $\log|\varphi|$ *is integrable then*

$$\sigma_p(U_\varphi^*) \subseteq \left\{\lambda : |\lambda| = \exp \int_0^{2\pi} \log|\varphi| \frac{d\theta}{2\pi}\right\}.$$

Proof. Let φ in L^∞ satisfy that $\log|\varphi|$ is integrable.Let $\lambda \neq 0$ be an eigen value of U_φ^*. Then there exists $0 \neq f$ in $L^2(T)$ such that

$$U_\varphi^* f = \lambda f.$$

That is $(W_k M_\varphi)^* f = \lambda f$.This implies that $\overline{\varphi}(\theta) f(k\theta) = \lambda f(\theta)$ a.e. Let

$$\hat{E} = \{\theta \in [0, 2\pi]; f(\theta) \neq 0\}.$$

Since $\lambda \neq 0$,this implies that whenever $f(\theta) \neq 0$, then $f(k\theta) \neq 0$ a.e. and which gives that $k\theta \in \hat{E}$ whenever $\theta \in \hat{E}$. Hence $\hat{E}$ is invariant under $\tau_k : e^{i\theta} \to e^{ki\theta}$.Since τ_k is ergodic, therefore $\mu(\hat{E}) = 0$ and $\mu(\hat{E}) = 1$, where μ is the normalized Lebesgue measure. Also since $f(\theta) \neq 0$,therefore $\mu(\hat{E}) \neq 0$ and thus $\mu(\hat{E}) = 1$. Thus $f(\theta) \neq 0$ a.e. on T and therefore $f \circ \tau_k \neq 0$ a.e. on T for all n as E is invariant under τ_k. Now we have $|\overline{\varphi}(\theta)||f(k\theta)| = |\lambda||f(\theta)|$ a.e. This implies that

$$|\overline{\varphi}(k\theta)||f(k^2\theta)| = |\lambda||f(k\theta)| = \frac{|\lambda|^2|f(\theta)|}{|\overline{\varphi}(\theta)||}$$

a.e . That is

$$|\overline{\varphi}(k\theta)||f(k^2\theta)||\overline{\varphi}(\theta)| = |\lambda|^2|f(\theta)|a.e.$$

This further implies that

$$|\varphi(\theta)||\varphi(k^2\theta)||\varphi(k\theta)||f(k^2\theta)| = |\lambda|^2||f(\theta)|a.e.$$

Continuing like this,we have

$$\prod_{i=0}^{n-1} |\varphi(k^i\theta)||f(k^n\theta)| = |\lambda|^n||f(\theta)|a.e.$$

Since $f \circ \tau_k^n \neq 0$ a.e. on T ,therefore we get

$$\prod_{i=0}^{n-1} |\varphi(k^i\theta)| = \frac{|\lambda|^n|f(\theta)|}{|f(k^n\theta)|} a.e.$$

Equivalently,we have

$$\left(\prod_{i=0}^{n-1}|\varphi(k^i\theta)|\right)^{\frac{1}{n}} = \frac{|\lambda||f(\theta)|^{\frac{1}{n}}}{|f(k^n\theta)|^{\frac{1}{n}}}. \tag{2.1}$$

Next we claim that $\frac{|f(\theta)|^{\frac{1}{n}}}{|f(k^n\theta)|^{\frac{1}{n}}} \to 1$ a.e. on T.
Since $\log|\varphi|$ is integrable and $|\lambda|$ is constant, therefore

$$\log\frac{|f(\theta)|}{|(f\circ\tau_k)(\theta)|} = \log\frac{|f(\theta)|}{|f(k\theta)|} = \log\frac{|\varphi|}{|\lambda|} = \log|\varphi| - \log|\lambda|$$

is integrable.So applying Birkhoff's ergodic theorem [13] to L^1-function $\log\frac{|f(\theta)|}{|(f\circ\tau_k)(\theta)|}$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\log\frac{|f(k^i\theta)|}{|f(k^{i+1}\theta)|} \to \int_T \log\frac{|f(\theta)|}{|(f\circ\tau_k)(\theta)|}\frac{d\theta}{2\pi}a.e. \tag{2.2}$$

Consider

$$\begin{aligned}\int_0^{2\pi}\log\frac{|f(\theta)|}{|(f\circ\tau_k)(\theta)|}\frac{d\theta}{2\pi} &= \int_0^{2\pi}\log|f(\theta)|\frac{d\theta}{2\pi} - \int_0^{2\pi}\log|f(k\theta)|\frac{d\theta}{2\pi}\\ &= \int_0^{2\pi}\log|f(\theta)|\frac{d\theta}{2\pi} - \frac{1}{k}\int_0^{2k\pi}\log|f(\theta)|\frac{d\theta}{2\pi}\\ &= \int_0^{2\pi}\log|f(\theta)|\frac{d\theta}{2\pi} - \frac{1}{k}\bigg(\int_0^{2\pi}+\int_{2\pi}^{4\pi}+...\\ &+\int_{(k-1)2\pi}^{2k\pi}\log|f(\theta)|\frac{d\theta}{2\pi}\bigg)\\ &= 0.\end{aligned}$$

Hence by using (2.2) we get that $\frac{1}{n}\sum_{i=0}^{n-1}\log\frac{|f(k^i\theta)|}{|f(k^{i+1}\theta)|} \to 0$ a.e.

Since $\log\frac{|f(\theta)|}{|f(k^n\theta)|} = \frac{1}{n}\sum_{i=0}^{n-1}\log\frac{|f(k^i\theta)|}{|f(k^{i+1}\theta)|}$ therefore $\log\frac{|f(\theta)|}{|f(k^n\theta)|} \to 0$ and so $\frac{|f(\theta)|}{|f(k^n\theta)|} \to e^0 = 1$ a.e. θ. This further implies that $\frac{|f(\theta)|^{\frac{1}{n}}}{|f(k^n\theta)|^{\frac{1}{n}}} \to 1$ a.e. .

Therefore by using (2.1) and [2,Lemma 8] we get that,

$$|\lambda| = \left(\prod_{i=0}^{n-1}|\varphi(k^i\theta)|\right)^{\frac{1}{n}} \to \exp\left(\int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi}\right)$$

a.e. on T. Finally we can see λ cannot be zero.For if $\lambda = 0$ and λ is an eigen value of U_φ^*,then $|\overline{\varphi}(k\theta)||f(k\theta)| = 0$ a.e. on θ. Since $\log|\varphi|$ is integrable,therefore $\varphi \neq 0$ a.e. on T. This yields that $f(k\theta) = 0$ a.e. on T and hence $f(\theta) = 0$ a.e. on T. This is a contradiction. Thus, 0 cannot be a eigen value of U_φ^*. Hence

$$\sigma_p(U_\varphi^*) \subseteq \left\{\lambda : |\lambda| = \exp\int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi}\right\}.$$

This completes the proof .

□

Corollary 2.1. *Let φ be an L^∞ function such that $\log|\varphi|$ is integrable. Then for any λ such that $|\lambda| \neq \exp\int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi}$, $U_\varphi - \lambda$ is invertible iff $U_\varphi^* - \overline{\lambda}$ is onto.*

Proof. Let $U_\varphi - \lambda$ be invertible. This implies $U_\varphi^* - \overline{\lambda}$ is invertible and hence onto.
Conversely let $U_\varphi^* - \overline{\lambda}$ be onto and suppose that $U_\varphi^* - \overline{\lambda}$ is not one-one, then there exists some non-zero f in L^2 such that $(U_\varphi^* - \overline{\lambda})f = 0$. That is,$U_\varphi^* f = \overline{\lambda}f$.It follows that

$$\overline{\lambda} \in \sigma_p(U_\varphi^*) \subseteq \left\{\lambda : |\lambda| = \exp\int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi}\right\}$$

which gives that

$$|\lambda| = |\overline{\lambda}| = \exp\int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi}.$$

This is a contradiction to the hypothesis.Thus $U_\varphi^* - \overline{\lambda}$ is one-one and hence invertible.The result follows. □

Theorem 2.2. *If $\sigma_{le}(U_\varphi)$ and $\sigma_{re}(U_\varphi)$ denote the left and the right essential spectrum respectively and if λ is an isolated point in $\sigma(U_\varphi)$ not in $\sigma_{le}(U_\varphi)\cap\sigma_{re}(U_\varphi)$, satisfying $|\lambda| \neq \exp\int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi}$,then $U_\varphi - \lambda$ is invertible.*

Proof. As λ is an isolated point in $\sigma(U_\varphi)$ such that λ does not belong to $\sigma_{le}(U_\varphi)\cap\sigma_{re}(U_\varphi)$,therefore from [12] it follows that $U_\varphi - \lambda I$ is Fredholm and $i(U_\varphi - \lambda) = 0$. Since $|\lambda| \neq \exp\int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi}$ and $\sigma_p(U_\varphi^*) \subseteq \left\{\lambda : |\lambda| = \exp\int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi}\right\}$,therefore λ must not belong to $\sigma_p(U_\varphi^*)$.This further gives that $Ker(U_\varphi^* - \overline{\lambda}) = \{0\}$ and hence $U_\varphi^* - \overline{\lambda}$ is one-one with $dimKer(U_\varphi^* - \overline{\lambda}) = 0$. Again since, $dimKer(U_\varphi - \lambda) - dimKer(U_\varphi^* - \overline{\lambda}) = i(U_\varphi - \lambda) = 0$, therefore $dimKer(U_\varphi - \lambda) = 0$ and $Ker(U_\varphi - \lambda) = \{0\}$. This implies that $\mathcal{R}(U_\varphi^* - \overline{\lambda}) = L^2$ (where $\mathcal{R}(U_\varphi^* - \overline{\lambda})$ is the range of $U_\varphi^* - \overline{\lambda}$), implying that $U_\varphi^* - \overline{\lambda}$ is onto.
This completes the proof. □

3 Relation between U_φ^* and the isometries

Let φ be continuous in L^∞ .Then define
$(L_\varphi f)(\theta) = \varphi\left(\frac{\theta}{k}\right) f\left(\frac{\theta}{k}\right) + \varphi\left(\frac{\theta+2\pi}{k}\right) f\left(\frac{\theta+2\pi}{k}\right) + ... + \varphi\left(\frac{\theta+(k-1)2\pi}{k}\right) f\left(\frac{\theta+(k-1)2\pi}{k}\right)$.
Then we can see that for each n in $\mathbb{Z}$,

$$\langle L_\varphi f, e^{in\theta}\rangle = \int_0^{2\pi}(L_\varphi f)(0)e^{-in\theta}\frac{d\theta}{2\pi} = k\langle U_\varphi f, e^{in\theta}\rangle.$$

That is $L_\varphi f = kU_\varphi f$ for all f in L^2 and hence $U_\varphi = \frac{1}{k}L_\varphi$.

Remark 3.1. *Let X be a compact metric space with differential structure and let $T : X \to X$ be an n to 1 covering map.For any $\alpha > 0$ and a g in $C^\infty(X)$[12].Consider the operator[12] on $C^\alpha(X)$ as*

$$\mathcal{L}_g f(x) = \sum_{y\in T^{-1}(x)} g(y)f(y).$$

for all f in $C^\alpha(X)$.In our case $\mathcal{L}_g = L_\varphi$ with X=T and $\tau_k : T \to T$ defined as $\tau_k(z) = z^k$.This yields to another definition of U_φ as follows

$$(U_\varphi f)(z) = \frac{1}{k} L_\varphi(f(z)) = \frac{1}{k} \sum_{W^k=Z} \varphi(w) f(w), z \in T.$$

Now we will be using this definition of U_φ and with this we try to get a relation between U_φ^* and the isometries.To prove the main result we need the following Lemma .

Lemma 3.1. *If there exists a $\lambda > 0$ and $h \in L^\infty(T), h > 0$ such that*

$$U_{|\varphi|^2} h = \lambda h$$

then U_φ^* is similar to a constant multiple of isometry.

Proof. Let there exist a $\lambda > 0$ and $h \in L^\infty(T), h > 0$ such that

$$U_{|\varphi|^2} h = \lambda h. \tag{3.1}$$

Consider

$$\psi = \frac{\varphi \sqrt{h}}{\sqrt{\lambda (h \circ \tau_k)}}.$$

Then we can see from (2) that

$$\begin{aligned}\left|\psi\left(\frac{\theta}{k}\right)\right|^2 &= \left|\frac{\varphi(\frac{\theta}{k}) \sqrt{h(\frac{\theta}{k})}}{\sqrt{\lambda \left(h \circ \tau_k(\frac{\theta}{k}\right)}}\right|^2 \\ &= \frac{\left|\varphi(\frac{\theta}{k})^2\right| |h(\frac{\theta}{k})|}{|\lambda||h(\theta)|} \\ &= 1.\end{aligned}$$

In a similar way we can observe that

$$\left|\psi\left(\frac{\theta + 2\pi}{k}\right)\right|^2 = \left|\psi\left(\frac{\theta + 4\pi}{k}\right)\right|^2 = ... = \left|\psi\left(\frac{\theta + (k-1)2\pi}{k}\right)\right|^2 = 1$$

and hence we get that

$$\left|\psi\left(\frac{\theta}{k}\right)\right|^2 + \left|\psi\left(\frac{\theta + 2\pi}{k}\right)\right|^2 + ... + \left|\psi\left(\frac{\theta + (k-1)2\pi}{k}\right)\right|^2 = k \; a.e.$$

By Theorem 8 in [1] we have that U_ψ^* is an isometry. Also in this case for any f in $L^2(T)$,

$$U_\psi^* f = \overline{\psi}(f \circ \tau_k) = \left(\overline{\frac{\varphi \sqrt{h}}{\sqrt{\lambda (h \circ \tau_k)}}}\right).(f \circ \tau_k)$$

$$= \frac{1}{\sqrt{\lambda}} \frac{\overline{\varphi} \sqrt{h}}{\sqrt{h \circ \tau_k}}.f \circ \tau_k.$$

Let $\sqrt{h} = g$ and $\sqrt{\lambda} = r$ we get that

$$U_{\psi}^{*} f = \frac{1}{r} g\overline{\varphi}(g \circ \tau_k)^{-1}.(f \circ \tau_k). \tag{3.2}$$

Again if we consider

$$\begin{aligned} \frac{1}{r} M_g U_{\varphi}^{*} M_g^{-1} f &= \frac{1}{r} g\overline{\varphi} W^{*}(g^{-1} f). \\ &= \frac{1}{r} g\overline{\varphi}(g \circ \tau_k)^{-1}.(f \circ \tau_k). \end{aligned} \tag{3.3}$$

Therefore by (3.2) and (3.3) we get that

$$U_{\psi}^{*} f = \frac{1}{r} M_g U_{\varphi}^{*} M_g^{-1} f.$$

Equivalently,

$$U_{\varphi}^{*} = r(M_g^{-1} U_{\psi}^{*} M_g)$$

implying that U_{φ}^{*} is similar to a constant multiple of the isometry U_{ψ}^{*}.This completes the proof. □

This shows that if such an eigen value and an eigen vector exist then U_{φ}^{*} is similar to a constant multiple of the isometry U_{ψ}^{*}.The next theorem would lead us to such a situation.
Firstly, we define the following :
A sequence $z_n : n \geq 0$ in T is called backward sequence if τ_k if $\tau_k(z_{n+1}) = z_n \forall n \geq 0$ and a backward sequence z_n is called non-vanishing if $\varphi(z_n) \neq 0$ for all $n \geq 0$

Theorem 3.1. *Let $\varphi = \sum_n C_n e^{in\theta} \neq 0$ be a non-negative function on T whose Fourier coefficients decay to 0 exponentially ($|C_n| \leq Ce^{-\gamma n}$ for some $C > 0$ and $\frac{1}{k} < \gamma < 1$).Then there exists $\lambda > 0$ and a non-zero continuous function $h \geq 0$ on T such that $U_{\varphi} h = \lambda h$.Further if given any z in T, there exists at least two φ non-vanishing backward sequences starting from z,then $h > 0$.*

Proof. Let us consider the set

$$E = \left\{ f \in C(T) : f = \sum_n a_n e^{in\theta} and \sum |a_n|^2 e^{2|n|} < \infty \right\}.$$

Then E is a Hilbert Space with inner product

$$\langle f, g \rangle_E = \sum_n a_n \overline{b_n} e^{2|n|}$$

where $f = \sum_n a_n e^{in\theta}$ and $g = \sum_n b_n e^{in\theta}$.Thus, $\{e^n = e^{-|n|} z^n : n \in \mathbb{Z}\}$ is an orthonormal basis of E.Again E is invariant under U_{φ}.We claim that the restriction of U_{φ} on E is compact. To do the needful, it is sufficient to show that U_φ on E is Hilbert Schmidt operator. Consider

$$\begin{aligned}
|\langle U_\varphi e_m, e_n\rangle_E| &= |\langle U_\varphi e^{-|m|}z^m, e^{-|n|}z^n\rangle_E| \\
&= e^{-|m|}e^{-|n|}|\langle W_k M_\varphi z^m, z^n\rangle_E| \\
&= e^{-|m|}e^{-|n|}|\langle W_k \sum_{i=-\infty}^{\infty} C_{i-m}z^m, z^n\rangle_E| \\
&= e^{-|m|}e^{-|n|}|C_{kn-m}|e^{2|n|} \\
&= e^{-|m|}e^{|n|}|C_{kn-m}| \\
&\leq e^{-|m|}e^{|n|}Ce^{-\gamma|kn-m|} \\
&\leq Ce^{-(1-\gamma)|m|}e^{-(k\gamma-1)|m|} \\
&< \infty.
\end{aligned}$$

if $\frac{1}{k} < \gamma < 1$. Therefore from the above calculations and as the Fourier coeficients decay to 0 exponentially, we have

$$\begin{aligned}
\sum_{n,m=1}^{\infty} |\langle U_\varphi e_m, e_n\rangle_E|^2 &\leq \sum_{n,m=1}^{\infty} \frac{C^2}{C^{2(1-\gamma)|m|}e^{2(k\gamma-1)|n|}} \\
&\leq \frac{k^2C^2}{(1-e^{2(1-\gamma)})(1-e^{2(k\gamma-1)})} < \infty.
\end{aligned}$$

Thus U_φ on E is Hilbert Schmidt and therefore compact. Now let

$$E_0 = \{f \in E : f(z) \in \mathbb{R}, z \in T\}$$

then $E = E_0 + iE_0$. Therefore by Krein Rutman Theorem [11] to the space E,there exists an eigen value $\lambda = r_E(U_\varphi)$ (the spectral radius of U_φ) with $r_E(U_\varphi) \geq 0$ and a corresponding eigen function $h \geq 0$.
Now since $\varphi \neq 0$,this implies that $||\varphi||^2 > 0$.Therefore

$$\begin{aligned}
\lambda = \gamma_E(U_\varphi) &= \lim_{n\to\infty} ||U_\varphi^n||_E^{\frac{1}{n}} \\
&\geq \limsup_{n\to\infty} ||U_\varphi^n 1||_E^{\frac{1}{n}} \\
&\geq ||\varphi||^2 > 0.
\end{aligned}$$

Again we can also easily verify that for any f in C(T)

$$(U_\varphi^n f)(z) = \frac{1}{k^n} \sum_{\{z_t\}\in\wedge_z} \prod_{t=1}^{n} \varphi(z_t)f(z_n)$$

where $\wedge_z$ is the collection of all backward sequences of τ_k with $z_0 = z$.By the given assumption,we can always find two φ-non vanishing backward sequences of τ_k,say $\{z_t\}$ and $\{z'_t\}$ in $\wedge_z$.Let $t_0 = max\{t : z_t = z'_t\}$,then consider z_{t_0+1} and z'_{t_0+1}.Again by the same reasons we can continue like this we can show that given any N,we can find atleast N distinct φ-non vanishing backward sequences of τ_k in $\wedge_z$.If z is a zero of h,then for every $n \geq 1$,$\prod_{t=1}^{n} \varphi(z_t)h(z_n) = 0$ for any $\{z_t\} \in \wedge_z$,since $\varphi, h \geq 0$.Also since f in E can be extended to be analytic in a neighbourhood say $\{\zeta : e^{-1} < |\zeta| < e\}$, containing T, therefore f has finitely many zeroes on T unless f=0.Also $h \neq 0$ we may assume that h has M zeroes.Again choose a large n such that there are at least M+1 elements in $\tau_k^{-n}(z)$ and each of these M+1 elements belongs to a φ-non vanishing backward sequences of τ_k in $\wedge_z$.This shows h must vanish at least M+1 points, a contradiction.Therefore $h > 0$. This completes the proof. □

CONCLUSION: In this paper , some light has been thrown on the spectrum of the Generalized Slant Toeplitz operator and its adjoint and also we also tried to get nature of its eigen values and eigen functions under some specific conditions.

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