Behaviour of Bilateral Mock Theta Functions of Order Nine at Infinite Number of Points

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Abstract

In January, 1920 before three months of his death, Ramanujan wrote about seventeen functions in his last letter to G. H. Hardy, called them Mock Theta functions and assigned them of order three, five and seven. In 2015, Ahmad obtained eight bilateral mock theta functions of order "nine" by using transformation due to Slater and proved that these functions satisfy the characteristic property of the mock theta function. In this paper we found that bilateral mock theta functions of order nine are irrational at infinite number of points $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

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1 Introduction

Mock theta functions were introduced first time by Ramanujan in his last letter to G. H. Hardy in January 1920. He gave no definition of mock theta functions but only a list of 17 examples and a qualitative description of the key property that he had noticed. Ramanujan assigned these 17 examples as mock theta functions of order three, five and seven. Watson [1936, 1937] studied these functions of order three and five in great details, and added three more functions of order three to this set of functions.

Mingarelli [2007] showed that Ramanujan's mock theta functions of order three, Watson's three additional mock theta functions, the Rogers-Ramanujan q-series, and six mock theta functions of order five take on irrational values at the points $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$. Recently **Singh** [2020] showed that bilateral mock theta functions of order "five" obtained by Srivastava [1999] take on irrational values at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

Ahmad [2015] obtained the following eight bilateral mock theta functions of order 'nine' by using the transformation (1.1.3) due to Slater [1996] for $r = 4$:

(1.1) $\quad f_0c_4(q) = \sum_{-\infty}^{\infty} \frac{q^{2n^2-n}}{(-q;q)_n}$,

(1.2) $\quad f_1c_4(q) = \sum_{-\infty}^{\infty} \frac{q^{2n^2}}{(-q;q)_n}$,

(1.3) $\quad F_0c_4(q^2) = \sum_{-\infty}^{\infty} \frac{q^{4n^2-2n}}{(q;q^2)_n}$,

(1.4) $\quad F_1c_4(q^4) = \sum_{-\infty}^{\infty} \frac{q^{8n^2}}{(q^6;q^4)_n}$,

(1.5) $\quad \Phi_0c_4(q^2) = \sum_{-\infty}^{\infty} \frac{q^{4n^2}}{(-q;q^2)_n}$,

(1.6) $\quad \Phi_1c_4(q^2) = \sum_{-\infty}^{\infty} q^{3n^2+6n}\left(-q;q^2\right)_n$, (1.7) $\Psi_0 c_4(q) = \sum_{-\infty}^{\infty} q^{3\frac{(n^2+3n)}{2}} (-q;q)_n$, and

(1.8) $\Psi_1 c_4(q) = \sum_{-\infty}^{\infty} \frac{q^{2n^2+2n}}{2(-q;q)_n}$.

He showed that the above functions are the limiting cases of the basic hypergeometric series $_5\Phi_4$ and proved that they satisfy the characteristic property of the mock theta function. In this paper, we discuss the behaviour of the functions (1.1) - (1.8) at infinite number of points. In the Section 4, a theorem for the behaviour of these functions at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$ has been stated and proved. For proving this theorem, we use two theorems given by Oppenheim's [1954, 1955] regarding the Cantor series which are stated in the Section 3.

2 Notations

We use the following q - notations and some standard results:

For $\left|q^k\right| < 1$, k a non-negative integer, then

$\left(a;q^k\right)_n = \prod_{j=0}^{n-1}\left(1-aq^{kj}\right), n \geq 1, \quad \left(a;q^k\right)_0 = 1, \quad \left(a;q^k\right)_\infty = \prod_{j=0}^{\infty}\left(1-aq^{kj}\right),$

$(a)_n = (a;q)_n = (1-a)(1-aq)\ldots\left(1-aq^{n-1}\right),$

$\left(a_1,a_2,\ldots,a_m;q^k\right)_n = \left(a_1;q^k\right)_n\left(a_2;q^k\right)_n\ldots\left(a_m;q^k\right)_n$,

A generalised basic hypergeometric series with base q^k is defined as

$_r\Phi_{r-1}\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,b_2,\ldots,b_{r-1}\end{matrix};q^k,z\right] = \sum_{n=0}^{\infty}\frac{\left(a_1,a_2,\ldots,a_r;q^k,z\right)_n}{\left(q^k;q^k\right)_n\left(b_1,b_2,\ldots,b_{r-1};q^k,z\right)_n}z^n,\ |z|<1$.

A bilateral basic hypergeometric series with base q^k is defined as

$_r\Psi_r\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,b_2,\ldots,b_r\end{matrix};q^k,z\right] = \sum_{n=-\infty}^{\infty}\frac{\left(a_1,a_2,\ldots,a_r;q^k\right)_n}{\left(b_1,b_2,\ldots,b_r;q^k\right)_n}z^n$ for $\left|\frac{b_1b_2\ldots b_r}{a_1a_2\ldots a_r}\right| < |z| < 1$, and $(a;q^k)_{-n} = \frac{(-q^k/a)^n}{(q^k/a;q^k)_n}q^{\frac{kn(n-1)}{2}}$.

3 Theorems of Oppenheim regarding the Cantor Series

This theory began with a seminal paper presented by Cantor [1869] in which he gave a necessary and sufficient condition for the series of the form

(3.1) $S = \sum_{n=1}^{\infty}\frac{b_n}{a_1a_2\cdots a_n}$,

where the $a_i,\ b_i$ are integers to have irrational sums. Cantor showed that S is irrational if and only if the $b_i > 0$ infinitely often and $a_i - 1 > b_i$ infinitely often with the basic conditions of the form $a_i \geq 2,\ a_i - 1 \geq b_i \geq 0$ and for every integer $k \geq 1$ there is an n such that $k \mid a_1a_2\cdots a_n$.

Oppenheim [1954, 1955] gave an extension of the theorem to the case where the b_i can have both signs and dropped the divisibility condition on the product of the first n, a's. Further development in the theory came in a paper by Hancl and Tijdeman [2004] in which the use of Cantor-Oppenheim a priori condition $a_i - 1 \geq b_i$ is avoided. The following two theorems given by Oppenheim regarding the Cantor series have been used in next section:

Theorem 3.1 *(Oppenheim [1954], Theorem 4)*: Let $(a_n),\ (b_n)$ be two sequences of integers with $a_n \geq 2,\ 0 \leq b_n \leq a_n - 1$. If $b_n > 0$ infinitely often and if there is a subsequence i_n such that $a_{i_n} \to \infty$ and $b_{i_n}/a_{i_n} \to 0$ as $n \to \infty$, then S as defined in (3.1) is irrational.

Theorem 3.2 *(Oppenheim [1954], Theorem 8)*: Let $(a_n),\ (b_n)$ be two sequences of integers with $a_n \geq 2,\ |b_n| \leq a_n - 1$. Furthermore, let $b_mb_n < 0$ for some $m > i,\ n > i$ for any assigned integer i. If there is a subsequence i_n such that $a_{i_n} \to \infty$ and $b_{i_n}/a_{i_n} \to 0$ as $n \to \infty$, then S given by (3.1) is irrational.

4 Behaviour of Bilateral Mock Theta Functions of Order Nine

Theorem 4.1. *Bilateral mock theta functions of order nine* (1.1) − (1.8) *are irrational at* $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

Proof of Theorem 4.1: The proof of the Theorem 4.1 is given for the functions (1.1) − (1.8) one by one separately:

4.1

We can write the function (1.1) as

(4.1.1) $$f_0c_4(q) = 1 + f_0S_4(q) + \sum_{n=1}^{\infty} q^{\frac{3n(n+1)}{2}} (-1;q)_n .$$

It follows that $f_0c_4(q)$ is irrational if and only if $f_0S_4(q)$ is irrational, and

(4.1.2) $$f_0S_4(q) = \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(-q;q)_n} = \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(1+q)(1+q^2)\cdots(1+q^n)} .$$

For $p, q \in \mathbb{C},\ q \neq 0$, we have

(4.1.3) $$f_0S_4\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{2n^2-n}q^{\frac{3n}{2}}}{q^{\frac{3n^2}{2}}(q+p)(q^2+p^2)\cdots(q^n+p^n)} .$$

Setting $p = 1$ in (4.1.3), we get

(4.1.4) $$f_0S_4\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{q^{\frac{3n}{2}}}{q^{\frac{3}{2}}(q+1)q^{\frac{9}{2}}(q^2+1)\cdots q^{\frac{3}{2}(2n-1)}(q^n+1)} .$$

Comparing $f_0S_4\left(\frac{1}{q}\right)$ with the Cantor series as given by (3.1), we get $a_n = q^{\frac{3}{2}(2n-1)}(q^n+1)$ and $b_n = q^{\frac{3n}{2}}$. For every integer $q \geq 2$, and any $n \geq 1,\ b_n > 0,\ a_n \geq 2,\ a_n - 1 > b_n > 0,\ a_n \to \infty$ & $b_n/a_n \to 0$ as $n \to \infty$, then by Theorem 3.1, $f_0S_4\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Again setting $p = -1$ in (4.1.3), we get

(4.1.5) $$f_0S_4\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n}{2}}}{q^{\frac{3}{2}}(q-1)q^{\frac{9}{2}}(q^2+1)\cdots q^{\frac{3}{2}(2n-1)}(q^n+(-1)^n)} .$$

Comparing $f_0S_4\left(-\frac{1}{q}\right)$ with the Cantor series as given by (3.1), we get
$a_n = q^{\frac{3}{2}(2n-1)}(q^n+(-1)^n)$ and $b_n = (-1)^n q^{\frac{3n}{2}}$. For every integer $q \geq 2$, and any $n \geq 1, |b_n| > 0,\ a_n \geq 2,\ a_n - 1 > |b_n| > 0,\ a_n \to \infty$ & $b_n/a_n \to 0$ as $n \to \infty$, then by Theorem 3.2, $f_0S_4\left(-\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

As it has been shown that $f_0S_4\left(\frac{1}{q}\right)$ and $f_0S_4\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, so $f_0S_4(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

4.2

The function (1.2) can be written as

(4.2.1) $$f_1c_4(q) = 1 + f_1S_4(q) + \sum_{n=1}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}} (-1;q)_n ,$$

where

(4.2.2) $$f_1S_4(q) = \sum_{n=1}^{\infty} \frac{q^{2n^2}}{(-q;q)_n} = \sum_{n=1}^{\infty} \frac{q^{2n^2}}{(1+q)(1+q^2)\cdots(1+q^n)} .$$

Whenever it is defined, $f_1c_4(q) \notin \mathbb{Q}$, iff $f_1S_4(q) \notin \mathbb{Q}$. Let $p, q \in \mathbb{C},\ q \neq 0$, then

$$f_1S_4\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{2n^2} q^{\frac{n}{2}}}{q^{\frac{3n^2}{2}}(q+p)\left(q^2+p^2\right)\cdots(q^n+p^n)}\ . \tag{4.2.3}$$

Setting $p = \pm 1$ in (4.2.3), we get

$$f_1S_4\left(\pm\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{q^{\frac{n}{2}}}{q^{\frac{3}{2}}(q\pm 1)q^{\frac{9}{2}}\left(q^2+1\right)\cdots q^{\frac{3}{2}(2n-1)}(q^n+(\pm 1)^n)}\ . \tag{4.2.4}$$

The sum $f_1S_4\left(\pm\frac{1}{q}\right)$ is a Cantor series as given by (3.1) with the identifications $a_n = q^{\frac{3}{2}(2n-1)}\left(q^n + (\pm 1)^n\right),\ b_n = q^{\frac{n}{2}}$. For every integer $q \geq 2$, and any $n \geq 1$ all the conditions of Theorem 3.1 are satisfied, so by Theorem 3.1, $f_1S_4\left(\pm\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$. Hence $f_1S_4(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

4.3

The function (1.3) can be written as

$$F_0c_4\left(q^2\right) = 1 + F_0S_4(q) + \sum_{n=1}^{\infty}(-1)^n q^{3n^2+2n}\left(q;q^2\right)_n\ . \tag{4.3.1}$$

It is followed that $F_0c_4\left(q^2\right)$ is irrational if and only if $F_0S_4(q)$ is irrational, and

$$F_0S_4(q) = \sum_{n=1}^{\infty} \frac{q^{4n^2-2n}}{\left(q;q^2\right)_n} = \sum_{n=1}^{\infty} \frac{q^{4n^2-2n}}{(1-q)\left(1-q^3\right)\cdots\left(1-q^{2n-1}\right)}\ . \tag{4.3.2}$$

Let $p, q \in \mathbb{C},\ q \neq 0$, then

$$F_0S_4\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{4n^2-2n} q^{2n}}{q^{3n^2}(q-p)\left(q^3-p^3\right)\cdots\left(q^{2n-1}-p^{2n-1}\right)}\ . \tag{4.3.3}$$

Taking $p = \pm 1$ in preceding expression, we get

$$F_0S_4\left(\pm\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{q^{2n}}{q^3(q\mp 1)q^9\left(q^3\mp 1\right)\cdots q^{3(2n-1)}\left(q^{2n-1}\mp 1\right)}\ . \tag{4.3.4}$$

Since for every $q \geq 2,\ q \in \mathbb{Z}$, the above expression is a Cantor series as given by (3.1) with $a_n = q^{3(2n-1)}\left(q^{2n-1} \mp 1\right)$ and $b_n = q^{2n}$ and all the conditions of Theorem 3.1 are satisfied, so $F_0S_4\left(\pm\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$. Thus $F_0S_4(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

4.4

The function (1.4) can be written as

$$F_1c_4\left(q^4\right) = 1 + F_1S_4(q) + \sum_{n=1}^{\infty}(-1)^n q^{6n^2+4n}\left(q^{-2};q^4\right)_n\ . \tag{4.4.1}$$

It is noted that $F_1c_4\left(q^4\right)$ is irrational if and only if $F_1S_4(q)$ is irrational, and

$$F_1S_4(q) = \sum_{n=1}^{\infty} \frac{q^{8n^2}}{\left(q^6;q^4\right)_n} = \sum_{n=1}^{\infty} \frac{q^{8n^2}}{\left(1-q^6\right)\left(1-q^{10}\right)\cdots\left(1-q^{4n+2}\right)}\ . \tag{4.4.2}$$

As before, for $p, q \in \mathbb{C},\ q \neq 0$, we have

$$F_1S_4\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{8n^2} q^{4n}}{q^{6n^2}\left(q^6-p^6\right)\left(q^{10}-p^{10}\right)\cdots\left(q^{4n+2}-p^{4n+2}\right)}\ . \tag{4.4.3}$$

Taking $p = \pm 1$ in (4.4.3), we obtain

$$F_1S_4\left(\pm\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{q^{4n}}{q^6\left(q^6-1\right)q^{18}\left(q^{10}-1\right)\cdots q^{6(2n-1)}\left(q^{4n+2}-1\right)} \ . \tag{4.4.4}$$

The above sum is a Cantor series as given by (3.1) with $a_n = q^{6(2n-1)}\left(q^{4n+2}-1\right)$ and $b_n = q^{4n}$ for every $n \geq 1$. All the conditions of Theorem 3.1 are satisfied, so $F_1S_4\left(\pm\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$, therefore $F_1S_4(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

4.5

In this case, we write the function $\Phi_0 c_4\left(q^2\right)$ given by (1.5) as

$$\Phi_0 c_4\left(q^2\right) = 1 + \Phi_0 S_4(q) + \sum_{n=1}^{\infty} q^{3n^2}\left(-q;q^2\right)_n \ . \tag{4.5.1}$$

It follows that $\Phi_0 c_4(q)$ is irrational if and only if $\Phi_0 S_4(q)$ is irrational, and

$$\Phi_0 S_4(q) = \sum_{n=1}^{\infty} \frac{q^{4n^2}}{\left(-q;q^2\right)_n} = \sum_{n=1}^{\infty} \frac{q^{4n^2}}{(1+q)\left(1+q^3\right)\cdots\left(1+q^{2n-1}\right)} \ . \tag{4.5.2}$$

If $p, q \in \mathbb{C},\ q \neq 0$, we see that

$$\Phi_0 S_4\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{4n^2}}{q^{3n^2}(q+p)\left(q^3+p^3\right)\cdots\left(q^{2n-1}+p^{2n-1}\right)} \ . \tag{4.5.3}$$

Now set $p = \pm 1$ in (4.5.3), we obtain

$$\Phi_0 S_4\left(\pm\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^3(q\pm 1)q^9\left(q^3\pm 1\right)\cdots q^{3(2n-1)}\left(q^{2n-1}\pm 1\right)} \ . \tag{4.5.4}$$

It is observed that the above sum is a Cantor series as given by (3.1) with the identifications $a_n = q^{3(2n-1)}\left(q^{2n-1}\pm 1\right)$ and $b_n = 1$ for every n. All the conditions of Theorem 3.1 are satisfied, so $\Phi_0 S_4\left(\pm\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$. Hence $\Phi_0 S_4(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

4.6

The function $\Phi_1 c_4(q^2)$ given by (1.6) can be expressed as

$$\Phi_1 c_4\left(q^2\right) = \sum_{n=0}^{\infty} q^{3n^2+6n}\left(-q;q^2\right)_n + \Phi_1 S_4(q) \ . \tag{4.6.1}$$

It is clear that $\Phi_1 c_4\left(q^2\right)$ is irrational if and only if $\Phi_1 S_4(q)$ is irrational, and

$$\Phi_1 S_4(q) = \sum_{n=1}^{\infty} \frac{q^{4n^2-6n}}{\left(-q;q^2\right)_n} = \sum_{n=1}^{\infty} \frac{q^{4n^2-6n}}{(1+q)\left(1+q^3\right)\cdots\left(1+q^{2n-1}\right)} \ . \tag{4.6.2}$$

Whenever $p, q \in \mathbb{C}, q \neq 0$, then

$$\Phi_1 S_4\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{4n^2-6n}q^{6n}}{q^{3n^2}(q+p)\left(q^3+p^3\right)\cdots\left(q^{2n-1}+p^{2n-1}\right)} \ . \tag{4.6.3}$$

Setting $p = \pm 1$ in (4.6.3), we obtain

$$\Phi_1 S_4\left(\pm\frac{1}{q}\right) = \frac{q^3}{q\pm 1} + \frac{1}{q\pm 1}\sum_{n=1}^{\infty} \frac{q^3}{q^3\left(q^3\pm 1\right)q^9\left(q^5\pm 1\right)\cdots q^{3(2n-1)}\left(q^{2n+1}\pm 1\right)} \ . \tag{4.6.4}$$

The sum on the right of (4.6.4) is a Cantor series with the identifications $a_n = q^{3(2n-1)}\left(q^{2n+1}\pm 1\right)$ and $b_n = q^3$ for all n, and these terms satisfy the conditions of Theorem 3.1 for every integer $q \geq 2$, so $\Phi_1 S_4(\pm\frac{1}{q})$ is irrational for $q \geq 2$. Thus $\Phi_1 S_4(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

4.7

As in the preceding cases, the function (1.7) can be written as

(4.7.1) $$\Psi_0 c_4(q) = \sum_{n=0}^{\infty} q^{\frac{3(n^2+3n)}{2}} (-q;q)_n + \Psi_0 S_4(q),$$

where

(4.7.2) $$\Psi_0 S_4(q) = \sum_{n=1}^{\infty} \frac{q^{2n^2-5n}}{(-1;q)_n} = \sum_{n=1}^{\infty} \frac{q^{2n^2-5n}}{(1+1)(1+q)\left(1+q^2\right)\cdots\left(1+q^{n-1}\right)}.$$

It is clear that $\Psi_0 c_4(q)$ is irrational if and only if $\Psi_0 S_4(q)$ is irrational.

For $p, q \in \mathbb{C}, q \neq 0$, then

(4.7.3) $$\Psi_0 S_4\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{2n^2-5n} q^{\frac{9n}{2}}}{q^{\frac{3n^2}{2}}(1+1)(q+p)\left(q^2+p^2\right)\cdots\left(q^{n-1}+p^{n-1}\right)}.$$

Substituting $p = 1$ in (4.7.3) and simplifying, we get

(4.7.4) $$\Psi_0 S_4\left(\frac{1}{q}\right) = \frac{q^3}{2} + \frac{q^3}{2(q+1)} + \frac{1}{2(q+1)} \sum_{n=1}^{\infty} \frac{q^3}{q^3\left(q^2+1\right)q^6\left(q^3+1\right)\cdots q^{3n}\left(q^{n+1}+1\right)}.$$

It is clear that $\Psi_0 S_4\left(\frac{1}{q}\right)$ is irrational if the sum on the right of (4.7.4) is irrational, and the sum on the right of (4.7.4) is a Cantor series as given by (3.1) with the identifications $a_n = q^{3n}\left(q^{n+1}+1\right)$, $b_n = q^3$. For every integer $q \geq 2$, a_n and b_n satisfy all the conditions of Theorem 3.1, so $\Psi_0 S_4\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Again substituting $p = -1$ in (4.7.3) and simplifying, we get

(4.7.5) $$\Psi_0 S_4\left(-\frac{1}{q}\right) = -\frac{q^3}{2} + \frac{q^3}{2(q-1)} + \frac{1}{2(q-1)} \sum_{n=1}^{\infty} \frac{(-1)^n q^3}{q^3\left(q^2+1\right)q^6\left(q^3-1\right)\cdots q^{3n}\left(q^{n+1}+(-1)^{n+1}\right)}$$

It is clear that $\Psi_0 S_4\left(-\frac{1}{q}\right)$ is irrational if the sum on the right of (4.7.5) is irrational, and the sum on the right of (4.7.5) is a Cantor series as given by (3.1) with the identifications $a_n = q^{3n}\left(q^{n+1}+(-1)^{n+1}\right)$, $b_n = (-1)^n q^3$. For every integer $q \geq 2$ and any integer $n \geq 1$, $|b_n| > 0, a_n \geq 2, a_n - 1 > |b_n| > 0, a_n \to \infty$, $\frac{b_n}{a_n} \to 0$ as $n \to \infty$. Hence by Theorem 3.2, $\Psi_0 S_4\left(-\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Since $\Psi_0 S_4\left(\frac{1}{q}\right)$ and $\Psi_0 S_4\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, therefore $\Psi_0 S_4(q)$ is irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

4.8

In this case, we write $\Psi_1 c_4(q)$ given by (1.8) in the form

(4.8.1) $$\Psi_1 c_4(q) = \frac{1}{2} + \frac{1}{2}\Psi_1 S_4(q) + \frac{1}{2}\sum_{n=1}^{\infty} q^{\frac{3}{2}n(n-1)} (-1;q)_n,$$

where

(4.8.2) $$\Psi_1 S_4(q) = \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(-q;q)_n} = \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(1+q)\left(1+q^2\right)\cdots\left(1+q^n\right)}.$$

Whenever it is defined, $\Psi_1 c_4(q) \notin \mathbb{Q}$, iff $\Psi_1 S_4(q) \notin \mathbb{Q}$. For $p, q \in \mathbb{C}$ and $q \neq 0$, we have

(4.8.3) $$\Psi_1 S_4\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{2n^2+2n}}{q^{\frac{3}{2}n(n+1)}(q+p)\left(q^2+p^2\right)\cdots\left(q^n+p^n\right)}.$$

Setting $p = \pm 1$ in (4.8.3), we get

(4.8.4) $$\Psi_1 S_4\left(\pm\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^3(q\pm 1)q^6\left(q^2+1\right)\cdots q^{3n}\left(q^n+(\pm 1)^n\right)}.$$

The sum $\Psi_1 S_4\left(\pm\frac{1}{q}\right)$ is a Cantor series as given by (3.1) with the identifications $a_n = q^{3n}\left(q^n + (\pm 1)^n\right)$, $b_n = 1$ for all n. For every integer $q \geq 2$ and any $n \geq 1$, all the conditions of Theorem 3.1 are satisfied, therefore $\Psi_1 S_4\left(\pm\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$. Hence $\Psi_1 S_4(q)$ is irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \cdots$.

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