A Study of $F_a(2K + S, S)$- Hsu-structure satisfying F^{2K+S}+$a^r F^s$=0 Nijenhuis tensor and CR-structure

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Abstract

We set the object to find conditions for $F_a(2K+S, S)$- Hsu-structure and CR-structures satisfying $F^{2K+S} + a^r F^S = 0$.We have also studied Nijenhuis tensor and relation between $F_a(2K + S, S)$-Hsu structure and CR-structure.

Subject Classification:[2020]Primary 53C12; Secondary 53C15.

Keywords: Hsu-structure, Hsu-metric structure, F-structure, CR-structure.

1 Introduction

The study of geometric structures on the differentiable manifold was initiated by Sasaki[16] (1960), Yano[19] (1963), Beklemishev[2] (1965), Shirokov[17] (1974) and Mishra and Singh[15] (1975). Chen [4, 5] studied certain structures on a differentiable manifold which are similar to quaternion structure and established integrability conditions of a ϕ-structure satisfying $\phi^4 \pm \phi^2 = 0$ Various properties of integrability and other conditions have been studied by many geometers [8, 11, 12].
The concept of CR-structure has been defined with the geometric structure induced on a real hypersurface of C_n, $n \geq 2$.CR-submanifolds of a Kahlerian manifold have been introduced by Bejancu[1]. Blair and Chen [3] have discussed and proved some basic results on CR-submanifolds of Hermitian manifold. Das has developed a relationship between CR-structures and F-structure satisfying$F^k + (-)^{K+1} F = 0$. Furthermore, Khan and Das[9] have determined the relationship between CR-structure and the general quadratic structure. They discussed integrability conditions on CR-structure and the general quadratic structure. Khan[10] has studied Nijenhuis tensor of tensor field F of type $(1, 1)$ and established integrability conditions of it. Numerous researchers made valuable contributions to CR-structures and various geometric structures including Yano and Kon[18], Das[8], Khan[13]. Motivated by above studies, we study $F_a(2K + S, S)$-Hsu structure satisfying $F^{2K+S} + a^r F^s = 0$ Nijenhuis tensor and CR-structure.The paper is organized as follows. In section-2 we prove some basic theorems of $F_a(2K + S, S)$ - Hsu-structure satisfying $F^{2K+S} + a^r F^s = 0$.
In section 3 we define CR-structure and some theorems. In section 4 we prove some interesting theorems of $F_a(2K+S, S)$-Hsu structure satisfying F^{2K+S}+$a^r F^S$=0 by the use of Nijenhuis tensor. In this section, we also state and prove necessary and sufficient conditions for integrability of $F_a(2K + S, S)$-Hsu-structure and two distributions D_l and D_m of operators$l$ and m respectively. In section 5 we establish relation between $F_a(2K + S, S)$- Hsu-structure and CR-structure.
The purpose of this paper is to establish a relationship between CR- structure and $F_a(2K + S, S)$-

Hsu-structure satisfying $F^{2K+S}+a^rF^S=0$ where K is a fixed integer greater than or equal to 1 and S is a fixed odd integer greater than or equal to 1. The rank of $(F)= r'$=constant. We also found relationship between CR- structure and $F_a(2K+S,S)$- Hsu-structure satisfying $F^{2K+S}+a^rF^S=0$, where K is a fixed integer greater than or equal to 1 and S is a fixed odd integer greater than or equal to 1.

Definition 1.1. *A structure on an n-dimensional manifold M of class C^∞ given by a non-null tensor field F satisfying*

$$F^2 = a^r I \tag{1.1}$$

is called Hsu-structure, where a is a non zero complex constant and I denotes the unit tensor field. Then M is called Hsu-structure manifold [6, 7].

2 CR-structure and $F_a(2K+S,S)$- Hsu-structure satisfying $F^{2K+S}+a^rF^S=0$

Let F be a non-zero tensor field of the type (1, 1) and of class C^∞ on an n-dimensional manifold M such that,

$$F^{2K+S} + a^rF^S = 0 \tag{2.1}$$

Where K is a fixed integer greater than or equal to 1 and S is a fixed odd integer greater than or equal to 1. The rank of (F)=r'=constant.

Let us define the operators on M as follows

$$l = -\frac{F^{2K}}{a^r}, m = l + \frac{F^{2K}}{a^r} \tag{2.2}$$

Where I denotes the identity operator[5].

Theorem 2.1. *Let M be an F-structure manifold satisfying the equation* (2.1)*, then*

$$\begin{cases} (a) l + m = I \\ (b) l^2 = l \\ (c) m^2 = m \\ (d) lm = ml = 0 \end{cases} \tag{2.3}$$

Thus, for a (1, 1) *tensor field F(≠0) satisfying* (2.1)*, there exist complementary distributions D_l and $D_m$corresponding to the projection operators l and m respectively, then dim $D_l = r'$ and dim $D_m = (n - r')$* [8, 14]

Theorem 2.2. *We have*

$$\begin{cases} (a) lF = Fl = F, mF = Fm = 0, \\ (b) \frac{F^{2K}}{a^r} m = 0, \frac{F^{2K}}{a^r} l = -l \end{cases} \tag{2.4}$$

Thus F^k acts on D_l as an almost complex structure and on D_m as a null operator.

3 CR-structure

Let M be a differentiable manifold and $T^C(M)$ be its complexified tangent bundle. A CR-Structure on M is a complex sub-bundle H of $T^C(M)$ such that $H_p \cap \widetilde{H}_p$=0, and H is involutive, that is for complex vector fields X and Y in H, [X,Y] is in H. In this case M is a CR-Submanifold, here $\widetilde{H}_p$ is complex conjugate of H_p .
Let $F_a(2K+S,S)$- Hsu-structure be integrable structure satisfying equation (2.1) of rank $r' = 2m$ on M. We define complex subbundle H of $T^C(M)$ by

$$H_p = \left\{X - \sqrt{-1}FX; X \in \chi(D_l)\right\}, \tag{3.1}$$

Where $\chi(D_l)$ is the $F(D_m)$ module of all differentiable sections of D_l then $R_\epsilon(H)$=D_l and $H_p \cap \widetilde{H}_p$=0

Theorem 3.1. *If P and Q are two elements of H, then the following relation holds*

$$[P,Q] = [X,Y] - [FX,FY] - \sqrt{-1}(-1)([X,FY] + [FX,Y]) \tag{3.2}$$

Proof. Let us define $P = X - \sqrt{(-1)}FX$ and $Q = Y - \sqrt{(-1)}FY$
Then by direct calculation and on simplifying, we obtain

$$[P,Q] = [X - \sqrt{(-1)}FX, Y - \sqrt{(-1)}FY],$$

$$[P,Q] = [X,Y] - [FX,FY] - \sqrt{(-1)}([X,FY] + [FX,Y]).$$

□

4 Nijenhuis tensor of $F_a(2K+S,S)$-Hsu-structure

Theorem 4.1. *If* $F_a(2K+S,S)$ *Hsu-structure satisfying* (2.1) *is integrable, then we have*

$$\left(-\frac{F^{2K-1}}{a^r}\right)([FX,FY] + F^2[X,Y]) = l([FX,Y] + [X,FY]) \tag{4.1}$$

Proof. The Nijenhuis tensor N(X,Y) of F in M is expressed as follows for every vector field X,Y on M

$$N(X,Y) = F[FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y] \tag{4.2}$$

The Nijenhuis tensor N(X,Y) of F satisfying (2.1) in M is given by equation (4.2) Since, $N(X,Y) = 0$ we obtain

$$[FX,FY] + F^2[X,Y] = F([FX,Y] + [X,FY]) \tag{4.3}$$

Operating (4.3) by $(-\frac{F^{2K-1}}{a^r})$
we get

$$\left(-\frac{F^{2K-1}}{a^r}\right)[FX,FY] + F^2[X,Y] = \left(-\frac{F^{2K-1}}{a^r}\right)F([FX,Y] + [X,FY]) \tag{4.4}$$

On making use of (2.2) in the above equation, we obtain (4.1), which proves the Theorem [6]. □

Theorem 4.2. *The following identities hold*

$$mN(X,Y) = m[FX,FY], \tag{4.5}$$

$$mN(\frac{F^{2K-1}}{a^r},Y) = m(\frac{F^{2K}}{a^r}X,FY). \tag{4.6}$$

Proof. The proof of (4.5) and (4.6) follows easily by virtue of theorems (2.1), (2.2) and equation (4.2). □

Theorem 4.3. *For any two vector fields X and Y, the following conditions are equivalent.*

$$\begin{cases} (a)mN(X,Y) = 0, \\ (b)m[FX,FY] = 0, \\ (c)mN(\frac{F^{2K-1}}{a^r}X,Y) = 0, \\ (d)mN(\frac{F^{2K}}{a^r}X,FY) = 0, \\ (e)mN(\frac{F^{2K}}{a^r}lX,FY) = 0, \end{cases} \tag{4.7}$$

Proof. In consequence of equations (2.1), (2.2), (4.2) and theorem 2.2, the above equation can be proved to be equivalent. □

Theorem 4.4. *If* $(\frac{F^{2K-1}}{a^r})$ *acts on* D_l *as an almost complex structure, then*

$$m(\frac{F^{2K}}{a^r}lX,FY) = m[-FX,FY] = 0 \tag{4.8}$$

Proof. In view of equation (2.4*a*), (2.4*b*), we see that F^k acts on D_l as an almost complex structure, then (4.2) follow in an obvious manner. To show that $m(\frac{F^{2K}}{a^r}lX,FY) = 0$ we use the definition of Lie bracket and in view of equation (2.4*a*), the result follows directly. □

5 Relation between $F_a(2K+S,S)$ - Hsu-structure and CR-structure

Theorem 5.1. *For* $X,Y \in \chi(D_l)$, *we have*

$$l([X,FY]+[FX,Y]) = [X,FY]+[FX,Y] \tag{5.1}$$

Proof. Since $[X,FY]$ and $[FX,Y] \in \chi(D_l)$ on making use of (2.4*a*), we obtain the required result. □

Theorem 5.2. *The integrable* $F_a(2K+S,S)$*- Hsu-structure satisfying equation* (2.1) *on M defines a CR-structure H on it such that*

$$ReH = D_l \tag{5.2}$$

Proof. In view of the fact that $[X,FY]$ and $[FX,Y] \in \chi(D_l)$ and on using equations 3.2, 4.1 and Theorem 5.1, we get

$$\begin{aligned} l[P,Q] &= l[X,Y] - l[FX,FY] - \sqrt{(-1)}(-1)l([X,FY]+[FX,Y]) \\ &= [X,Y] - [FX,FY] - \sqrt{(-1)}(-1)([X,FY]+[FX,Y]) \\ &= [P,Q] \end{aligned}$$

Hence $[P, Q] \in \chi(D_l)$, Then $F_a(2K+S, S)$ - Hsu-structure satisfying equation (2.1) on M defines a CR-structure. □

Definition 5.1. *Let $\bar{K}$ be the complementary distribution of ReH to T(M). We define a morphism of vector bundles $F : T(M) \to T(M)$, given by F(X)=0 for all $X \in \chi(\bar{K})$ such that*

$$FX = \frac{1}{2}\sqrt{(-1)}(-1)(P - \widetilde{P}) \tag{5.3}$$

where $P = X + \sqrt{-1}(-1)Y \in \chi(H_p)$ and $\widetilde{P}$ is a complex conjugate of P

Corollary 5.1. *If $P = X + iY$ and $\widetilde{P}$=$X - iY$ belongs to H_p and $F(X)$=$\frac{1}{2}\sqrt{(-1)}$, $F(Y)$=$\frac{1}{2}(P + \widetilde{P})$ and $F(-Y)$=$-\frac{1}{2}(P + \widetilde{P})$ and $F(X) = -Y$, $F^2(X) = -X$ and $F(-Y) = -X$*

Theorem 5.3. *If M has a CR-structure H then we have $F^{2K+S} + a^r F^S = 0$ and consequently $F_a(2K+S, S)$ - Hsu-structure is defined on M such that the distribution D_l and D_m coincide with $R_e(H)$ and $\bar{K}$ respectively.*

Proof. Let M has a CR-structure, Then in view of definition (5.1) and corollary (5.1), we get

$$F(X) = -Y \tag{5.4}$$

operating equation (6.4) by $(F^{2K} + a^r F)$ we have,

$$(F^{2K} + a^r F)F(X) = (F^{2K} + a^r F)(-Y)$$

On making use of corollary (5.1), the R.H.S of the above equation becomes

$$\begin{aligned}(F^{2K+3} + a^r F^4)(X) &= (F^{2K+1} + a^r F^2)F^2(X)\\ &= (F^{2K+1} + a^r F^2)(-X)\\ &= -(F^{2K-1} + a^r)F^2(X)\\ &= -(F^{2K-1} + a^r)(-X)\\ &= (F^{2K-1} + a^r)(X).\end{aligned}$$

We continue simplifying in same manner and obtain

$$(F^{2K+S} + a^r F^S)(X) = 0,$$

Which is indeed

$$F^{2K+S}(X) + a^r F^S(X) = 0.$$

□

7. Examples

7.1 $F_a(3, 1)$-Strucure

Let $F(\neq 0)$ a tensor field of type (1,1) in M satisfying $F^3 + a^r F = 0$ with rank r'. The projection operators are defined as
$l = -\frac{F^2}{a^r}$, $m = I + \frac{F^2}{a^r}$.
we have
$l + m = I$, $l^2 = l$, $m^2 = m$, $Fl = lF = F$, $Fm = mF = 0$,
$\frac{F^2}{a^r} = -l$, $\frac{F^2 l}{a^r} = -l$, $\frac{F^2 m}{a^r} = 0$. Thus F acts on D_l as an almost complex structure and on D_m as null operator. We can easily verified that a relationship between CR-structures and $F_a(3, 1)$ structure by applying a similar device.

7.2 F_a(3,-1)-Structure

Let$F(\neq 0)$ a tensor field of type (1,1) in M satisfying $F^3 - a^r F = 0$ with rank r'. The projection operators are defined as $l = \frac{F^2}{a^r}$, $m = I - \frac{F^2}{a^r}$.
we have,
$l + m = I$, $l^2 = l$, $m^2 = m$,
$Fl = lF = F$, $Fm = mF = 0$,
$\frac{F^2}{a^r} = l$, $\frac{F^2 l}{a^r} = l$, $\frac{F^2 m}{a^r} = 0$.
Thus F acts on D_l as an almost product structure and on D_m as null operator.

8. Conclusion

In the present work, we have studied $F_a(2K+S, S)$ -Hsu-structure satisfying $F^{2K+S} + a^r F^s = 0$, Nijenhuis tensor of $F_a(2K+S, S)$- Hsu-structure and integrability conditions are determined. Furthermore, we have established a relationship between CR-structures and $F_a(2K+S, S)$ Hsu-structure. Future studies could fruitfully analyze this issue by taking the polynomial structure $P(F) = F^n + a_n F^{n-1} + ... + a_2 F + a_1 I$ where $a_n, a_{n-1}, ...a_2, a_1$ are constants, I is unit tensor field and F is a tensor field of type (1,1) on M.

Acknowledgement: The reviewer's suggestions to make it easier to read are really appreciated by the authors.

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