Fractional Integral and Differentiation of the j-generalized p - k Mittag-Leffler Function

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Abstract

The aim of this paper is to obtain certain properties of j-generalized p-k Mittag-Leffler function via Riemann-Liouville fractional integrals and derivatives (valid under specific conditions). Some remarkable paticular/special cases also been discussed in detailed by taking suitable values of parameters.

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1 Introduction

1.1 Definition [9]

Let $x \in \mathbb{C}; k, p \in \mathbb{R}^+ \setminus \{0\}$ and $Re(x) > 0, n \in \mathbb{N}$, the p - k Pochhammer Symbol (i.e. Two Parameter Pochhammer Symbol), ${}_p(x)_{n,k}$ is given by

$$ {}_p(x)_{n,k} = (\frac{xp}{k})(\frac{xp}{k} + p)(\frac{xp}{k} + 2p).........(\frac{xp}{k} + (n-1)p) \tag{1.1}$$

and the two parameter Gamma function is given by [4], some of it's result define as ,

1.2 Definition [4]

For $x \in C \setminus k\mathbb{Z}^-; k, p \in \mathbb{R}^+ \setminus \{0\}$ and $Re(x) > 0, n \in \mathbb{N}$, the p - k Gamma Function (i.e. Two Parameter Gamma Function) ${}_p\Gamma_k(x)$ as

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n!p^{n+1}(np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}} \tag{1.2}$$

or

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n!p^{n+1}(np)^{\frac{x}{k}-1}}{{}_p(x)_{n,k}} \tag{1.3}$$

Some properties of p - k Gamma Function are listed below,

$$(1.4) \qquad {}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}} t^{x-1} dt$$

$$(1.5) \qquad {}_p\Gamma_k(x) = (\frac{p}{k})^{\frac{x}{k}} \Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k} \Gamma(\frac{x}{k})$$

$$(1.6) \qquad {}_p(x)_{n,k} = (\frac{p}{k})^n (x)_{n,k} = (p)^n (\frac{x}{k})_n$$

Some facts of Generalized p - k Pochhammer Symbol are mentioned below as,

$$(1.7) \qquad {}_p(x)_{nq,k} = (\frac{p}{k})^{nq} \, (x)_{nq,k} = (p)^{nq} \, (\frac{x}{k})_{nq} = (pq)^{nq} \, \prod_{r=1}^{q} (\frac{\frac{x}{k} + r - 1}{q})_n$$

$$(1.8) \qquad {}_p(x)_{n,k} = \frac{{}_p\Gamma_k(x + nk)}{{}_p\Gamma_k(x)}$$

$$(1.9) \qquad {}_p\Gamma_k(x + k) = \frac{xp}{k} \, {}_p\Gamma_k(x)$$

$$(1.10) \qquad np \, {}_p(x)_{n-1,k} = \, {}_p(x)_{n,k} - \, {}_p(x - k)_{n,k}$$

and

$$(1.11) \qquad {}_p(x)_{n+j,k} = \, {}_p(x)_{j,k} \times \, {}_p(x + jk)_{n,k}$$

Gehlot and Bhandari [9] introduced the j-generalized p- k Mittag-Leffler function, that is denoted by ${}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined for $k, p \in R^+ \setminus \{0\}; \alpha, \beta, \gamma \in \mathbb{C} \setminus k\mathbb{Z}^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0,$ $j \in \mathbb{N}_0$ and $q > 0$.
The j-generelized p - k Mittag-Leffler function denoted by ${}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined below as

$$(1.12) \qquad {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha + \beta)} \frac{z^n}{(n + j)!}, \qquad z \in \mathbb{C}$$

where ${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol given by equation (1.1) and ${}_p\Gamma_k(x)$ is the two parameter Gamma function given (1.3).

The Fractional Integral operators ([11], Definition 2.1, Page 33) are defined as,

$$(1.13) \qquad (I^{\vartheta}_{0+} f)(z) = \frac{1}{\Gamma(\vartheta)} \int_0^z \frac{f(t)}{(z - t)^{1-\vartheta}} dt, (z > 0)$$

and

$$(1.14) \qquad (I^{\vartheta}_{-} f)(z) = \frac{1}{\Gamma(\vartheta)} \int_z^\infty \frac{f(t)}{(t - z)^{1-\vartheta}} dt, (z > 0)$$

The Fractional Derivative ([11], Definition 2.2, Page 35) are defined as,

$$(D^{\vartheta}_{0+}f)(z) = (\frac{d}{dz})^{[Re(\vartheta)]+1}(I^{1-\vartheta+[Re(\vartheta)]}_{0+}f)(z)$$

$$= \frac{1}{\Gamma(1-\vartheta+[Re(\vartheta)])}(\frac{d}{dz})^{[Re(\vartheta)]+1}\int_0^z \frac{f(t)}{(z-t)^{\vartheta-[Re(\vartheta)]}}dt(z>0), \tag{1.15}$$

and

$$(D^{\vartheta}_{-}f)(z) = (-\frac{d}{dz})^{[Re(\vartheta)]+1}(I^{1-\vartheta+[Re(\vartheta)]}_{-}f)(z)$$

$$= \frac{1}{\Gamma(1-\vartheta+[Re(\vartheta)])}(-\frac{d}{dz})^{[Re(\vartheta)]+1}\int_z^{\infty} \frac{f(t)}{(t-z)^{\vartheta-[Re(\vartheta)]}}dt(z>0), \tag{1.16}$$

where $\vartheta \in C(Re(\vartheta) > 0)$

2 Fractional Integral and Differentiation of the j-generalized p - k Mittag-Leffler Function

In this section, we discuss Riemann-Liouville fractional integral and derivative of the j-generalized p - k Mittag-Leffler Function.

Theorem 2.1 The left-side Riemann-Liouville Fractional Integral Operator $I^{\vartheta}_{0^+}$ of the j-generalized p - k Mittag-Leffler Function is

$$(I^{\vartheta}_{0^+}[t^{\frac{\beta}{k}-1}\ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(t^{\frac{\alpha}{k}})])(z) = p^{\vartheta}z^{\frac{\beta}{k}+\vartheta-1}\ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k\vartheta}(z^{\frac{\alpha}{k}}), \tag{2.1}$$

where

$$k,p \in \mathbb{R}^+ \setminus \{0\}; \alpha,\beta,\gamma \in \mathbb{C} \setminus k\mathbb{Z}^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, j \in \mathbb{N}_0,\ q > 0$$

and $Re(\vartheta) > 0$.

Proof: Consider left hand side of (2.1)

$$I \equiv (I^{\vartheta}_{0^+}[t^{\frac{\beta}{k}-1}\ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(t^{\frac{\alpha}{k}})])(z),$$

by virtue of equation (1.12) and (1.13), we get

$$I \equiv \frac{1}{\Gamma(\vartheta)}\int_0^z \frac{t^{\frac{\beta}{k}-1}}{(z-t)^{1-\vartheta}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{t^{\frac{n\alpha}{k}}}{(n+j)}dt$$

On interchanging the order of integration and summation and evaluate the inner integral by substitute $t = zu$ and using the beta function, gives,

$$I \equiv \frac{z^{\frac{\beta}{k}+\vartheta-1}}{\Gamma(\vartheta)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{z^{\frac{n\alpha}{k}}}{(n+j)!}\frac{\Gamma(\frac{n\alpha+\beta}{k})\Gamma(\vartheta)}{\Gamma(\frac{n\alpha+\beta+k\vartheta}{k})},$$

using the equation (1.5), we have

$$I \equiv p^{\vartheta}z^{\frac{\beta}{k}+\vartheta-1}\ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k\vartheta}(z^{\frac{\alpha}{k}})$$

Hence, we get the desired result.

Theorem 2.2 The right-side Riemann-Liouville Fractional Integral Operator I_{-}^{ϑ} of the j-generalized p - k Mittag-Leffler Function is

$$(I_{-}^{\vartheta}[t^{-\frac{\beta}{k}-\vartheta}\ {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(t^{-\frac{\alpha}{k}})])(z) = p^{\vartheta} z^{-\frac{\beta}{k}}\ {}_{p}^{j}E_{k,\alpha,\beta+k\vartheta}^{\gamma,q}(z^{-\frac{\alpha}{k}}), \tag{2.2}$$

where

$$k, p \in \mathbb{R}^{+} - \{0\}; \alpha, \beta, \gamma \in \mathbb{C}/k\mathbb{Z}^{-}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, j \in \mathbb{N}_0,\ q > 0$$

and $Re(\vartheta) > 0$.
Proof: Consider left hand side,

$$A \equiv (I_{-}^{\vartheta}[t^{-\frac{\beta}{k}-\vartheta}\ {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(t^{-\frac{\alpha}{k}})])(z),$$

by virtue of equation (1.12) and (1.14), we have

$$A \equiv \frac{1}{\Gamma(\vartheta)} \int_{z}^{\infty} \frac{t^{-\frac{\beta}{k}-\vartheta}}{(t-z)^{1-\vartheta}} \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_{k}(n\alpha+\beta)} \frac{t^{-\frac{n\alpha}{k}}}{(n+j)!} dt.$$

On interchanging the order of integration and summation and evaluate the inner integral by substitute $t = \frac{z}{u}$ and using the beta function formula[4], it gives

$$A \equiv \frac{z^{-\frac{\beta}{k}}}{\Gamma(\vartheta)} \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_{k}(n\alpha+\beta)} \frac{z^{-\frac{n\alpha}{k}}}{(n+j)!} \frac{\Gamma(\frac{n\alpha+\beta}{k})\Gamma(\vartheta)}{\Gamma(\frac{n\alpha+\beta+k\vartheta}{k})},$$

using the equation (1.5), we have

$$A \equiv p^{\vartheta} z^{-\frac{\beta}{k}}\ {}_{p}^{j}E_{k,\alpha,\beta+k\vartheta}^{\gamma,q}(z^{-\frac{\alpha}{k}}).$$

Hence, we get the desired result.

Theorem 2.3 The left-side Riemann-Liouville fractional derivative operator $D_{0^{+}}^{\vartheta}$ of the j-generalized p - k Mittag-Leffler Function is

$$(D_{0^{+}}^{\vartheta}[t^{\frac{\beta}{k}-1}\ {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(t^{\frac{\alpha}{k}})])(z) = p^{-\vartheta} z^{\frac{\beta}{k}-\vartheta-1}\ {}_{p}^{j}E_{k,\alpha,\beta-k\vartheta}^{\gamma,q}(z^{\frac{\alpha}{k}}), \tag{2.3}$$

where

$$k, p \in \mathbb{R}^{+} \setminus \{0\}; \alpha, \beta, \gamma \in \mathbb{C}/k\mathbb{Z}^{-}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, j \in \mathbb{N}_0,\ q > 0$$

and $Re(\vartheta) > 0$.
Proof: Consider left hand side,

$$A \equiv (D_{0^{+}}^{\vartheta}[t^{\frac{\beta}{k}-1}\ {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,q}(t^{\frac{\alpha}{k}})])(z),$$

by virtue of equation (1.12) and (1.15), we have

$$A \equiv \frac{1}{\Gamma(1-\vartheta+[Re(\vartheta)])} (\frac{d}{dz})^{[Re(\vartheta)]+1} \int_{0}^{z} \frac{t^{\frac{\beta}{k}-1}}{(z-t)^{\vartheta-Re(\vartheta)}} \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_{k}(n\alpha+\beta)} \frac{t^{\frac{n\alpha}{k}}}{(n+j)!} dt.$$

On interchanging the order of integration and summation and evaluate the inner integral by substitute $t = zu$ and using the beta function formula, it gives

$$A \equiv \frac{z^{\frac{\beta}{k}-\vartheta-1}}{\Gamma(\vartheta)} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^{\frac{n\alpha}{k}}}{(n+j)!} \frac{\Gamma(\frac{n\alpha+\beta}{k})\Gamma(\vartheta)}{\Gamma(\frac{n\alpha+\beta-k\vartheta}{k})}.$$

Now using the equation (1.5), we get

$$A \equiv p^{-\vartheta} z^{\frac{\beta}{k}-\vartheta-1}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta-k\vartheta}(z^{\frac{\alpha}{k}}).$$

Hence, we get the desired result.

Theorem 2.4 The right-side Riemann-Liouville fractional derivative operator D^{ϑ}_{-} of the j-generalized p - k Mittag-Leffler function is

$$(D^{\vartheta}_{-}[t^{-\frac{\beta}{k}+\vartheta}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta}(t^{-\frac{\alpha}{k}})])(z) = p^{-\vartheta} z^{-\frac{\beta}{k}}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta-k\vartheta}(z^{-\frac{\alpha}{k}}), \tag{2.4}$$

where

$$k, p \in \mathbb{R}^+ \setminus \{0\}; \alpha, \beta, \gamma \in \mathbb{C}/k\mathbb{Z}^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, j \in \mathbb{N}_0,\ q > 0$$

and $Re(\vartheta) > 0$.
Proof: Consider left hand side,

$$A \equiv (D^{\vartheta}_{-}[t^{-\frac{\beta}{k}+\vartheta}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta}(t^{-\frac{\alpha}{k}})])(z).$$

By virtue of equation (1.12) and (1.16), we have

$$A \equiv \frac{1}{\Gamma(1-\vartheta+[Re(\vartheta)])}(-\frac{d}{dz})^{[Re(\vartheta)]+1} \int_z^{\infty} \frac{z^{-\frac{\beta}{k}+\vartheta}}{(t-z)^{\vartheta-[Re(\vartheta)]}} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{t^{-\frac{n\alpha}{k}}}{(n+j)!} dt.$$

On interchanging the order of integration and summation and evaluate the inner integral by substitute $t = \frac{z}{u}$ and using the beta function formula, it gives

$$A \equiv \frac{z^{-\frac{\beta}{k}}}{\Gamma(\vartheta)} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^{-\frac{n\alpha}{k}}}{(n+j)!} \frac{\Gamma(\frac{n\alpha+\beta}{k})\Gamma(\vartheta)}{\Gamma(\frac{n\alpha+\beta-k\vartheta}{k})}.$$

Now using the equation (1.5), we get

$$A \equiv p^{-\vartheta} z^{-\frac{\beta}{k}}\, {}_p^jE^{\gamma,q}_{k,\alpha,\beta-k\vartheta}(z^{-\frac{\alpha}{k}}).$$

Hence, we get the desired result.

Particular cases : For some particular values of the parameters $j, p, q, k, \alpha, \beta, \gamma$ we can obtain certain defined and undefined Mittag-Leffler functions:

(a) For $j = 0$ equation (2.1), reduces in the p-k Mittag-Leffler functions defined by [5],

$${}_p^0E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!}, \qquad z \in C. \tag{2.5}$$

(b) For $q = 1$ equation (2.1), reduces in j form of p-k Mittag-Leffler functions defined as,

$$ {}_{p}^{j}E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{(n+j),k}}{{}_{p}\Gamma_{k}(n\alpha+\beta)} \frac{z^n}{(n+j)!}, \qquad z \in C. \tag{2.6} $$

(c) For $q = 1, p = k$ equation (2.1), reduces in j form of k- Mittag-Leffler functions defined as,

$$ {}_{k}^{j}E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{(n+j),k}}{\Gamma_{k}(n\alpha+\beta)} \frac{z^n}{(n+j)!}, \qquad z \in C. \tag{2.7} $$

(d) For $q = 1, j = 0$ equation (2.1), reduces in generalized form of k- Mittag-Leffler functions defined as.

$$ {}_{p}E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{{}_{p}(\gamma)_{n,k}\, z^n}{{}_{p}\Gamma_{k}(n\alpha+\beta)(n!)}. \tag{2.8} $$

(e) For $p = k, j = 0$ equation (2.1), reduces in Generalized k- Mittag-Leffler functions defined by [3].

$$ {}_{k}E_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{{}_{k}(\gamma)_{nq,k}\, z^n}{{}_{k}\Gamma_{k}(n\alpha+\beta)(n!)} = GE_{k,\alpha,\beta}^{\gamma,q}(z). \tag{2.9} $$

(f) For $p = k, q = 1, j = 0$ equation (2.1), reduces in k - Mittag-Leffler functions defined by [2].

$$ {}_{k}E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\, z^n}{\Gamma_{k}(n\alpha+\beta)(n!)} = E_{k,\alpha,\beta}^{\gamma}(z). \tag{2.10} $$

(g) For $p = k$ and $k = 1, j = 0$ equation (2.1), reduces in Mittag-Leffler functions defined by [12].

$$ {}_{1}E_{1,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}\, z^n}{\Gamma(n\alpha+\beta)(n!)} = E_{\alpha,\beta}^{\gamma,q}(z). \tag{2.11} $$

(h) For $p = k = q = 1$ equation (2.1), reduces in L-Mittag-Leffler functions defined by [11].

$$ {}_{1}E_{1,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j}\, z^n}{\Gamma(n\alpha+\beta)(n+j)!} = L_{\alpha,\beta}^{\gamma,j}(z). \tag{2.12} $$

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