Some trace inequalities of a special type of Rhotrix

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Abstract

Matrices play an important role in various branches of mathematics such as coding theory, combinatorics and cryptography. Rhotrices are represented by coupled matrices and have wide range of applications in graph theory, cryptography and coding theory. Here, we discuss some trace inequalities of $n-$dimensional special type of rhotrix. Also, we derive some results related to the eigen values of special type of rhotrix.

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1 Introduction

In 2003, Ajibade [2] introduced rhotrix which is a mathematical object some way between 2×2-dimensional and 3×3- dimensional matrices. Dimension of a rhotrix is always an odd number. A rhotrix of 3-dimension is defined as

$$R_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle,$$

where a, b, c, d, e are real numbers. The heart of the rhotrix is c, denoted by $H(c)$. Let

$$Q_3 = \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle$$

be the another rhotrix of same dimension and addition of two rhotrices is defined as

$$R_3 + Q_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a+f & \\ b+g & c+h & d+j \\ & e+k & \end{array} \right\rangle.$$

Sani [8] extended rhotrix of $n-$dimension to any odd number $n \geq 3$ and gave the row-column multiplication. Sani [7] introduced heart oriented rhotrix multiplication and inverse of a rhotrix as

$$R_3 \; o \; Q_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & ch & aj+dk \\ & bj+ek & \end{array} \right\rangle$$

and

$$R_3^{-1} = \left\langle \begin{array}{ccc} & \frac{c}{ac-bd} & \\ \frac{-b}{bd-ac} & \frac{1}{e} & \frac{-d}{bd-ac} \\ & \frac{a}{ac-bd} & \end{array} \right\rangle.$$

Determinant of rhotrix is $c(ae - bd) \neq 0$. Representation of a rhotrix into a coupled matrix is discussed by Sani [9]. For instance, for a 5–dimensional rhotrix

$$R_5 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle = \langle A, C \rangle,$$

we have two coupled matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

Sharma et al. [14] discussed trace of 3-dimensional and 5-dimensional rhotrices of positive integral power and the equation of rhotrix system was discussed by Aminu [5]. The different structure and operation of rhotrices are discussed by several authors, see [1, 3, 6, 10-19]. The trace of a rhotrix R_n is the sum of its diagonal entries and also the sum of all eigen values of that rhotrix. The trace is denoted by Tr and defined as

$$Tr(R_n) = \sum_{i=1}^{n} r_{ii}.$$

Trace inequalities for matrices are discussed by several authors, see [4, 20, 21]. In the present paper, we discuss some inequalities of rhotrices related with trace which satisfy the following condition

$$P : \alpha_{ij} = \left\{ \begin{array}{ll} 1 & if\ i+j = even \\ 0 & if\ i+j = odd \end{array} \right\}.$$

We also obtain some results related to the eigen values of special type of rhotrix.

2 Main Results

Theorem 2.1 Let R_n be the rhotrix of n-dimension satisfying the condition P. Then the trace of the rhotrix is same as the order of the rhotrix.

Proof: Let R_n be a rhotrix of $n-$ dimension defined as

$$R_n = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & a_{nn-2} & c_{n-1n-2} & a_{n-1n-1} & c_{n-2n-1} & a_{n-2n} & \\ & & a_{nn-1} & c_{n-1n-1} & a_{n-1n} & & \\ & & & a_{nn} & & & \end{array} \right\rangle,$$

where the couple matrices a_{ij} and c_{lk} are written as

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & \dots & a_{nn} \end{bmatrix}$$

and

$$c_{lk} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & \dots & c_{1(n-1)} \\ c_{21} & c_{22} & c_{23} & \dots & \dots & c_{2(n-1)} \\ c_{31} & c_{32} & c_{33} & \dots & \dots & c_{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{(n-1)1} & c_{(n-1)2} & c_{(n-1)3} & \dots & \dots & c_{(n-1)(n-1)} \end{bmatrix}.$$

By applying the condition P on rhotrix R_n, we get

$$R_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & 1 & 0 & 1 & 0 & 1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & a_{n(n-2)} & 0 & a_{(n-1)(n-1)} & 0 & a_{(n-2)n} & \\ & & 0 & c_{(n-1)(n-1)} & 0 & & \\ & & & a_{nn} & & & \end{array} \right\rangle = \left\langle a_{ij}, c_{lk} \right\rangle,$$

$$a_{ij} = \begin{bmatrix} 1 & 0 & 1 & \dots & \dots & a_{1n} \\ 0 & 1 & 0 & \dots & \dots & a_{2n} \\ 1 & 0 & 1 & \dots & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & \dots & a_{nn} \end{bmatrix}$$

and

$$c_{lk} = \begin{bmatrix} 1 & 0 & 1 & \dots & \dots & c_{1(n-1)} \\ 0 & 1 & 0 & \dots & \dots & c_{2(n-1)} \\ 1 & 0 & 1 & \dots & \dots & c_{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{(n-1)1} & c_{(n-1)2} & c_{(n-1)3} & \dots & \dots & c_{(n-1)(n-1)} \end{bmatrix}.$$

The eigen values of matrices a_{ij} and c_{lk} are given by

$$a_{ij}X = \lambda_p X$$

and

$$c_{lk}X = \lambda_q X,$$

where λ_p and λ_q are eigen values of a_{ij} and c_{lk} respectively for $X \neq 0$.
Now, the characteristic equation of a_{ij} is as follows

$$\left|a_{ij} - \lambda I\right| = 0.$$

Let the eigen values of a_{ij} are $\lambda_p = \lambda_1, \lambda_2, \lambda_3, \lambda_4, \cdots, \lambda_t$ and the eigen values of c_{lk} are $\lambda_q = \lambda_{t+1}, \lambda_{t+2}, \lambda_{t+3}, \lambda_{t+4}, \cdots, \lambda_n$.

Therefore, the trace of rhotrix R_n is

$$\begin{aligned} Tr(R_n) &= \text{Sum of all eigen values of rhotrix } R_n = \lambda_p + \lambda_q \\ &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_t + \lambda_{t+1} + \lambda_{t+2} + \lambda_{t+3} + \lambda_{t+4} + \cdots + \lambda_n \\ &= \sum_{i=1}^{t} \lambda_i + \sum_{i=t+1}^{n} \lambda_i \\ &= \sum_{i=1}^{n} \lambda_i = n. \end{aligned}$$

Therefore, the order of any rhotrix is same as the trace of that rhotrix R_n.

Example 2.1 Let R_3 be a rhotrix of 3-dimension satisfying the condition P as follows:

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle.$$

From the characteristic equation of R_3, that is,

$$|R_3 - \lambda I| = 0,$$

we have $\lambda = 1, 1, 1$ as the eigen values of R_3. Now,

$$Tr(R_3) = \sum_{i=1}^{3} \lambda_i = 3,$$

which is same as the order of the rhotrix R_3.

For $n = 5$,

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle.$$

The eigen values of R_5 are 0, 1, 1, 1, 2. Now,
$Tr(R_5) = \sum_{i=1}^{5} \lambda_i = 0 + 1 + 1 + 1 + 2 = 5$. Therefore, the trace of rhotrix R_5 is same as the order of the rhotrix R_5.

For $n = 7$,

$$R_7 = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & 1 & 0 & 1 & 0 & 1 & \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 & 0 & 1 & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle.$$

The eigen values of R_7 are 0, 0, 0, 1, 2, 2, 2 and
$Tr(R_7) = \sum_{i=1}^{7} \lambda_i = 7$, which is same as the order of the rhotrix R_7.

For $n = 9$,

$$R_9 = \left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & 1 & 0 & & & \\ & & 1 & 0 & 1 & 0 & 1 & & \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \\ & & 1 & 0 & 1 & 0 & 1 & & \\ & & & 0 & 1 & 0 & & & \\ & & & & 1 & & & & \end{array} \right\rangle.$$

The eigen values of R_9 are 0, 0, 0, 0, 0, 2, 2, 2, 3. So, $Tr(R_9) = 0+0+0+0+0+2+2+2+3 = 9$. Therefore, the trace of rhotrix R_9 is same as the order of the rhotrix R_9.

For $n = 13$,

$$R_{13} = \left\langle \begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 0 & 1 & 0 & & & & & \\ & & & & 1 & 0 & 1 & 0 & 1 & & & & \\ & & & 0 & 1 & 0 & 1 & 0 & 1 & 0 & & & \\ & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & & \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \\ & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & & \\ & & & 0 & 1 & 0 & 1 & 0 & 1 & 0 & & & \\ & & & & 1 & 0 & 1 & 0 & 1 & & & & \\ & & & & & 0 & 1 & 0 & & & & & \\ & & & & & & 1 & & & & & & \end{array} \right\rangle.$$

The eigen values of R_{13} are 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3, 3, 4.
Now, $Tr(R_{13}) = 0+0+0+0+0+0+0+0+0+3+3+3+4 = 13$.
Therefore, the trace of rhotrix R_{13} is same as the order of the rhotrix R_{13}.

Theorem 2.2 Let R_n be the rhotrix of n-dimension satisfying the condition P. Then the determinant of rhotrix is zero, when the order of rhotrix is greater than or equal to 5.
Proof: Let R_n be an $n-$ dimensional rhotrix with condition P defined as

$$R_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & 1 & 0 & 1 & 0 & 1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & a_{n(n-2)} & 0 & a_{(n-1)(n-1)} & 0 & a_{(n-2)n} & \\ & & 0 & c_{(n-1)(n-1)} & 0 & & \\ & & & a_{nn} & & & \end{array} \right\rangle = \left\langle a_{ij}, c_{lk} \right\rangle,$$

where

$$a_{ij} = \begin{bmatrix} a_{11} & 0 & a_{13} & \dots & \dots & a_{1n} \\ 0 & a_{22} & 0 & \dots & \dots & a_{2n} \\ a_{31} & 0 & a_{33} & \dots & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & \dots & a_{nn} \end{bmatrix}$$

and

$$c_{lk} = \begin{bmatrix} c_{11} & 0 & c_{11} & \cdots & \cdots & c_{1(n-1)} \\ 0 & c_{22} & 0 & \cdots & \cdots & c_{2(n-1)} \\ c_{31} & 0 & c_{33} & \cdots & \cdots & c_{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{(n-1)1} & c_{(n-1)2} & c_{(n-1)3} & \cdots & \cdots & c_{(n-1)(n-1)} \end{bmatrix}.$$

Let $\lambda_p = \lambda_1, \lambda_2, \lambda_3, \lambda_4, \cdots, \lambda_t$ are the eigen values of a_{ij} and $\lambda_q = \lambda_{t+1}, \lambda_{t+2}, \lambda_{t+3}, \lambda_{t+4}, \cdots, \lambda_n$ are the eigen values of c_{lk}.
One of these eigen values will be zero. Therefore, the determinant of R_n rhotrix is

$$\begin{aligned} det.(R_n) &= \text{Product of eigen values of matrices } a_{ij} \text{ and } c_{lk} \\ &= \lambda_1\lambda_2\lambda_3\lambda_4\cdots\lambda_t\lambda_{t+1}\lambda_{t+2}\lambda_{t+3}\lambda_{t+4}\cdots\lambda_n \\ &= \prod_{i=1}^{t}\lambda_i \prod_{i=t+1}^{n}\lambda_i \\ &= \prod_{i=1}^{n}\lambda_i = 0. \end{aligned}$$

Since one or more eigen values will be zero whenever $n \geq 5$. Therefore, determinant of the rhotrix R_n is equal to zero for $n \geq 5$.

Example 2.2 Let R_n be a rhotrix satisfying the condition P. For $n = 5$

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle.$$

The eigen values of R_5 are 0, 1, 1, 1, 2. Therefore, $det.(R_5) = 0.1.1.1.2 = 0$.

For $n = 9$

$$R_9 = \left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & 1 & 0 & & & \\ & & 1 & 0 & 1 & 0 & 1 & & \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \\ & & 1 & 0 & 1 & 0 & 1 & & \\ & & & 0 & 1 & 0 & & & \\ & & & & 1 & & & & \end{array} \right\rangle.$$

The eigen values of R_9 are 0, 0, 0, 0, 0, 2, 2, 2, 3. Therefore, $det.(R_9) = 0.0.0.0.0.2.2.2.3 = 0$.

Theorem 2.3 Let R_n and S_n be two n-dimensional rhotrices satisfying the condition P. Then both the rhotrices R_n and S_n have the same trace.

Proof: Given R_n and S_n are two $n-$dimensional rhotrices satisfying the condition P. Let Q_n be an invertible rhotrix. Then $R_n = Q_n^{-1} S_n Q_n$, is a diagonalizable rhotrix with the rhotrix S_n. We have,

$$\begin{aligned} Tr(R_n - \lambda I_n) &= Tr(Q_n^{-1} S_n Q_n - \lambda I_n) \\ &= Tr(Q_n^{-1} S_n Q_n - Q_n^{-1} \lambda I_n Q_n) \\ &= Tr(Q_n^{-1}(S_n - \lambda I_n) Q_n) \\ &= Tr(Q_n^{-1}) Tr(S_n - \lambda I_n) Tr(Q_n) \end{aligned}$$

$$= Tr(Q_n^{-1}) Tr(Q_n) Tr(S_n - \lambda I_n). \tag{2.1}$$

As the rhotrix Q_n is an invertible rhotrix, this gives $Tr(Q_n^{-1})Tr(Q_n) = 1$. Therefore, equation (2.1) becomes,

$$Tr(R_n - \lambda I_n) = Tr(S_n - \lambda I_n).$$

Hence, R_n and S_n have same trace.

Theorem 2.4 Let R_n and S_n be two n-dimensional rhotrices satisfying the condition P. Then for positive semi-definite rhotrices

$$0 \leq Tr(R_n S_n)^{2m} \leq (Tr R_n)^2 (Tr R_n^2)^{m-1} (Tr S_n^2)^m, m = 1, 2, 3, \ldots \tag{2.2}$$

Proof: Given R_n and S_n are two $n-$dimensional rhotrices satisfying the condition P. Now,

$$Tr(R_n S_n)^{2m} = Tr(R_n (S_n R_n)^{2m-1} S_n)$$

$$\leq Tr(R_n) Tr((S_n R_n)^{2m-1} S_n). \tag{2.3}$$

Also,

$$\begin{aligned} Tr((S_n R_n)^{2m-1} S_n) &= Tr(S_n (R_n S_n)^{2(m-1)} R_n S_n) \\ &= Tr((R_n S_n)^{2(m-1)} R_n S_n^2) \\ &\leq Tr((R_n S_n)^{2(m-1)} R_n) Tr S_n^2 \\ &= Tr(R_n (S_n R_n)^{2(m-1)-1} S_n R_n) Tr S_n^2 \\ &= Tr((S_n R_n)^{2(m-1)-1} S_n R_n^2) Tr S_n^2 \end{aligned}$$

$$\leq Tr((S_n R_n)^{2(m-1)-1} S_n) Tr R_n^2 Tr S_n^2. \tag{2.4}$$

From equations (2.3) and (2.4), we obtain

$$Tr((S_n R_n)^{2m-1} S_n) \leq Tr((S_n R_n)^{2(m-1)-1} S_n) Tr R_n^2 Tr S_n^2.$$

This gives,

$$\begin{aligned} Tr((S_n R_n)^{2(m-1)} S_n) &\leq Tr(S_n R_n S_n)(Tr R_n^2)^{m-1} (Tr S_n^2)^{m-1} \\ &= Tr(R_n S_n^2)(Tr R_n^2)^{m-1} (Tr S_n^2)^{m-1} \end{aligned}$$

$$= Tr R_n (Tr R_n^2)^{m-1} (Tr S_n^2)^m. \tag{2.5}$$

Hence, the equations (2.3) and (2.5) give the desired result.

Example 2.3 Let two rhotrices R_5 and S_5 satisfying the condition P. Let

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle$$

and

$$S_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle.$$

Now,

$$(R_5S_5)^2 = \left[\left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle \right]^2$$

$$= \left\langle \begin{array}{ccccc} & & 8 & & \\ & 0 & 1 & 0 & \\ 8 & 0 & 1 & 0 & 8 \\ & 0 & 1 & 0 & \\ & & 8 & & \end{array} \right\rangle.$$

Therefore,

$$Tr(R_5S_5)^2 = \sum_{i=1}^{5} d_{ii} \text{ (Sum of all diagonal entries)}$$

(2.6) $$= 8 + 1 + 1 + 1 + 8 = 19.$$

Now, for $m = 1$ in (2.2), we get

$$(TrR_5)^2(TrR_5^2)^0(TrS_5)^2 = \left[Tr\left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle \right]^2 (1)\, Tr\left\langle \begin{array}{ccccc} & & 2 & & \\ & 0 & 1 & 0 & \\ 2 & 0 & 1 & 0 & 2 \\ & 0 & 1 & 0 & \\ & & 2 & & \end{array} \right\rangle$$

(2.6A) $$= (5)^2\, 7 = 175.$$

By substituting (2.6) and (2.6A) in (2.2), we get

$$0 \leq 19 \leq 175.$$

Therefore, the inequality (2.2) is satisfied for two positive semi-definite rhotrices.

Theorem 2.5 Let R_n and S_n be two n-dimensional rhotrices satisfying the condition P. Then

(2.7) $$0 \leq Tr(R_nS_n)^{2m+1} \leq (TrR_n)(TrS_n)(TrR_n^2)^m(TrS_n^2)^m, m = 1, 2, 3, \ldots \in \mathbb{Z}^+$$

Proof: Here,

$$Tr(R_nS_n)^{2m+1} = Tr(R_n(S_nR_n)^{2m}S_n)$$

(2.8) $$\leq TrR_n(Tr(S_nR_n)^{2m}S_n).$$

Now,

$$Tr((S_nR_n)^{2m}S_n) = Tr(S_n(R_nS_n)^{2m-1}R_nS_n)$$
$$= Tr((R_nS_n)^{2m-1}R_nS_n^2)$$
$$\leq Tr((R_nS_n)^{2m-1}R_n)TrS_n^2$$
$$= Tr(R_n(R_nS_n)^{2(m-1)}S_nR_n)TrS_n^2$$
$$= Tr((S_nR_n)^{2(m-1)}S_nR_n^2)TrS_n^2$$

(2.9) $$= Tr((S_nR_n)^{2(m-1)}S_n)TrR_n^2TrS_n^2.$$

This gives,

$$Tr((S_nR_n)^{2m}S_n) = Tr((S_nR_n)^{2(m-1)}S_n)TrR_n^2TrS_n^2.$$

So, we have

(2.10) $$Tr((S_nR_n)^{2m}S_n) = TrS_n(TrR_n^2)^m(TrS_n^2)^m.$$

Hence, from equations (2.8) and (2.10), we obtain the desired inequality.

Theorem 2.6 Let R_n and S_n be two n-dimensional rhotrices satisfying the condition P. Then the inequality

(2.11) $$Tr(R_nS_n)^m \leq (Tr(R_n^{2m})Tr(S_n^{2m}))^{1/2},$$

holds, for $m \in \mathbb{Z}^+$.

Proof: Given R_n and S_n are two rhotrices satisfying the condition P. Now,

$$Tr(R_nS_n)^m = Tr(R_n^mS_n^m)$$
$$= \sum_{i=1}^{n} \lambda_i[(R_n^mS_n^m)]$$
$$\leq \sum_{i=1}^{n} \lambda_i(R_n^m)\lambda_i(S_n^m)$$
$$\leq \left[\sum_{i=1}^{n} \lambda_i{}^2(R_n^m) \sum_{i=1}^{n} \lambda_i{}^2(S_n^m)\right]^{1/2}$$
$$\leq \left[\sum_{i=1}^{n} \lambda_i{}^2[(R_n^m)(S_n^m)]\right]^{1/2}$$
$$= \left[Tr(R_n^{2m})Tr(S_n^{2m})\right]^{1/2}.$$

Hence proved.

Example 2.4 Let two rhotrices R_5 and S_5 satisfying the condition P. Put $m = 1$ in (2.11), we have

$$\left[Tr(R_5^2)Tr(S_5^2)\right]^{1/2} = \left[Tr\left\langle \begin{array}{ccccc} & & 2 & & \\ & 0 & 1 & 0 & \\ 2 & 0 & 1 & 0 & 2 \\ & 0 & 1 & 0 & \\ & & 2 & & \end{array} \right\rangle Tr\left\langle \begin{array}{ccccc} & & 2 & & \\ & 0 & 1 & 0 & \\ 2 & 0 & 1 & 0 & 2 \\ & 0 & 1 & 0 & \\ & & 2 & & \end{array} \right\rangle\right]^{1/2}$$

$$= [7 * 7]^{1/2}$$

$$= (7^2)^{1/2} = 7. \tag{2.12}$$

From (2.11) and (2.12), the inequality is verified.

Theorem 2.7 Let R_n be an n-dimensional rhotrix and R_n^* be its conjugate rhotrix satisfying the condition P, then

$$Tr(R_n)^{2m} \le Tr(R_nR_n^*)^m \le Tr[(R_n)^m(R_n^*)^m] \tag{2.13}$$

and

$$Tr(R_n)^{2m} \le [Tr(R_n)^{1/2}]^2[Tr(R_n)]^{(m-1)}[Tr(R_n^*)]^m. \tag{2.14}$$

Proof: Here, the rhotrix R_n is satisfying the condition P. Now,

$$Tr(R_n)^{2m} \le \sum_{i=1}^{n} \lambda_i[(R_nR_n^*)^m]$$

$$= Tr[(R_nR_n^*)^m]$$

$$\le \sum_{i=1}^{n} \lambda_i[(R_n)^m(R_n^*)^m]$$

$$= Tr[(R_n)^m(R_n^*)^m] \tag{2.15}$$

and

$$Tr[(R_n)^m(R_n^*)^m] \le Tr(R_n)Tr(R_n)^{(m-1)}Tr(R_n^*)^m$$

$$\le [Tr(R_n)^{1/2}]^2[Tr(R_n)]^{(m-1)}[Tr(R_n^*)]^m. \tag{2.16}$$

Therefore, equations (2.15) and (2.16) give us the desired results.

3 Conclusion

In the present paper, we have discussed some trace inequalities of special type of rhotrix satisfying the condition P. We have also obtained results related to the eigen values of $n-$ dimensional rhotrices.

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References

[1] Absalom, E. E., Sani, B., Sahalu, J. B.: The concept of heart-oriented rhotrix multiplication. Global J. Sci. Fro. Research. 11(2), 35-42 (2011).

[2] Ajibade, A. O.: The concept of rhotrices in mathematical enrichment. Int. J. Math. Educ. Sci. Tech. 34 (2), 175-179 (2003).

[3] Alfred, J. M., Paul, V. O., Scott, A. V.: Hand book of Applied Cryptography. CRC Press, (1996).

[4] Ames, W. F.: A Matrix Trace Inequality. J. Maths. Anal. & Appl. 250, 372-374 (2000).

[5] Aminu, A.: A note on the rhotrix system of equation. Journal of the Nigerian association of Mathematical Physics 21, 289-296 (2012).

[6] Mohammed, A., Ezugwu, E.A. and Sani, B.: On generalization and algorithmatization of heart-based method for multiplication of rhotrices. Int. J. Comp. Inform. Syst. 2, 46-49 (2011).

[7] Sani, B.: An alternative method for multiplication of rhotrices. Int. J. Math. Educ. Sci. Technol. 35(5), 777-781 (2004).

[8] Sani, B.: The row-column multiplication for high dimensional rhotrices. Int. J. Math. Educ. Sci. Technol. 38, 657-662 (2007).

[9] Sani, B.: Conversion of a rhotrix to a coupled matrix. Int. J. Math. Educ. Sci. Technol. 39, 244-249 (2008).

[10] Sharma, P. L., Gupta, S., Dhiman, N.: Sylvester rhotrices and their properties over finite field.Bull. Pure Appl. Sci.36E (1)(2017).

[11] Sharma, P. L., Gupta, S., Rehan, M.: On circulant like rhotrices over finite fields. App. Appl. Math.: An Int. J. (AAM) 12 (1), 506-520 (2017).

[12] Sharma, P.L., Kanwar, R. K.: A Note on relationship between invertible rhotrices and associated invertible matrices. Bull. Pure Appl. Sci. 30E (2), 333-339 (2011).

[13] Sharma, P. L., Kanwar, R. K.: Adjoint of a rhotrix and its basic properties. Int. J. Math. Sci. 11(3-4), 337-34 (2012).

[14] Sharma, P. L., Kumar, A., Gupta, S.: Trace of the positive integral powers of three and five dimensional rhotrices. Bull. Pure Appl. Sci. Sect. E Math. Stat. 39E(1), 165–175 (2020).

[15] Sharma, P. L., Kumar, A., Sharma, A. K.: On the characteristic roots and heart of a class of rhotrices over a finite field. Bull. Pure Appl. Sci. Sect. E Math. Stat. 39E(2), 277–288 (2020).

[16] Sharma, P. L., Kumar, S., Rehan, M.: On Hadamard rhotrix over finite field.Bull. Pure Appl. Sci. 32E (2), 181-190 (2013).

[17] Sharma, P. L., Kumar, S., Rehan, M.: On construction of Hadamard codes using Hadamard rhotrices. Int. J. Theo.& Appl. Sci.6 (1), 102-111 (2014).

[18] Tudunkaya, S. M.: Rhotrix polynomial and polynomial rhotrix. Pure Appl. Math. J.2,38-41 (2013).

[19] Tudunkaya, S.M., Makanjuola, S.O.: Rhotrices and the construction of finitefields, Bull. Pure Appl. Sci. 29E (2), 225-229 (2010).

[20] Ulukak, Z., Turmen, R.: On Some Matrix Trace Inequalities. J. Ineq. & Appl. 2010, 1-8 (2010).

[21] Yang, Z. P., Feng, X. X.: A note on the trace inequality for products of Hermitian matrix power. Journal Ineq. Pure Appl. Maths. 3(5), 1-26 (2002).