Effect of Ricci Flow on flat Friedmann-Lemaitre-Roberstson-Walker metric and Vaidya metric :A Computational Perspective

Shouvik Datta Choudhury [1] and Arindam Bhattacharyya [2]

[1] *Department of Mathematics,*
Jadavpur University,
Kolkata 700032 India
shouvikdc8645@gmail.com

[2]*Department of Mathematics,*
Jadavpur University,,
Kolkata 700032 India
aribh22@hotmail.com

Abstract

In this paper we have attempted to demonstrate the effect of Ricci flow on flat Friedmann-Lemaitre-Robertson-Walker metric and Vaidya metric using Maple.
A graphical demonstration is also provided.

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1 Introduction

Cosmology is the study of universe at a macroscopic level. Many observational outcomes connote that the overall regularities found are global rather than local. The important objective behind the cosmological studies is to develop a coherent, consistent and undeviating picture of the universe established on observational data and develop theoretical models which attune to the picture and support them the best and impart us qualitative as well as quantitative information about the past and future of the universe.

The most imperative observation is that of the expansion of the universe. The most preeminent observations about the expansion of the universe were discovered in late 1920s by Hubble and Humason. They observed that the nature of the light spectra which are emanating or regressing from galaxies at a distance has resemblance to that of the known elements of the earth as observed in the terrestrial laboratory, but the wavelengths of the lines in the galactic spectra show to be larger and more encompassing when compared with the corresponding lines in the terrestrial laboratory. This phenomenon is known as the Hubble shift and is a conspicuous proof of shift or recession of distant galaxies from our positions in the universe. Elucidating the above idea, the distance to the remote galaxies in the universe are increasing which is informally expressed or enunciated as that the distant galaxies are ”receding” from us. But in a stringent and rigorous perspective, individual galaxies are not travelling through space from us. The individual galaxies move around at random within clusters, which are at rest. The universe does not need a centre to expand away nor any empty space to contract into.This is the reason behind propounding the Robertson Walker metric.

The simplest assumption or surmise is that the universe is homogeneous and isotropic. It means that the universe has resembling properties when viewed from any position within itself. These are called the cosmological principles with respect to the Robertson-Walker metric and are also supported by observational information in the macroscopic level. In an exactly homogeneous model, there exists no center or boundary. So there remains no significance behind the statement that the universe is expanding to and or from somewhere.

Spatial homogeneity requires that space-time should admit group of motions with three indepe-ndent space -like generators. The orbits of this group prescribe a geodesically parallel group of space-like hyper surfaces orthogonal to time lines. From the above mentioned concept, it can be now said that the generic line element in the Friedmann-Lemaitre-Robertson-Walker metric can be expressed in the followi -ng form

$$ds^2 = dt^2 + g_{ik} dx^i dx^j \tag{1.1}$$

where the second term on the R.H.S is negative definite. The isotropy of the space means that it must have space properties in all directions and the three space must be of constant curvature. The metric can now expressed as

$$ds^2 = dt^2 - \frac{R^2(t)(dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2)}{(1 + kr^2/4)^2} \tag{1.2}$$

where $R(t)$ is a function of time and is also called the scale factor, k is a tri-valued constant $(0, 1 or - 1)$. The value of k depends on the curvature of the space satisfying the metric. θ varies between 0 and π (both inclusive) and φ varies between 0 and 2π (both inclusive). Equation (2) is the celebrated Friedmann-Lemaitre-Robertson-Walker metric.The effect of Ricci flow on the metric at $k = 0$ is the main objective of our paper.

In the perspective of geometric topology, Friedmann-Lemaitre-Robertson-Walker metric has an interesting implication [1]. The warped product $\overline{M} \times_F N$, of a l-dimensional manifold $(\overline{M}, \overline{g}), \overline{g}_{11} = -1$, with a warping function F and a 3-dimensional Riemannian manifold $(N, \tilde{g})$ is said to be a generalized Robertson-Walker space time. In particular, when the manifold $(N, \tilde{g})$ is a Riemannian space of constant curvature, the warped product $\overline{M} \times_F N$ is called a Robertson-Walker space time.

Einstein's theory of general relativity forecasts that matter can go through a catastrophical collapse to a point-like region where both the density of matter and the curvature of space time diverge (tend towards infinite values) ,given immense enough densities. This is connoted as a singularity, and also describes the formation of a black hole. For a black hole, the singularity is concealed from view by an event horizon, but general relativity has no such implication. In other words, general relativity does not include the existence of naked singularities. This is very disconcerting on both theoretical and observational perspective because as physical conditions become increasingly utmost, and ultimately the laws of physics succumb or disintegrate entirely, as one approaches a singularity. In the 1960s Roger Penrose proposed that there be some physical principle, as yet not fathomed, that excludes naked singularities as solutions to the equations of general relativity. In other words, every singularity must acquire an event horizon that hides the singularity from view. This is characterized as the 'cosmic censorship conjecture'.It is worth mentioning that it can be showed that Vaidya metric (also known as the radiating Schwarzschild space time) breaks the cosmic censorship principle. So Vaidya metric is the earliest counter illustration of cosmic censorship violation.

Considering spherically symmetric structure of inter-universal bodies as approximation, Schwa -rzschild (1916) gave the first symmetric solution of Einstein's field equation. Spherical symmetry signifies the invariance of the metric under rotation about an arbitrary axis at a particular point. This particular point is called the center of symmetry. The line element can expressed in such a way such that it is compatible for future calculations as

$$ds^2 = (1 - 2\frac{GM}{r})dt^2 - (1 - 2\frac{GM}{r})^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.3}$$

This is Schwarzschild solution of Einstein's field equation or the Schwarzschild metric. The Vaidya (1951) metric is a non - static generalization of the Schwarzschild metric. This is a simple yet engrossing generalization of the Schwarzschild metric called the Vaidya metric, which can be explained as a space time with an outgoing spherically symmetric radiation of massless particles. As a prototype, consider a spherically symmetric body ('star') that emanate a unceasing and uninterrupted stream of photons with each photon travelling radially outwards. The metric will now have as its source both the energy-momentum tensor of the star as well as the energy-momentum tensor of the radially outgoing null rays. The Vaidya metric is proficient enough to describe this situation and provides an fascinating model for a time dependent spherically symmetric metric. To realize this metric, we shall first make a coordinate transformation from the Schwarzschild time coordinate t to another coordinate u such that

$dt = du + dr/(1-2GM/r)$. (The coordinate u has a simple physical meaning. From the above transformation we construe that $u = constant$ corresponds to curves which satisfy the equation $dr/dt = +(1-2GM/r)$. The radial null lines in the Schwarzschild geometry will correspond to the curves with $d\theta = 0$, $d\varphi = 0$ and $ds = 0$. This directly gives $(dr/dt)^2 = (1-2GM/r)^2$. The positive root of this equation will depict a radially outgoing null ray. (The negative root will give a radially ingoing null ray.) Thus we see that $u = constant$ along the radially outgoing null rays.With this transformation, the Schwarzschild metric becomes

$ds^2 = (1-2\frac{GM}{r})dt^2 + 2dudr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2)$ which is the Vaidya metric.

2 Motivation

Considering the Robertson-Walker space-time as four dimensional Riemannian manifold (M, g), a Ricci soliton is said to be a generalisation of a Einstein metric and is defined by $(\pounds_V g)(X, Y)+ 2Ric(X, Y) + 2\lambda g(X, Y) = 0$ where $\pounds_V g$ signifies the Lie derivative of g along a vector field V, λ a constant, and arbitrary vector fields X, Y on M. The Ricci soliton is said to be shrinking, steady,and expanding accordingly as λ is negative, zero, and positive respectively. This equation tells how to explicitly compute Ricci soliton and know its nature. We can study whether Ricci solitons are shrinking,steady or expanding by observing the graphical solutions instead of explicitly computing them as an alternative, if exact solutions are available.

3 Effect of Ricci flow on Friedmann-Lemaitre-Robertson-Walker metric

Ricci flow is an intrinsic geometric flow having a signature of a parabolic partial differential equation similar to heat diffusion or a evolution equation. It processes the deformation of a metric. In this chapter we investigate the nature of Ricci flow's effect on Robertson -Walker metric considering the flat case where the scale factor k becomes 0 [14]. The scale factor k is an invariant having three values only $(-1, 0, +1)$. It has been observed that the observational data which is recorded at various distinguished and significant observatories on our planet mostly comply with and is consistent with the flat Robertson -Walker model bestowed with the Robertson -Walker metric. So we have selected $k = 0$ for this purpose.

The main idea or algorithm behind our computational method is as follows . We at first state the Ricci tensors formed by the four dimensional space- time satisfying the Robertson-Walker metric which are standard equations as well as the four metric components , which are standard results. Then we set the Ricci flow equation for each Ricci tensor and its associated metric component. We then put the symbolic metric component values and the associated symbolic Ricci tensor and produce a series of ordinary differential equations in t(time). Ordinary differential equations are generated instead of partial differential equations because of the existence of a single variable t as dependent variable. An approach or practise prevails that to treat the series of ordinary differential equations as system of ordinary differential equations (same case exists for partial differential equations) and perform stipulate mathematical manipulations and cancellations to bring it down to a single ordinary differential equation which has the possibility of describing the model. But this method, although suitable in some cases to portray the intrinsic model of the existing scenario, has few significant frailty. The individual partial differential equation or ordinary differential equations may represent more explicitly and articulate the intrinsic behaviour of the system sectionally for many cases , as in our case, so that the exact solution of each single partial differential equation or ordinary differential equation if possible to derive can give us a more coherent and a complete perspective to analysis along with their graphical testimony. Moreover, the differential equation generated by mathematical manipulation of a system of differential equations may not exhibit the accurate and authentic nature of a stochastic system or a dynamical system which our method may give better result. In our paper, we attempt to give an exact solution by analytical methods or computer generated methods using Maple and study and observe whether the graphical demonstrations generated gives us a Ricci soliton structure without explicitly computing for them.

For the Robertson-Walker metric satisfied by (2), the corresponding metric components can be stated as follows :-

Metric Components:-

$g_{00} = 1, g_{11} = -\frac{R^2}{1-kr^2}, g_{22} = -r^2R^2, g_{33} = -sin^2\theta^2, g^{00} = 1, g^{11} = -\frac{1-kr^2}{R^2}, g^{22} = 1/r^2R^2, g^{33} = -1/sin^2\theta^2$

The corresponding Ricci tensors can be stated as follows:-

$R_{00} = -3(\frac{d^2R}{dt^2})/R$

$R_{11} = (R\frac{d^2R}{dt^2} + 2(\frac{dR}{dt})^2 + 2ck^2)^2(1 - kr^2)$

$R_{22} = (R\frac{d^2R}{dt^2} + 2(\frac{dR}{dt})^2 + 2ck^2)^2r^2$

$R_{33} = (R\frac{d^2R}{dt^2} + 2(\frac{dR}{dt})^2 + 2ck^2)^2r^2sin^2\theta$, considering $(x_0, x_1, x_2, x_3) = (ct, r, \theta, \phi)$.

The meaning of all the notation denoted above are explained in the fourth paragraph of Introduction as well as in [2].

The series of generic Ricci flow equations are

$$\frac{\partial g_{00}}{\partial t} = -2R_{00},$$

$$\frac{\partial g_{11}}{\partial t} = -2R_{11},$$

$$\frac{\partial g_{22}}{\partial t} = -2R_{22},$$

,

$$\frac{\partial g_{33}}{\partial t} = -2R_{33}$$

Associating the metric components and corresponding Ricci tensor we obtain a series of ordinary differential equations as follows : -

(3.1) $$\frac{d(1)}{dt} = -6(\frac{d^2R}{dt^2})$$

(3.2) $$\frac{d(-R^2)}{dt} = -2(R(\frac{d^2R}{dt^2}) + 2(\frac{dR}{dt})^2)^2$$

(3.3) $$\frac{d(-r^2R^2)}{dt} = -2(R(\frac{d^2R}{dt^2}) + 2(\frac{dR}{dt})^2)^2r^2$$

(3.4) $$\frac{d(-sin^2\theta)}{dt} = -2(R(\frac{d^2R}{dt^2}) + 2(\frac{dR}{dt})^2)^2r^2sin^2\theta$$

A trivial solution is that $R(t) = 0$ which has no cosmological implication as the continua of space-time collapses since R(t) is the scale factor. An exact solution of the first Ricci flow equation is $R(t) = At + B$ where A and B are arbitrary constants. Some exact solution of the second Ricci flow equation is $R(t) = 0, \frac{\sqrt{6t+9A}(2t+3B)}{21}$,
$-\frac{\sqrt{6t+9A}(2t+3B)}{21}$ where A and B are arbitrary constants. Some exact solution of the third Ricci flow

equations are $R(t) = 0$, $\frac{\sqrt{6t+9A}(2t+3B)}{21}$,
$-\frac{\sqrt{6t+9A}(2t+3B)}{21}$ where A and B are arbitrary constants. The graphical plot is same as the second equation as both of the second and third Ricci flow equations are equivalent. An exact solution of the fourth Ricci flow equation $R(t) = (3At + 3B)^{1/3}$ where A and B are arbitrary constants.

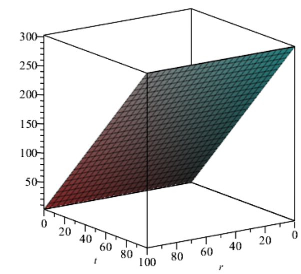

Fig. 1: $R(t) = At + B$,Graphical plot of solution of 1st Ricci flow equation

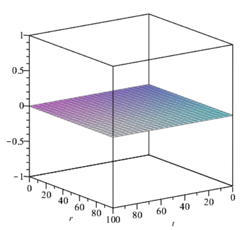

Fig. 2: $R(t) = 0$

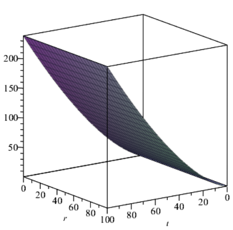

Fig. 3: $R(t) = \frac{\sqrt{6t+9A}(2t+3B)}{21}$

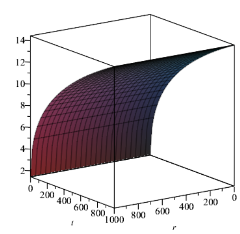

Fig. 4: $R(t) = -\frac{\sqrt{6t+9A}(2t+3B)}{21}$

Fig. 5: Graphical plots of solutions of second Ricci flow equation

The graphical demonstration has been based on taking suitable values of constants A and B. Solutions involving complex numbers are not provided as they cannot provide graphical demonstration.

4 Effect of Ricci flow on Vaidya metric

Since the only surviving Ricci tensor for Vaidya metric is
$-\frac{2GM'}{r^2}$.we have the only Ricci flow equation as

$$\frac{d}{du}(1 - \frac{2GM}{r}) = -\frac{4GM'}{r^2} \tag{4.1}$$

where $g_{00} = (1 - \frac{2GM}{r})$ and $R_{00} = \frac{2GM'}{r^2}$, M' indicates derivative of M with respect to u.
The solution is $M(u) = constant$.The graphical plot is
The implication of the graphical solutions are stated in Conclusion.

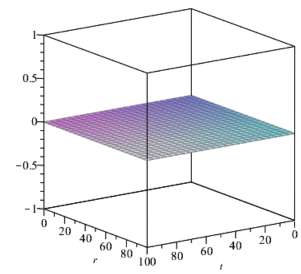

Fig. 6: $M(u) = constant$

5 Conclusion

From cosmological perspective,

$R(t) = (3At + 3B)^{1/3}$, $\frac{\sqrt{6t+9A}(2t+3B)}{21}$, $R(t) = At + B$ are consistent with cosmological principles that the universe is expanding even if the Ricci flow is acting on the metric because $R(t)$ is the scale factor.

$M(u) = constant$ signifies that under the effect of Ricci flow, the non static radiation emitting Vaidya metric transforms into spherically symmetric static metric, an example of which is Schwarzschild metric which is obviously independent of function t.

It can be stated that any compact steady or expanding Ricci soliton must be Einstein. In dimension $n \leq 3$, there are no compact shrinking Ricci solitons other than those of constant positive curvature.

Our case is four dimensional one. So by virtue of the above statement from [3] we can state that there exists gradient shrinking non compact solitons for $n \geq 3$ and there is also existence of complete noncompact Ricci solitons (steady, shrinking and expanding) that are not Einstein. So the first ,second and fourth soliton may or may not be Einstein.It can observed that all are steady Ricci soliton.

Exact solutions of Ricci flow if exists can assist us to observe and inspect the nature of Ricci solitons. In our paper, the metric evolves with respect to time, so the metric structure does not remain the same. The metric equation also changes with respect to time. So the space which initially satisfied the Robertson-Walker metric or the Vaidya metric does not remain the same due to the effect of Ricci flow.

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