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ON THE DEGREE OF APPROXIMATION OF CONJUGATE FUNCTIONS USING GENERALIZED NÖRLUND-EULER SUMMABILITY METHOD

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Abstract

In this paper, we have established a very interesting result for the degree of approximation of conjugate functions belonging to the $W[L_r, \xi(t)]$ class by generalized Nörlund-Euler product summability method of conjugate series of Fourier series. The results presented in this paper is the generalization of many known and unknown results.

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1. Introduction

Summability theory plays a significant role to study area of Fourier Analysis, Wavelet Analysis, Fixed point theory and many other fields. The well known theorem of Weierstrass is the origin of theory of approximation. The degree of approximation of functions belonging to various classes have been determined by various investigators Mishra [9], Mishra and Mishra [10], Mishra et al. ([12], [13], [15], [16]), Deepmala et al. [14], Mishra [17], Mishra et al.[19], Psarakis and Moustakides [22], Krasniqi ([23], [24]) and many others (See also [1], [11], [18]). Recently Kushwaha et al [4], Kushwaha and Kumar [5], Zafarov [6], Mishra et al. [15], Sahani and Mishra [8] have determined the degree of approximation by product summability method of Fourier series. Using product summability means Pradhan et al. [21] have determined the degree of approximation of function belonging to weighted class. But no work seems to have been done so far to find the degree of approximation of functions of weighted class by $(\overline{N}, p_n, q_n)(E, s)$ product means of conjugate series of Fourier series.Working in this direction, we have determined the degree of approximation of conjugate of functions belonging to weighted class by $(\overline{N}, p_n, q_n)(E, s)$-product summability method of conjugate series of Fourier series which is the generalization of several known and unknown results. Therefore, this result will be useful for researchers in future.

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2. Definition and Preliminaries

Let $\sum u_n$ be an infinite series with the sequence of partial sums $\{s_n\}$. The Euler's means of the sequence $\{s_n\}$ is defined by

$$E_n^s = \frac{1}{(1+s)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} s^{n-\nu} s_\nu. \tag{2.1}$$

If $E_n^s \to s$ as $n \to \infty$, then the series $\sum u_n$ is summable to s with respect to (E, s) summability and (E, s) means is regular (Hardy, [2]). Let $\{p_n\}$ and $\{q_n\}$ be sequence of positive real numbers such that

$$P_n = \sum_{k=0}^{n} p_k \quad \text{and} \quad Q_n = \sum_{k=0}^{n} q_k$$

and let $R_n = p_0 q_n + p_1 q_{n-1} + ... + p_n q_0 \neq 0, p_{-1} = q_{-1} = R_{-1} = 0$. The sequence to sequence transformation

$$t_n^N = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k s_k \tag{2.2}$$

defines the sequence $\{t_n^N\}$ of the $(\overline{N}, p_n, q_n)$ mean of the sequence $\{s_n\}$ generated by the sequence of coefficients p_n and q_n. Similarly, we define the extended mean.

$$t_n^{\overline{N}} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k s_k \tag{2.3}$$

where $R_n = p_0 q_0 + p_1 q_1 + ... + p_n q_n \neq 0, p_{-1} = q_{-1} = R_{-1} = 0$. If $t_n^{\overline{N}} \to s$ as $n \to \infty$, then the series $\sum u_n$ is $(\overline{N}, p_n, q_n)$ summable to s. The Riesz summability method is said to be regular if

(i) $\frac{p_k q_k}{R_n} \to 0$, for each integer $k \geq 0$ as $n \to \infty$.
(ii) $|\sum_{k=0}^{n} p_k q_k| < C|R_n|$, where C is any positive integer independent of n.

Now we define a new product summability method $(\overline{N}, p_n, q_n)(E, s)$ of $\{s_n\}$ as

$$T_n^{\overline{N}E} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k \{E_k^s\} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k \left\{ \frac{1}{(1+s)^k} \sum_{\nu=0}^{k} \binom{n}{\nu} s^{n-\nu} s_\nu \right\} \tag{2.4}$$

If $t_n^{\overline{N}E} \to s$ as $n \to \infty$, then $\sum u_n$ is summable to s by $(\overline{N}, p_n, q_n)(E, s)$ method. Let f be 2π periodic and integrable over $(-\pi, \pi)$ in Lebesgue sense, then its Fourier series be given by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n cosnx + b_n sinnx) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x). \tag{2.5}$$

The conjugate series of Fourier Series (2.5) is given by

$$\sum_{k=1}^{\infty}(a_n sinnx - b_n cosnx) \tag{2.6}$$

A function $f \in Lip\alpha$, if

$$f(x+t) - f(x) = O\left(|t^{\alpha}|\right) \text{ for } 0 < \alpha \leq 1, t > 0.$$

A function $f \in Lip\,(\alpha, r)$ for $a \leq x \leq b$ if

$$\left\{\int_a^b |f(x+t) - f(x)|^r dx\right\}^{1/r} \leq M\left(|t^{\alpha}|\right), r \geq 1,\ 0 < \alpha \leq 1$$

where M is absolutely constant.
We have $f \in Lip\,(\xi(t), r)$, if

$$\left\{\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right\}^{1/r} = O\left(\xi(t)\right), r \geq 1, t > 0.$$

A function $f \in W(L^r, \xi(t))$, Khan [3] if

$$\left\{\int_0^{2\pi} |f(x+t) - f(x)|^r sin^{\beta r}(x/2) dx\right\}^{1/r} = O\left(\xi(t)\right), \beta \geq 0, r \geq 1, t > 0,$$

where, $\xi(t)$ is increasing function of t.
If $\beta = 0$ then the generalized weighted Lipschitz $W\left(L^r, \xi(t)\right)$ $(r \geq 1)$ class reduces to $Lip\left(\xi(t), r\right)$ class. If $\xi(t) = t^{\alpha}$ then, $Lip\left(\xi(t), r\right)$ class coincides with the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ converted to $Lip\alpha$ class.
The L_{∞}-norm of a function $f : R \to R$ is defined by

$$||f||_{\infty} = ess\sup\left\{|f(x)| : x \in R\right\}$$

The L^r-norm of a function is defined by

$$||f||_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}}, 1 \leq r \leq \infty.$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of order n under sup norm $||.||_{\infty}$ is defined by MacFadden [20]

$$||t_n - f||_{\infty} = \sup_{x \in R}\left\{|t_n(x) - f(x)|\right\}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_r$ is defined by

$$E_n(f) = \min_{\tau_n} ||\tau_n - f||_r.$$

We use following notations through out the paper:

$$\psi(t) = f(x+t) - f(x-t)$$

$$\overline{f}(x) = -\frac{1}{2\pi}\int_0^{\pi} \psi(t) cot(t/2) dt$$

$$\overline{M_n}(t) = \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^n}\sum_{\nu=0}^{k}\binom{k}{\nu} s^{k-\nu}\frac{cos(\nu + 1/2)t}{sin(t/2)}\right\}.$$

3. Main Theorem

Let f be a 2π-periodic function which is integrable in Lebesgue sense in $[0, 2\pi]$. If $f \in W(L^r, \xi(t))$ class, then the degree of approximation of conjugate of function is given by

$$\|\tau_n^{\overline{N}E} - \overline{f}\|_r = O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \tag{3.1}$$

where $\tau_n^{\overline{N}E}$ is the $(\overline{N}, p_n, q_n)(E, s)$ transform of $\{s_n\}$, provided $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{be decreasing function} \tag{3.2}$$

$$\left\{\int_0^{\frac{1}{(n+1)}}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^r sin^{\beta r}t dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{3.3}$$

$$\left\{\int_{\frac{1}{(n+1)}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left\{(n+1)^{\delta}\right\} \tag{3.4}$$

where δ is an arbitrary number such that $(\beta - \delta)q - 1 > 0$, $r^{-1} + q^{-1} = 1$, $1 \leqslant r \leqslant \infty$, and conditions (3.3) and (3.4) hold uniformly in x.

4. Lemma

To prove the theorem, we need the following lemma:

$$|\overline{M}_n(t)| = O\left(\frac{1}{t}\right), \text{for } \frac{1}{(n+1)} \leqslant t \leqslant \pi.$$

Proof. For $\frac{1}{(n+1)} \leqslant t \leqslant \pi$, $sin(t/2) \geq \frac{t}{\pi}$ (Jordan's Lemma), so

$$\begin{aligned}
\left|\overline{M_n}(t)\right| &= \left|\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s^{k-\nu}\frac{cos(\nu+1/2)t}{sin(t/2)}\right\}\right| \\
&\leq \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s^{k-\nu}\frac{e^{i(\nu+1/2)t}}{(t/\pi)}\right\}\right| \\
&= \frac{1}{2tR_n}\left|\sum_{k=0}^{n} p_k q_k \frac{|s+e^{it}|^k}{(1+s)^k}\right| \\
&= \frac{1}{2tR_n}\left|\sum_{k=0}^{n} p_k q_k \frac{(1+s^2+2scost)^{k/2}}{(1+s)^k}\right| \\
&\leq \frac{1}{2tR_n}\left|\sum_{k=0}^{n} p_k q_k e^{\frac{-2st^2k}{\pi^2(1+s)^2}}\right|
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2tR_n}\sum_{k=0}^{n} p_k q_k \\ &= O\left(\frac{1}{t}\right). \end{aligned}$$

5. Proof of main Theorem

The k^{th} partial sum of the conjugate series of the Fourier series (2.5) is given by

$$\overline{s_n}(x) = \frac{-1}{2\pi}\int_0^{\pi} cot(t/2)\psi(t)dt + \frac{1}{2\pi}\int_0^{\pi}\frac{cos(n+1/2)t}{sin(t/2)}\psi(t)dt$$

$$\overline{s_n}(x) - \left(\frac{-1}{2\pi}\int_0^{1/n+1} cot(t/2)\psi(t)dt - \frac{-1}{2\pi}\int_{1/n+1}^{\pi} cot(t/2)\psi(t)dt\right)$$

$$= \frac{1}{2\pi}\left(\int_0^{1/n+1} + \int_{1/n+1}^{\pi}\right)\frac{cos(n+1/2)t}{sin(t/2)}\psi(t)dt$$

$$\overline{s_n}(x) - \overline{f_n}(m) = \frac{1}{2\pi}\left(\int_0^{1/n+1}\frac{cos(n+1/2)t}{sin(t/2)} - cott/2\right)\psi(t)dt + \frac{1}{2\pi}\int_{1/n+1}^{\pi}\frac{cos(n+1/2)t}{sin(t/2)}\psi(t)dt$$

taking $(\overline{N}, p_n, q_n)(E, s)$ transformation, we get

$$\begin{aligned} \overline{\tau}^{\overline{N}E} - \overline{f} &= \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_k \int_0^{1/n+1}\frac{\psi(t)}{(1+s)^k}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}s^{k-\nu}\left(\frac{cos(\nu+1/2)t}{sin(t/2)} - cott/2\right)dt\right\} \\ &+ \frac{1}{2\pi R_n}\sum_{\nu=0}^{n} p_k q_k \int_{1/n+1}^{\pi}\frac{\psi(t)}{(1+s)^k}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}s^{k-\nu}\left(\frac{cos(\nu+1/2)t}{sin(t/2)}\right)dt\right\} \\ &= \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_k \int_0^{1/n+1}\frac{\psi(t)}{(1+s)^k}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}s^{k-\nu}\left(\frac{2\sin(\nu+1)t/2 sin(-\nu t)/2}{sin(t/2)}\right)\right\}dt \\ &+ \int_{1/(n+1)}^{\pi}\psi(t)\overline{m}_n(t)dt \end{aligned} \quad (5.1)$$

$$\begin{aligned} \left|\overline{\tau}^{\overline{N}E} - \overline{f_n}\right| &\leq \frac{1}{\pi R_n}\sum_{\nu=0}^{n} p_k q_k \int_0^{1/n+1}\frac{\psi(t)}{(1+s)^k}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}s^{k-\nu}\left(\frac{(\nu+1)sin(t/2)\left|sin\frac{\nu t}{2}\right|}{sin(t/2)}\right)dt\right\} + I_2 \\ &= \frac{1}{\pi R_n}\sum_{\nu=0}^{n} p_k q_k \int_0^{1/n+1}\frac{\psi(t)}{(1+s)^k}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}s^{k-\nu}(\nu+1)\right\}dt + I_2 \end{aligned}$$

$$
\begin{aligned}
&= O\,(n+1)\int_0^{1/(n+1)} \psi(t)dt + I_2 \\
&= I_1 + I_2.(say)
\end{aligned}
\tag{5.2}
$$

Now, $|I_1| \leq \int_0^{1/(n+1)} |\psi(t)|\, O\,(n+1)\, dt$. Further $f \in W\,(L_r, \xi(t))$ implies $\psi \in W\,(L_r, \xi(t))$, thus $|I_1| \leq \int_0^{1/(n+1)} \left| \frac{t\psi(t)sin^\beta t}{\xi(t)} . \frac{\xi(t)O(n+1)}{tsin^\beta t} \right| dt$.
Now, by Hölder's inequality, we have

$$
\begin{aligned}
|I_1| &\leq \left(\int_0^{1/(n+1)} \left| \frac{t\psi(t)sin^\beta t}{\xi(t)} \right|^r dt \right)^{1/r} \times \left(\lim_{\epsilon \to 0} \int_\epsilon^{1/(n+1)} \left| \frac{\xi(t)O\,(n+1)}{tsin^\beta t} \right|^q dt \right)^{1/q} \\
&= O\left(\frac{1}{n+1}\right) \left[\lim_{\epsilon \to 0} \int_\epsilon^{1/(n+1)} \left(\frac{\xi(t)O(n+1)}{tsin^\beta t} \right)^q dt \right]^{1/q} \text{ by (3.3)} \\
&= \left[\lim_{\epsilon \to 0} \int_\epsilon^{1/(n+1)} \left(\frac{\xi(t)}{tsin^\beta t} \right)^q dt \right]^{1/q} \\
&= \xi\left(\frac{1}{n+1}\right) \left[\left(\frac{t^{-q-\beta q+1}}{-q-\beta q+1} \right)_0^{\frac{1}{n+1}} \right]^{1/q} \\
&= O\left\{ \xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+1/r} \right\} \text{ since } r^{-1} + q^{-1} = 1
\end{aligned}
\tag{5.3}
$$

Now by Hölder's inequality and Lemma, we have

$$
\begin{aligned}
|I_2| &\leq \left(\int_{1/(n+1)}^\pi \left| \frac{t^{-\delta}|\psi(t)|sin^\beta t}{\xi(t)} \right|^r dt \right)^{1/r} \left(\int_{1/(n+1)}^\pi \left| \frac{\xi(t)\overline{M}_n(t)}{t^{-\delta}\,sin^\beta t} \right|^q \right)^{1/q} \\
&= O\{(n+1)^\delta\} \left\{ \int_{1/(n+1)}^\pi \left(\frac{\xi(t)}{t^{1-\delta+\beta}} \right)^q dt \right\}^{1/q} \text{ by(3.4)} \\
&= \{(n+1)^\delta\} \left\{ \int_{(n+1)}^{1/\pi} \left(\frac{\xi(1/y)}{y^{\delta-\beta-1}} \right)^q \frac{dy}{y^2} \right\}^{1/q} \text{ by(3.2)}
\end{aligned}
$$

Again by using second Mean Value theorem, we get

$$
\begin{aligned}
|I_2| &= O\left\{ (n+1)^\delta \xi\left(\frac{1}{n+1}\right) \right\} \left(\int_{(n+1)}^{1/\pi} \frac{dy}{y^{q(\delta-\beta-1)+2}} \right)^{1/q} \\
&= O\left\{ (n+1)^{\beta+1/r} \xi\left(\frac{1}{n+1}\right) \right\} \text{ since } r^{-1} + q^{-1} = 1
\end{aligned}
\tag{5.4}
$$

Now, by using (5.3) and (5.4) in (5.2), we get

$$
\|\tau_n^{\overline{N}E} - \overline{f}\|_r = O\left\{ (n+1)^{\beta+1/r} \xi\left(\frac{1}{n+1}\right) \right\}
$$

This completes the proof of the theorem.

6. Corollaries

Corollary 1. If we put $\beta = 0$, then the weighted class reduces to $Lip(\xi(t), r)$.If f is 2π periodic and belonging to class $Lip(\xi(t), r)$ then the degree of approximation by $(\overline{N}, p_n, q_n)(E, s)$ method of conjugate series of Fourier series (2.5) is given by

$$\|\overline{\tau}_n^{\overline{N}E} - \overline{f}\| = O\left\{(n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right\} \tag{6.1}$$

Corollary 2. If we put $\beta = 0$ and $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$, then weighted class $W(L^r, \xi(t))$ reduces to $Lip(\alpha, r)$, then the degree of approximation of 2π-periodic function$f$ belonging to $Lip(\alpha, r)$ by $(\overline{N}, p_n, q_n)(E, s)$ method of conjugate series of Fourier series (2.5) is given by

$$\|\overline{\tau}_n^{\overline{N}E} - \overline{f}\| = O\left\{\frac{1}{(n+1)^{\alpha-1/r}}\right\}, 0 < \alpha \leq 1, r \geq 1. \tag{6.2}$$

Corollary 3. If we put $\beta = 0$, $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$ and $r \to \infty$. Then weighted class $W(L^r, \xi(t))$ reduces to $Lip\alpha$, then the degree of approximation of 2π-periodic function by $(\overline{N}, p_n, q_n)(E, s)$ method of conjugate series of Fourier series (2.5) is given by

$$\|\overline{\tau}_n^{\overline{N}E} - \overline{f}\| = O\left\{\frac{1}{(n+1)^{\alpha}}\right\}, 0 < \alpha \leq 1, r \geq 1. \tag{6.3}$$

7. Conclusion

In the present work we have used generalized Nörlund-Euler product summability method.If we consider $q_n = 1$, then it reduces to $(\overline{N}, p_n)(E, s)$ product summability method. Similarly, if we consider $s = 1$ in above then it reduces to $(\overline{N}, p_n)(E, 1)$ product summability means. In this way we can see that our result is superior to many other results.

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