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DEGREE OF APPROXIMATION OF CONTINUOUS FUNCTION IN THE HÖLDER METRIC BY $(C,1)F(a,q)$ MEANS OF ITS FOURIER SERIES

H. L. RATHORE, U. K. SHRIVASTAVA and L. N. MISHRA

Abstract

A new estimate for the $(C,1)[F,d_n]$ product summability with Hölder metric recently has been determined by Rathore, Shrivastava and Mishra [On approximation of continuous function in the Hölder metric by $(C,1)[F,d_n]$ means of its Fourier series. Jnanabha, 51(2),(2021)161-167], which is the family of $F(a,q)$ mean includes of $[F,d_n]$ mean. Further we extend the under weaker condition of Hölder metric using Cesàro mean and $F(a,q)$ mean the associate with Fourier series. We obtained a theorem on approximation of continuous function in the Hölder metric by $(C,1)F(a,q)$ means of its Fourier series.

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1. Introduction

Chandra [P. Chandra, "On the generalized Fej'er mean in the metric of Hölder Metric," Mathematics nachrichten, vol.109, no.1(1982), 39-45] was first to extend the result of Prössdorf [S. Prössdorf "Zur konvergenz der Fourierreihen Hölder Stetiger Funktionen" Mathematische Nachrichten, vol. 69, No. 1, (1975), 7-14]. In 1983 Mohapatra and Chandra [R. N. Mohapatra and P. Chandra, "Degree of approximation of function in the Hölder metric" Acta Math. Hung; 41(1983), 67-76] result to find the degree of approximation in the Hölder metric using matrix transform. Later on Das, Ghosh and Ray obtained a number of interesting result on the degree of approximation of function by their Fourier series in the generalized Hölder Metric. In 2001 Lal and Yadav was most interesting result on degree of approximation of function belonging to the Lipschitz class by $(C,1)(E,1)$ means of its Fourier series. In 2008 Singh and Mahajan was deduced many previous result to extend the error bound of periodic signal in Hölder Metric. Later on Shrivastava and Rathore to find the degree of approximation of continuous function in the Hölder Metric by $F(a,q)$ mean. In 2009 Lal and Kushuwaha defined $(C,1)(E,q)$ summability of Fourier series. In 2021 Rathore, Shrivastava and Mishra defined $(C,1)[F,d_n]$. It is known that $F(a,q)$ mean includes like several mean (E,q), $[F,d_n]$ mean. So in the present paper we have defined

$(C,1)F(a,q)$ mean of Fourier series and generalizing the above two result obtain the degree of approximation of continuous function in the Hölder metric by $(C,1)F(a,q)$ mean.

2. Definition and Notation

Let f be a periodic function with period- 2π and integrable in the Lebesgue sense over $[-\pi,\pi]$. Let the Fourier series associated with f at x be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx). \tag{2.1}$$

Let $C_{2\pi}$ denote the Banach Spaces of all $2\pi-$ periodic continuous function under "sup" norm for $0 < \alpha \leq 1$ and some positive constant K the function space H_α is given by the following

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq K|x-y|^\alpha\}. \tag{2.2}$$

The space H_α is a Banach space (see Prössdorf [11]) with the norm $||.||_\alpha$ defined by

$$||f||_\alpha = ||f||_c + \sup_{x,y} \Delta^\alpha[f(x,y)], \tag{2.3}$$

where

$$||f||_c = \sup_{-\pi\leq x\leq\pi} |f(x)| \tag{2.4}$$

and

$$\Delta^\alpha\{f(x,y)\} = |x-y|^{-\alpha}|f(x) - f(y)|, (x \neq y). \tag{2.5}$$

We shall use the convention that $\Delta^0 f(x,y) = 0$. The metric induced (2.3) on the H_α is called the Hölder metric. It can be seen that $||f||_\beta \leq (2\pi)^{\alpha-\beta}||f||_\alpha$ for $0 \leq \beta < \alpha \leq 1$. Thus $\{(H_{\alpha,}||.||_\alpha)\}$ is a family of Banach Spaces which decreases as α increases.

We write

$$\varphi_x(t) = \{f(x+t) + f(x-t) - 2f(x)\}. \tag{2.6}$$

The family $F(a,q)$ of summability method was introduced by Meir [9]. The summability matrix $\{C_{nk}\}$ belong to $F(a,q)$, if it satisfies the condition: n is a discrete or continuous parameter: $q = q(n)$ is a positive increasing function which tend to infinity as $n \to \infty$ 'a' is a positive constant: for every fixed $\delta : \frac{1}{2} < \delta < \frac{2}{3}$

$$C_{nk} = \frac{\sqrt{a}}{\pi q}\exp(-aq)^{-1}(k-q)^2\left\{1 + O\{\frac{|k-q|+1}{q}\} + O\{\frac{|k-q|^3}{q^2}\}\right\}, \tag{2.7}$$

as $n \to \infty$ uniformly in k for $|k-q| \leq q^\delta$ and

$$\sum_{|k-q|>q^\delta}(k+1)C_{nk} = O\{\exp(-q^\mu)\}, \tag{2.8}$$

where μ is some positive number independent of n.

Let

$$F_n^{(a,q)}(f;x) = \sum_{k=0}^{\infty} C_{nk} S_k(f;x). \tag{2.9}$$

Denote the $F(a,q)$ mean of the Fourier series (2.1) of f, where $S_k(f;x)$ is the kth partial sum of (2.1). The family $F(a,q)$ contains the summability method of generalised Borel, Euler, Taylor, S_α and Valiron.

It is known (see Kuttner, Rajagopal and Rangachari [6]) that

$$\sum_{k=0}^{\infty} C_{pk} = 1 + O(q^{-1/2}). \tag{2.10}$$

The summability method of Euler, Taylor, S_α and Borel satisfy (2.10) in the stronger from

$$\sum_{k=0}^{\infty} C_{pk} = 1. \tag{2.11}$$

The series $\sum_{k=0}^{\infty} u_k$ is said to be $(C,1)$ summable to S.

If

$$\frac{1}{n+1}\sum_{k=0}^{n} S_k \to S \text{ as } n \to \infty. \tag{2.12}$$

The $(C,1)$ transform of the $F(a,q)$ transform is defined as the $(C,1)$ $F(a,q)$ transform of the partial sum S_n of the $\sum_{k=0}^{\infty} u_k$

$$(C,F)_n^{(a,q)} = \frac{1}{n+1}\sum_{k=0}^{n} F_k^{(a,q)} \to S \text{ as } n \to \infty. \tag{2.13}$$

where $F_n^{(a,q)}$ denotes the $F(a,q)$ transform of S_n then the series $\sum_{k=0}^{\infty} u_k$ is said to be summable $(C,1)F(a,q)$ means or simply summable $(C,1)F(a,q)$ to S.

3. Some Theorem

In 1928 **Alexits** [1] proved the following theorem

Theorem A: If $f \in C_{2\pi} \cap Lip\alpha, (0 < \alpha \le 1)$ then

$$\|\sigma_n^\delta - f\| = O(n^{-\alpha}\log n). \tag{3.1}$$

The case $\alpha = \delta = 1$ was proved by Bernstein [2]. Theorem A was extended by several workers such as **Holland, Sahney and Tzimbalario** [5]. Replacing (C,δ) mean by $(E,q)(q>0)$ **Chandra** [3] obtained the following result:

Theorem B: Let $0 \le \beta < \alpha \le 1$ and let $f \in H_\alpha$. Then

$$\|E_n^q(f) - f\|_\beta = O\{n^{\beta-\alpha}\log n\}. \tag{3.2}$$

where $E_n^q(f,x)$ denotes(E,q) transform of $S_n(f;x)$.

Shrivastava and Rathore [13] extended above result for $F(a, q)$ means. They proved the following theorem.
Theorem C: Let $[q]$ denote the integral part of $q = q(p)$ and $m = [q] + 1$. If $0 \leq \beta < \alpha \leq 1$ and $f \in H_\alpha$ then

$$\|t_p(f; x) - f(x)\| = O(m^{\beta - \alpha/2}). \tag{3.3}$$

Lal and Kushwaha [8] obtained the degree of approximation of function of $Lip\alpha$ class by product summability mean of the form $(C, 1)F(a, q)$. Their Theorem is as follows:
Theorem D : If $f : R \to R$ is 2π-periodic Lebesgue integrable on $[-\pi, \pi]$ and belonging to the Lipschitz class then the degree of approximation of f by the $(C, 1)(E, q)$ product means of its Fourier series satisfies for $n = 0, 1, 2...$

$$\|(C, E)_n^q(x) - f(x)\|_\infty = O[(n + 1)^{-\alpha}] \text{ for } 0 < \alpha < 1. \tag{3.4}$$

4. Main Theorem

In this paper we extend the above result by proving following approximation result for product summability mean $(C, 1)F(a, q)$ means.
Theorem: Let n be the integral part of $q = q(n)$. If $0 \leq \beta < \alpha \leq 1$ and $f \in H_\alpha$ then

$$\|(C, F)_n^{(a,q)} - f(x)\|_\beta = O((n + 1)^{\beta - \alpha} \log(n + 1)), \tag{4.1}$$

where $(C, F)_n^{(a,q)}$ is the product summability mean $(C, 1)F(a, q)$ mean of $S_n(f; x)$. We will use following lemmas.

Lemma 4.1. *Let $\varphi_x(t)$ be defined in (2.6) then for $f \in H_\alpha$, we have*

$$|\varphi_x(t) - \varphi_y(t)| \leq 4k|x - y|^\alpha, \tag{4.2}$$

and also

$$|\varphi_x(t) - \varphi_y(t)| \leq 4k|t|^\alpha. \tag{4.3}$$

It is easy to verify.

Lemma 4.2. *If $q = q(n)$, is an integer valued function of n, then for $\frac{1}{2} < \delta < \frac{2}{3}$ we have*

$$\int_0^\pi \frac{|\varphi(t)|}{\sin t/2} \sum_{|k-q| \leq^\delta} \frac{\sqrt{a}}{\pi q} \exp(-aq^{-1}(k - q)^2) \sin(k + \frac{1}{2})tdt$$
$$= \int_0^\pi \frac{|\varphi(t)|}{\sin \frac{t}{2}} \exp(\frac{-qt^2}{4a}) \sin(q + \frac{1}{2})tdt + O(q \exp(-aq^{2\delta - 1})). \tag{4.4}$$

Lemma 4.3. *Let*

$$M_n(t) = \frac{1}{(n + 1)\pi} \sum_{k=0}^{n} \frac{\exp(\frac{-qt^2}{4a}) \sin(q + \frac{1}{2})t}{\sin t/2}$$
$$= O(n + 1) \, for \; 0 \leq t \leq \frac{\pi}{n + 1}$$

PROOF. Using $\sin nt \le n \sin t$ for $0 \le t \le \frac{\pi}{n+1}$.
Then

$$M_n(t) \le \frac{1}{(n+1)\pi} \sum_{k=0}^{n} \exp(\frac{-qt^2}{4a}) \frac{(2q+1)\sin t/2}{\sin t/2}$$
$$= \sum_{k=0}^{n} (2q+1)$$
$$= O(n+1). \tag{4.5}$$

□

LEMMA 4.4. *Let*

$$M_n(t) = \frac{1}{(n+1)\pi} \sum_{k=0}^{n} \frac{\exp(\frac{-qt^2}{4a}) \sin(q+\frac{1}{2})t}{\sin t/2}.$$

Then

$$M_n(t) = O(\frac{1}{t}) \, for \, \frac{\pi}{n+1} \le t \le \pi.$$

PROOF. Using $\sin \frac{t}{2} \ge (\frac{t}{\pi})$ and $|\sin(q+\frac{1}{2})t| \le 1$ for $\frac{\pi}{n+1} \le t \le \pi$

$$M_n(t) = \frac{1}{(n+1)\pi} \sum_{k=0}^{n} \exp(\frac{-qt^2}{4a}) \frac{1}{t/\pi}$$
$$= O(\frac{1}{t}). \tag{4.6}$$

□

5. Proof of the main theorem:

Following Zygmund [15] we have

$$S_k(f;x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \frac{1}{\sin t/2} \varphi_x(t) \sin(k+\frac{1}{2}) t dt. \tag{5.1}$$

We have

$$F_n^{(a,q)} - f(x) = \frac{1}{\pi} \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{k=0}^{\infty} C_{nk} \sin(k+\frac{1}{2}) t dt + O(q^{-1/2}). \tag{5.2}$$

The $(C,1)F(a,q)$ transform of $S_k(f;x)$ by $(C,F)_n^{(a,q)}$ we have

$$(C,F)_n^{(a,q)} - f(x) = \frac{1}{(n+1)\pi} \sum_{k=0}^{n} \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{k=0}^{\infty} C_{nk} \sin(k+\frac{1}{2}) t dt \tag{5.3}$$

writing $I_n(x) = (C,F)_n^{(a,q)} - f(x)$.

We have

$$|I_n(x)| = |(C,F)_n^{(a,q)} - f(x)| \leq \frac{1}{(n+1)\pi} |\sum_{k=0}^{n} \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{k=0}^{\infty} C_{nk} \sin(k+\frac{1}{2})tdt|. \quad (5.4)$$

We have

$$\begin{aligned}
|I_n(x) - I_n(y)| &= |\frac{1}{(n+1)\pi} \sum_{k=0}^{n} \int_0^{\pi} \frac{\varphi_x(t) - \varphi_y(t)}{\sin t/2} \sum_{k=0}^{\infty} C_{nk} \sin(k+\frac{1}{2})tdt| \\
&= \left| \frac{1}{(n+1)\pi} \sum_{k=0}^{n} \int_0^{\pi} \frac{\varphi(t)}{\sin t/2} [(\sum_{|k-q|\leq q^{\delta}} + \sum_{|k-q|>q\delta})] C_{nk} \sin(k+\frac{1}{2})tdt \right| \\
&\leq \left| \frac{1}{(n+1)\pi} \sum_{k=0}^{n} \int_0^{\pi} \frac{\varphi(t)}{\sin t/2} \sum_{|k-q|\leq q^{\delta}} C_{nk} \sin(k+\frac{1}{2})tdt \right| \\
&+ \left| \frac{1}{(n+1)\pi} \sum_{k=0}^{n} \int_0^{\pi} \frac{\varphi(t)}{\sin t/2} \sum_{|k-q|>q^{\delta}} C_{nk} \sin(k+\frac{1}{2})tdt \right| \\
&= |S_1| + |S_2| + O(q^{-\frac{1}{2}}). \quad (5.5)
\end{aligned}$$

$$\begin{aligned}
|S_2| &\leq \frac{1}{(n+1)\pi} \sum_{k=0}^{n} [\int_0^{\pi} \frac{|\varphi(t)|}{\sin t/2} \sum_{|k-q|>q^{\delta}} C_{nk} \sin(k+\frac{1}{2})tdt] \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^{n} [\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi} .] \frac{|\varphi(t)|}{\sin t/2} \sum_{|k-q|>q^{\delta}} C_{nk} \sin(k+\frac{1}{2})tdt \\
|S_2| &= |S_{2.1} + S_{2.2}|. \quad (5.6)
\end{aligned}$$

Now

$$|S_{2.1}| \leq \frac{1}{(n+1)\pi} \sum_{k=0}^{n} [\int_0^{\frac{\pi}{n+1}} |\varphi(t)| \sum_{|k-q|>q^{\delta}} C_{nk} \frac{1}{\sin t/2} \sin(k+\frac{1}{2})tdt]$$

Using $\sin nt \leq n \sin t$, for $0 < t < \frac{\pi}{n+1}$

$$\begin{aligned}
&= \frac{1}{(n+1)\pi} \sum_{k=0}^{n} [\int_0^{\frac{\pi}{n+1}} |\varphi(t)| \sum_{|k-q|>q^{\delta}} C_{nk} \frac{(2k+1)}{\sin t/2} \sin t/2dt] \\
&= \frac{1}{(n+1)} \sum_{k=0}^{n} (2k+1)[\int_0^{\frac{\pi}{n+1}} |\varphi(t)|dt] \text{ by (2.10)} \\
&= O(n+1) \int_0^{\frac{\pi}{n+1}} |t|^{\alpha} dt \text{ by Lemma (4.1) \& (4.3)}
\end{aligned}$$

$$= O(n+1)^{-\alpha}. \tag{5.7}$$

Now

$$|S_{2.2}| = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}[\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi(t)|}{\sin t/2}\sum_{|k-q|>q^{\delta}}C_{nk}\sin(k+\frac{1}{2})tdt]$$

$$|S_{2.2}| = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}[\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi(t)|}{\sin t/2}\sum_{|k-q|>q^{\delta}}C_{nk}\sin(k+\frac{1}{2})tdt]$$

Using

$$\sin(t/2) \geq t/\pi \text{ and } |\sin(k+\frac{1}{2})t| \leq (k+\frac{1}{2})t \text{ for } \frac{\pi}{n+1} \leq t \leq \pi$$

$$\begin{aligned}
|S_{2.2}| &\leq \frac{1}{(n+1)\pi}\sum_{k=0}^{n}[\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi(t)|}{t/\pi}\sum_{|k-q|>q^{\delta}}C_{nk}(k+\frac{1}{2})tdt] \\
&= \frac{1}{(n+1)}O(\exp(-q^{\mu})\int_{\frac{\pi}{n+1}}^{\pi}|\varphi(t)|dt \text{ by (2.8)} \\
&= \frac{1}{(n+1)}\int_{\frac{\pi}{n+1}}^{\pi}|t|^{\alpha}dt \\
&= O(n+1)^{-\alpha}.
\end{aligned} \tag{5.8}$$

So

$$|S_2| = O(n+1)^{-\alpha}. \tag{5.9}$$

Now

$$\begin{aligned}
|S_1| &= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{|\varphi(t)|}{\sin t/2}\sum_{|k-q|\leq q^{\delta}}C_{nk}\sin(k+\frac{1}{2})tdt \\
&= \frac{1}{n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{|\varphi(t)|}{\sin t/2}\sum_{|k-q|\leq q^{\delta}}\frac{\sqrt{a}}{\pi q}\exp(-aq^{-1}(k-q)^2) \\
&\times\{1+O(\frac{|k-q|+1}{q})+O(\frac{|k-q|^3}{q^2})\}\sin(k+\frac{1}{2})tdt \\
&= |S_3+S_4+S_5|.
\end{aligned} \tag{5.10}$$

We estimate S_4 as follows,

$$\begin{aligned}
|S_4| \leq \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{|\varphi(t)|}{t} \\
\times\sum_{|k-q|\leq q^{\delta}}\frac{\sqrt{a}}{\pi q}\exp(-aq^{-1}(k-q)^2)(\frac{|k-q|+1}{q})\sin(k+\frac{1}{2})tdt
\end{aligned}$$

$$= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}[\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}]\frac{|\varphi(t)|}{t}$$

$$\times \sum_{|k-q|\le q^{\delta}} \frac{\sqrt{a}}{\pi q}\exp(-aq^{-1}(k-q)^2)(\frac{|k-q|+1}{q})\sin(k+\frac{1}{2})tdt$$

$$= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\frac{\pi}{n+1}}\frac{|\varphi(t)|}{t}$$

$$\times \sum_{|k-q|\le q^{\delta}} \frac{\sqrt{a}}{\pi q}\exp(-aq^{-1}(k-q)^2)(\frac{|k-q|+1}{q})(|k-q|+q+\frac{1}{2})tdt$$

$$+ \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi(t)|}{t}$$

$$\times \sum_{|k-q|\le q^{\delta}} \frac{\sqrt{a}}{\pi q}\exp(-aq^{-1}(k-q)^2)(\frac{|k-q|+1}{q})tdt$$

$$= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}[O(\sqrt{q}\int_0^{\frac{\pi}{n+1}}|t|^{\alpha}dt) + O(\frac{1}{\sqrt{q}}\int_{\frac{\pi}{n+1}}^{\pi}|t|^{\alpha-1}dt)]$$

$$|S_4| = O(n+1)-^{\alpha}. \tag{5.11}$$

Similarly

$$|S_5| = O(n+1)^{-\alpha}. \tag{5.12}$$

Now

$$|S_3| = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{|\varphi(t)|}{\sin t/2}\sum_{|k-q|\le q^{\delta}}\frac{\sqrt{a}}{\pi q}\exp(-aq^{-1}(k-q)^2)\sin(k+\frac{1}{2}).$$

Applying Lemma 4.2

$$|S_3| = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{|\varphi(t)|}{\sin t/2}\left[\exp(\frac{-qt^2}{4a})\sin(q+\frac{1}{2})tdt\right] + O(q\exp(-aq^{2\delta-1})).$$

Now

$$|S_3| = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{|\varphi(t)|}{\sin t/2}\left[\exp(\frac{-qt^2}{4a})\sin(q+\frac{1}{2})tdt\right]$$

$$= \left(\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}.\right)|\varphi(t)|M_n(t)dt \text{ by Lemma (4.3) and (4.4)}$$

$$|S_3| = |S_{3.1} + S_{3.2}|. \tag{5.13}$$

Then

$$|S_{3.1}| = \int_0^{\frac{\pi}{n+1}}|\varphi(t)|M_n(t)dt$$

$$= O(n+1)\int_0^{\frac{\pi}{n+1}} |t|^\alpha dt \text{ by Lemma (4.3)}$$
$$= O(n+1)^{-\alpha} \tag{5.14}$$

and

$$\begin{aligned}|S_{3.2}| &= \int_{\frac{\pi}{n+1}}^{\pi} |\varphi(t)| M_n(t) dt \\ &= \int_{n+1}^{\pi} O(\frac{1}{t})|t|^\alpha dt \text{ by Lemma (4.4)} \\ &= O(n+1)^{-\alpha}.\end{aligned} \tag{5.15}$$

From (5.14) and (5.15) combining we have

$$|S_3| = O(n+1)^{-\alpha}. \tag{5.16}$$

Now using

$$\begin{aligned}|\varphi(t)| &= |\varphi_x(t) - \varphi_y(t)| \\ &= O(|x-y|^\alpha).\end{aligned}$$

We obtain

$$|S_2| = O(n+1)^{-\alpha} \tag{5.17}$$

and

$$\begin{aligned}|S_4| \leq& \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{|\varphi(t)|}{t} \\ &\times \sum_{|k-q|\leq q^\delta}\frac{\sqrt{a}}{\pi q}\exp(-aq^{-1}(k-q)^2)(\frac{|k-q|+1}{q})|\sin(k+\frac{1}{2})t|dt \\ =& \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\frac{\pi}{n+1}}\frac{|\varphi(t)|}{t} \\ &\times \sum_{|k-q|\leq q^\delta}\frac{\sqrt{a}}{\pi q}\exp(-aq^{-1}(k-q)^2)(\frac{|k-q|+1}{q})(|k-q|+q+\frac{1}{2})tdt \\ &+ \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi(t)|}{t}\sum_{|k-q|\leq q^\delta}\frac{\sqrt{a}}{\pi q}\exp(-aq^{-1}(k-q)^2)(\frac{|k-q|+1}{q}) \\ =& \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[O(\sqrt{q}\int_0^{\frac{\pi}{n+1}}|\varphi(t)|dt) + O(\frac{1}{\sqrt{q}}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi(t)|}{t}dt)\right] \\ =& O(|x-y|^\alpha) + O(|x-y|^\alpha)\log(n+1) \\ |S_4| =& O(|x-y|^\alpha)\log(n+1).\end{aligned} \tag{5.18}$$

Similarly

$$|S_5| = O(|x-y|^{\alpha}) \tag{5.19}$$

and applying Lemma 4.2 and $q = q(n)$ is an integer valued of n we get

$$|S_3| = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{|\varphi(t)|}{\sin t/2}[\exp(\frac{-qt^2}{4a})\sin(q+\frac{1}{2})tdt].$$

Applying Lemma 4.3 and Lemma 4.4

$$\begin{aligned}
|S_3| &= (\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}.)|\phi(t)|M_n(t)dt \\
&= O(n+1)\int_0^{\frac{\pi}{n+1}}|\varphi(t)|dt + \int_{\frac{\pi}{n+1}}^{\pi}O(\frac{1}{t})|\varphi(t)|dt \\
&= O(|x-y|^{\alpha}) + O(|x-y|^{\alpha})\log(n+1) \\
|S_3| &= O(|x-y|^{\alpha})\log(n+1).
\end{aligned} \tag{5.20}$$

Now for $k = 2, 3, 4, 5$ and for $0 \le \beta < \alpha \le 1$, We observe that

$$|S_k| = |S_k|^{1-\beta/\alpha}|S_k|^{\beta/\alpha}. \tag{5.21}$$

By using (5.9) and (5.17) respectively in the first and the second factor on the right of the above identify (5.21) for $k = 2$ we obtain that

$$|S_2| = O\{|x-y|^{\beta}(n+1)^{\beta-\alpha}\}. \tag{5.22}$$

Again using (5.16) and (5.20) in the first and second factor on the right of the identify (5.21) for $k = 3$ we have

$$|S_3| = O\{|x-y|^{\beta}(n+1)^{\beta-\alpha}\log(n+1)\}. \tag{5.23}$$

By using (5.11) and (5.18) in the first and second factor on the right of the identify (5.21) for $k = 4$ we have

$$|S_4| = O\{|x-y|^{\beta}(n+1)^{\beta-\alpha}\log(n+1)\}. \tag{5.24}$$

Also replacing S_5 by estimates in the first factor from (5.12) and second factor from (5.19) on the right of (5.21) for $k = 5$ we get

$$|S_5| = O\{|x-y|^{\beta}(n+1)^{\beta-\alpha}\}. \tag{5.25}$$

Thus from (5.22), (5.23), (5.24) and (5.25) we get

$$\begin{aligned}
\sup_{\substack{x,y\\x\neq y}}|\Delta^{\beta}I_n(x,y)| &= \sup_{\substack{x,y\\x\neq y}}|I_n(x) - I_{n(y)}| \\
&= O\{(n+1)^{\beta-\alpha}\log(n+1)\}
\end{aligned} \tag{5.26}$$

Now using the fact that $f \in H_{\alpha} \Rightarrow \phi_x(t) = O(t^{\alpha})$

Proceeding as above we obtain

$$\|I_n\|_c = \sup_{-\pi\le x\le\pi} \|(CF)_n^{(a,q)} - f(x)\| = O\{(n+1)^{-\alpha}\log(n+1)\} \quad (5.27)$$

Combining the result of (5.26) and (5.27) and using (5.23) we get

$$\|(CF)_n^{(a,q)} - f(x)\|_\beta = O\{(n+1)^{\beta-\alpha}\log(n+1)\}$$

This completes the proof of the main theorem.

6. Application

The following result can be easily derived from main theorem (4.1).

COROLLARY 6.1. *If $f \in Lip\alpha$, when $0 < \alpha \le 1$ then for $n > 1$,*

$$\|(CF)_n^{(a,q)} - f(x)\| = O(n)^{-\alpha} logn$$

we put $\beta = 0$ then theorem B is particular case of main theorem (4.1).

Conclusion

We would like to mention that the space H_α and its norm given by Chandra P was generalized in 1982. Further our result is influencing more people and makes them to more efficiently obtain result in generalizing Hölder metric, Abel summation method and our main result in above mentioned special cases as well.

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H. L. Rathore, Department of Mathematics, Govt. College Pendra, 495019 Bilaspur, Chhattisgarh, India. *Corresponding author :
e-mail: hemlalrathore@gmail.com

U. K. Shrivastava, Department of Mathematics, Govt. E. R. Rao Science PG College Bilaspur, 495001, Chhattisgarh, India
e-mail: profumesh18@yahoo.co.in

L. N. Mishra, Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT), Vellore 632 014, Tamil Nadu, India
e-mail: lakshminarayanmishra04@gmail.com