

## APPROXIMATION OF LINEAR INITIAL VALUE PROBLEMS USING BERNOULLI POLYNOMIAL AND OPERATIONAL MATRIX

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### Abstract

In this work, we have developed mathematical model to find approximate solution of linear initial value problems (IVPs) in form of closed form series. The model is based on the orthonormalization of classical Bernoulli polynomials. An operational matrix has been formed with the orthonormal set of polynomials which is used to transform the derivatives into algebraic expressions. Thus, an IVP is converted into a set of linear equations. The technique has been demonstrated through three problems. Approximate solutions have been compared with available exact or other numerical solutions. High degree of accuracy has been noted in numerical values of solutions for considered problems.

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### 1. Introduction

While dealing with many scientific or engineering problems, initial value problems (IVPs) for ordinary differential equations are encountered naturally. The problems of fluid flow and continuum mechanics, electricity and magnetism, heat conduction, wave propagation, nuclear physics, atomic structures, diffusion problems, gas dynamics, population genetics communication theory, electricity and magnetism, geophysics, antenna, synthesis problem, economic modeling, radiation problems, and astrophysics are some examples which lead to IVPs. Requirements of high precision accuracy of approximations increase the complexity of problems which results in the unavailability of analytic solutions. This motivates researchers and scientists to model advanced and faster methods of approximate solutions. A bulk of literature is available to explore exact and numerical solutions of initial and boundary value problems [1–4]. For decades past, researchers have focused their attention on such approximation techniques for differential and integral equations. Cheon [5] discussed possible applications of Bernoulli polynomials and functions in numerical analysis. Xu [6] adopted the method of variational iteration. Pandey, et. al. [7] applied homotopic

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perturbation and method of collocation. Some latest investigations include uses of Legendre wavelets [8], Chebyshev polynomials [9], Laguerre polynomials and Wavelet Galerkin method [10], Legendre polynomials [11], and the operational matrix [12].

Many investigators have also worked to find numerical and approximate solutions to boundary value problems [13–17]. Many researchers also discussed Bernoulli polynomials and their properties contributing to the numerical solution of IVPs [18, 19]. Tohidi et. al. [20] obtained numerical approximation for generalized pantograph equation using the Bernoulli matrix method. Tohidi and Khorsand [21] solved a second-order linear system of partial differential equations. Mohsenyazadeh [22] used Bernoulli polynomials to solve Volterra type integral equations. Recently, Singh [23, 24] used Bernoulli polynomials to develop a trigonal operational matrix to solve Volterra-type integral equations.

In this work, it is proposed to solve linear initial value problems using orthogonal polynomials derived from Bernoulli polynomials with the operational matrix [23].

## 2. Introduction to Bernoulli Polynomials

The Bernoulli polynomials were introduced by Jakob Bernoulli in 1690 in his book “*ArsConjectandi*” [25], however, the name “Bernoulli Polynomial” was given by J. L. Raabe in 1851 while he was discussing and analysing the formula  $\sum_{n=0}^{m-1} B_n \left( x + \frac{k}{m} \right) = m^{-(n+1)} B_n(mx)$ . In this formula,  $B_n(x)$  represent Bernoulli polynomials of degree  $n$ . A thorough study of these polynomials was first done by Leonhard Euler in 1755, who showed in his book “Foundations of differential calculus” that these polynomials satisfy the finite difference relation:

$$B_n(\zeta + 1) - B_n(\zeta) = n\zeta^{n-1}, \quad n \geq 1 \quad (2.1)$$

and proposed the method of generating function to calculate  $B_n(x)$ . Following Leonhard Euler, recently Costabile and Dell’Accio [25] showed that Bernoulli Polynomials are monic which can be extracted from its generating function

$$\frac{\gamma e^{\zeta\gamma}}{e^\gamma - 1} = \sum_{n=0}^{\infty} B_n(\zeta) \frac{\gamma^n}{n!} \quad (|\zeta| < 2\pi) \quad (2.2)$$

and represented in the simple form:

$$B_n(\zeta) = \sum_{j=0}^n \binom{n}{j} B_j(0) \zeta^{n-j}, \quad n = 0, 1, 2, \dots \quad 0 \leq \zeta \leq 1 \quad (2.3)$$

where  $B_n(0)$  are the Bernoulli numbers which can also be calculated with Kronecker’s formula  $B_n(0) = - \sum_{j=1}^{n+1} \frac{(-1)^j}{j} \binom{n+1}{j} \sum_{k=1}^j k^n$ ;  $n \geq 0$  [26]. Thus, first few Bernoulli

polynomials can be written as  $B_0(\zeta) = 1, B_1(\zeta) = \zeta - \frac{1}{2}, B_2(\zeta) = \zeta^2 - \zeta + \frac{1}{6}, B_3(\zeta) = \zeta^3 - \frac{3}{2}\zeta^2 + \frac{1}{2}\zeta, B_4(\zeta) = \zeta^4 - 2\zeta^3 + \zeta^2 - \frac{1}{30}$ .

These Bernoulli Polynomials form a complete basis over  $[0, 1]$  and show many interesting properties [27–29] out of which some used in this work are as under:

$$\left. \begin{aligned} B'_n(\zeta) &= nB_{n-1}(\zeta), \quad n \geq 1 \\ \int_0^1 B_n(z)dz &= 0, \quad n \geq 1 \\ B_n(\zeta + 1) - B_n(\zeta) &= n\zeta^{n-1}, \quad n \geq 1 \end{aligned} \right\}. \quad (2.4)$$

### 3. The Orthonormal Polynomials

Using the property that Bernoulli Polynomials form a complete basis over  $[0, 1]$  with respect to standard inner product, a set of  $n + 1$  orthonormal polynomials can be derived from first  $n$  Bernoulli polynomials using Gram-Schmidt orthogonalization with respect to standard inner product on  $L^2 \in [0, 1]$ . For the sake of better understanding of a reader, first few such orthonormal polynomials are given here.

$$\phi_0(\zeta) = 1 \quad (3.1)$$

$$\phi_1(\zeta) = \sqrt{3}(-1 + 2\zeta) \quad (3.2)$$

$$\phi_2(\zeta) = \sqrt{5}(1 - 6\zeta + 6\zeta^2) \quad (3.3)$$

$$\phi_3(\zeta) = \sqrt{7}(-1 + 12\zeta - 30\zeta^2 + 20\zeta^3) \quad (3.4)$$

$$\phi_4(\zeta) = 3(1 - 20\zeta + 90\zeta^2 - 140\zeta^3 + 70\zeta^4) \quad (3.5)$$

$$\phi_5(\zeta) = \sqrt{11}(-1 + 30\zeta - 210\zeta^2 + 560\zeta^3 - 630\zeta^4 + 252\zeta^5) \quad (3.6)$$

### 4. Approximation of Functions

**Theorem.** Let  $H = L^2[0, 1]$  be a Hilbert space and  $Y = \text{span}\{y_0, y_1, y_2, \dots, y_n\}$  be a subspace of  $H$  such that  $\dim(Y) < \infty$ , then each  $f \in H$  has a unique best approximation out of  $Y$  [27], that is,  $\forall y(t) \in Y, \exists \hat{f}(t) \in Y$  s.t.  $\|f(t) - \hat{f}(t)\|_2 \leq \|f(t) - y(t)\|_2$ . This implies that,  $\forall y(t) \in Y, \langle f(t) - \hat{f}(t), y(t) \rangle = 0$ , where  $\langle, \rangle$  is standard inner product on  $L^2 \in [0, 1]$  (c.f. Theorems 6.1-1 and 6.2-5, Chapter 6 [27]).

**Remark.** Let  $Y = \text{span}\{\phi_0, \phi_1, \phi_2, \dots, \phi_n\}$ , where  $\phi_k \in L^2[0, 1]$  are orthonormal Bernoulli polynomials. Then the above theorem concludes that for any function  $f \in L^2[0, 1]$ ,

$$f \approx \hat{f} = \sum_{k=0}^n c_k \phi_k, \quad (4.1)$$

where  $c_k = \langle f, \phi_k \rangle$  and  $\langle, \rangle$  is the standard inner product on  $L^2 \in [0, 1]$ . For numerical approximation, series (5) can be written as:

$$f(\zeta) \approx \sum_{k=0}^n c_k \phi_k = C^T \phi(\zeta) \quad (4.2)$$

where  $C = (c_0, c_1, c_2, \dots, c_n)$  and  $\phi(\zeta) = (\phi_0, \phi_1, \phi_2, \dots, \phi_n)$  are column vectors. The number of polynomials  $n$  can be chosen to meet required accuracy.

### 5. Construction of operational matrix

The orthonormal polynomials, thus obtained, can be expressed as:

$$\int_0^\zeta \phi_o(\eta) d\eta = \frac{1}{2} \phi_o(\zeta) + \frac{1}{2\sqrt{3}} \phi_1(\zeta) \quad (5.1)$$

$$\begin{aligned} \int_0^\zeta \phi_i(x) dx = & \frac{1}{2\sqrt{(2i-1)(2i+1)}} \phi_{i-1}(\zeta) \\ & + \frac{1}{2\sqrt{(2i+1)(2i+3)}} \phi_{i+1}(\zeta), \quad (\text{for } i = 1, 2, \dots, n) \end{aligned} \quad (5.2)$$

Relations (5.1-5.2) can be represented in closed form as:

$$\int_0^\zeta \phi(\eta) d\eta = \Theta \phi(\zeta) \quad (5.3)$$

where  $\zeta \in [0, 1]$  and  $\Theta$  is operational matrix of order  $(n+1)$  given as :

$$\Theta = \frac{1}{2} \begin{bmatrix} 1 & \frac{1}{\sqrt{1.3}} & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{1.3}} & 0 & \frac{1}{\sqrt{3.5}} & \cdots & 0 \\ 0 & -\frac{1}{\sqrt{3.5}} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{\sqrt{(2n-1)(2n+1)}} \\ 0 & 0 & \cdots & -\frac{1}{\sqrt{(2n-1)(2n+1)}} & 0 \end{bmatrix} \quad (5.4)$$

### 6. Solution of Initial Value Problems

Consider the linear IVP:

$$\frac{d^2 y}{d\zeta^2} + P(\zeta) \frac{dy}{d\zeta} + Q(\zeta)y = r(\zeta), \quad y(0) = \alpha, \left( \frac{dy}{d\zeta} \right)_{\zeta=0} = \beta \quad (6.1)$$

where  $y, p, q$  and  $r$  are continuous for  $\zeta \in [0, 1]$ . It is further assumed that eq. (6.1) admits a unique solution on  $[0, 1]$ , otherwise, a suitable transformation  $z = z(x)$  may be applied to change the domain of  $y, p, q$  and  $r$ .

**6.1. Case 1 : Coefficients of  $y$  and  $\frac{dy}{d\zeta}$  are constant** Taking  $p$  for  $P(x)$  and  $q$  for  $Q(x)$ , eq. (6.1) can be written as:

$$\frac{d^2 y}{d\zeta^2} + p \frac{dy}{d\zeta} + q(\zeta)y = r(\zeta), \quad y(0) = \alpha, \left( \frac{dy}{d\zeta} \right)_{\zeta=0} = \beta \quad (6.2)$$

Let  $R = (r_0, r_1, \dots, r_n)$  be a real column vector such that the function  $r(\zeta)$  can be approximated in terms of first  $n+1$  orthonormal Bernoulli polynomials as:

$$r(\zeta) = R \phi(\zeta). \quad (6.3)$$

Let  $C = (c_o, c_1, c_2, \dots, c_n)$  be a column vector of  $n + 1$  unknown quantities. Taking

$$\frac{d^2 y}{d\zeta^2} = C^T \phi(\zeta), \quad (6.4)$$

eq. (6.1) can be re-written as:

$$C^T \phi(\zeta) + p C^T \Theta \phi(\zeta) + q C^T \Theta^2 \phi(\zeta) = R^T \phi(\zeta) \quad (6.5)$$

which gives:

$$C^T = [I + p \Theta + q \Theta^2]^{-1} R^T \quad (6.6)$$

Substituting eq. (6.6) back into eq. (6.4), an approximation for  $y(\zeta)$  can be obtained as:

$$y(\zeta) = C^T \Theta^2 \phi(\zeta). \quad (6.7)$$

## 6.2. Case 2 : Coefficients of $y$ and $\frac{dy}{d\zeta}$ are functions of independent variable

Taking  $P(\zeta) = a^T \phi(\zeta)$  and  $Q(\zeta) = b^T \phi(\zeta)$  together with eqs. (6.3-6.4), eq. (6.1) can be written as:

$$C^T \phi(\zeta) + (a^T \phi(\zeta)) (C^T \Theta \phi(\zeta)) + (b^T \phi(\zeta)) (C^T \Theta^2 \phi(\zeta)) = R^T \phi(\zeta) \quad (6.8)$$

Because  $a^T \phi(\zeta)$  and  $b^T \phi(\zeta)$  in second and third terms, respectively on left side of eq. (6.8) are just the polynomials or degree  $2n$ , eq. (6.8) can be re-written as:

$$C^T \phi(\zeta) + C^T \Theta [\phi(\zeta) (a^T \phi(\zeta))] + C^T \Theta^2 [\phi(\zeta) (b^T \phi(\zeta))] = R^T \phi(\zeta) \quad (6.9)$$

Here,  $\phi(\zeta) (a^T \phi(\zeta))$  is a vector of type

$$\left( \phi_o(\zeta) \sum_{k=0}^{k=n} a_k \phi_k(\zeta), \phi_1(\zeta) \sum_{k=0}^{k=n} a_k \phi_k(\zeta), \dots, \phi_n(\zeta) \sum_{k=0}^{k=n} a_k \phi_k(\zeta) \right) \\ \equiv (\psi_o(\zeta), \psi_1(\zeta), \dots, \psi_n(\zeta)) = \psi(\zeta), \quad (\text{say!}) \quad (6.10)$$

In eq. (6.10), each  $\psi_k(\zeta)$  can be approximated as a linear combination of orthonormal polynomials in the form  $\psi_k(\zeta) = A_k^T \phi(\zeta)$ , where  $A_k^T$  are vectors of form  $1 \times (n + 1)$  for  $k = 1, 2, \dots, n$ . Therefore,  $\phi(\zeta) (a^T \phi(\zeta)) = \psi(\zeta) = A \phi(\zeta)$ , where  $A = (A_o^T, A_1^T, \dots, A_n^T)_{(n+1 \times 1)}$ . Similarly,  $\phi(\zeta) (b^T \phi(\zeta))$  can be approximated as  $B^T \phi(\zeta)$  for some vector  $B^T = (B_o, B_1, \dots, B_n)_{(n+1 \times 1)}$  such that  $B_k^T$  are real vectors of form  $1 \times (n + 1)$ . With these intermediate approximations, eq. (6.9) can be written as :

$$C^T \phi(\zeta) + C^T \Theta A \phi(\zeta) + C^T \Theta^2 B \phi(\zeta) = R^T \phi(\zeta) \quad (6.11)$$

From eq. (6.11), the required coefficient vector  $C$  is obtained as:

$$C^T = R^T (I + \Theta A + \Theta^2 B)^{-1}, \quad (6.12)$$

where  $I$  is identity matrix of order  $n$ . The expression for  $y(\zeta)$  is obtained as:

$$y(\zeta) = C^T \Theta^2 \phi(\zeta) \quad (6.13)$$

## 7. Numerical Examples

In order to discuss and establish the accuracy and efficacy of the present method, following examples have been taken.

**Example 1:** Let us consider the IVP

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = e^{-x}; \quad y(0) = \left(\frac{dy}{dx}\right)_{x=0} = 0 \quad (7.1)$$

which has exact solution  $y(x) = e^{-\frac{5}{2}x} \left( \cosh\left(\frac{\sqrt{13}}{2}x\right) + \frac{3}{\sqrt{13}} \sinh\left(\frac{\sqrt{13}}{2}x\right) \right) - e^{-x}$ .

Comparing eq.(7.1) with eq. (6.1) and taking  $m = 6$ , equations (6.3 - 6.6) yield

$$B^T = \begin{pmatrix} 0.08086, & -0.02459, & -0.00156, & -0.00240, \\ -0.001046, & -0.000425, & -0.00017, & 0 \end{pmatrix} \quad (7.2)$$

Using value of  $B^T$  in eq. (6.7), an approximate solution is obtained as:

$$y(x) \approx -0.00009x + 0.25057x^2 - 0.08655x^3 + 0.07415x^4 \\ - 0.10716x^5 + 0.0930x^6 - 0.03382x^7 \quad (7.3)$$

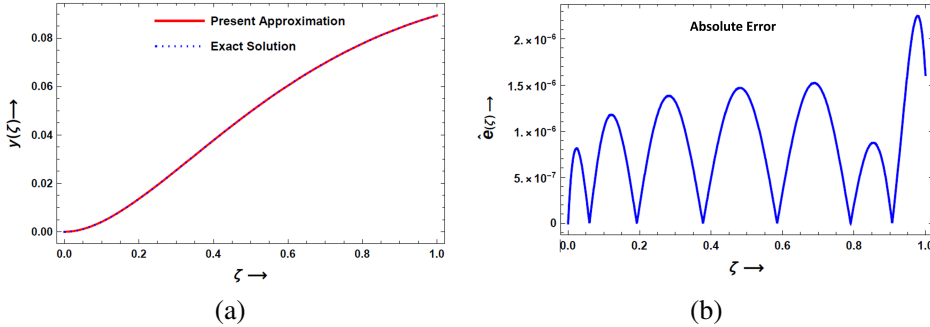


FIGURE 1. (a) Comparison of exact and present solution for example 1. (b) Absolute error between exact and approximate solutions of example 1

**Example 2:** Consider the IVP

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = \tan(x); \quad y(0) = \left(\frac{dy}{dx}\right)_{x=0} = 0 \quad (7.4)$$

which is linear in nature but its not easy to find an analytic solution for the same. We will compare the present solution of this IVP with the one generated by Mathematica.

Comparing eq.(7.4) with eq. (6.1) and taking  $m = 9$ , equations (6.3 - 6.6) yield:

$$B^T = \begin{pmatrix} 5.1220, 5.5181, 2.9304, 1.0668, 0.2958, \\ 0.0663, 0.0125, 0.0020, 0.0003, 0, 0 \end{pmatrix} \quad (7.5)$$

Using value of  $B^T$  in eq. (6.7), an approximate solution is obtained as:

$$y(x) \approx 0.0001x - 0.0025x^2 + 0.1942x^3 + 0.04799x^4 + 0.7521x^5 - 0.9599x^6 + 1.5043x^7 - 0.9351x^8 + 0.3669x^9 \quad (7.6)$$

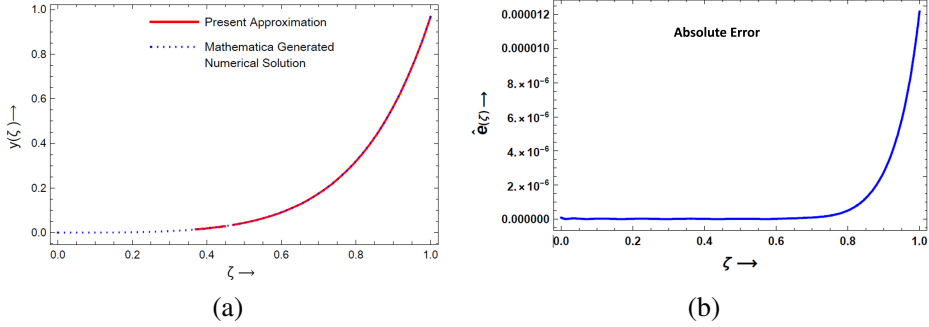


FIGURE 2. (a) Comparison of present approximation and *Mathematica* generated numerical solutions to example 2. (b) Absolute error between present approximation and *Mathematica* generated numerical solutions to example 2.

**Example 3:** Let us take the IVP

$$\frac{d^2y}{dx^2} + \tan(x)\frac{dy}{dx} + 2\cos^2(x)y = 2\cos^4(x); \quad y(0) = \left(\frac{dy}{dx}\right)_{x=0} = 0 \quad (7.7)$$

The exact solution of this IVP is  $y(x) = 2 - 2\cos(\sqrt{2}\sin^2 x) - \sin x$ .

Using the method discussed in section 6.2, coefficient vector  $C^T$  and approximate solution  $y(x)$  of example (7.7) is obtained for  $n = 6$  as:

$$C^T = (0.78730, 0.62821, 0.10352, -0.02660, 0.00101, 0.00136, 0.00011) \quad (7.8)$$

$$y(x) \approx -0.00025 + 0.00718x + 2.94625x^2 + 0.16696x^3 - 1.57951x^4 + 0.17065x^5 + 0.33315x^6 \quad (7.9)$$

## 8. Conclusion

In this work, a new method was presented and demonstrated to find a fast and approximate solution to linear initial value problems using orthogonal Bernoulli polynomials. The method includes the derivation of a set of  $n$  orthonormal polynomials and an operational matrix from the first  $n$  Bernoulli polynomials. The present method converts a given initial value problem into a system of algebraic equations with unknown coefficients, which are easily obtained with the help of the operational matrix. Finally, an approximate solution is obtained in form of a polynomial of degree  $n$ . The method has been demonstrated with three examples. The main features of this method can be summarized as:

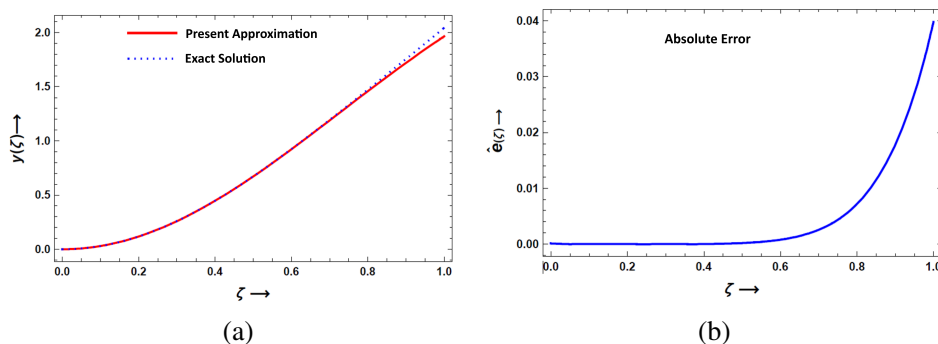


FIGURE 3. (a) Comparison of exact solution and present approximation to example 3. (b) Absolute error between exact solution and present approximation to example 3.

- the method is programmable.
- solution is obtained in form of a polynomial of degree  $n$  which can be easily used for various applications.
- error can be minimized up to the required accuracy because error decreases quickly with an increase of  $n$  (the degree of Bernoulli polynomials).
- error is negligible for simple IVPs with constant coefficients.

#### Author contributions:

*Conceptualisation:* J. P. Tripathi, U. P. Singh; *Software:* J. P. Tripathi, U. P. Singh; *Writing-Original Draft:* J. P. Tripathi, U. P. Singh, S. G. Rao

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