

ON CERTAIN CLASSES OF STARLIKE AND CONVEX FUNCTIONS INVOLVING POISSON DISTRIBUTION SERIES

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Abstract

In this paper, we obtain sufficient conditions for a convolution of analytic univalent functions and the Poisson distribution series to belong to the families of uniformly starlike functions and uniformly convex functions in the open unit disk. Further, we consider the properties of integral operator related to Poisson distribution series. Several corollaries and consequences of the main results are also considered.

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1. Introduction

Let \mathcal{A} denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Further, we denote by S the subclass of \mathcal{A} consisting of functions of the form (1.1) which are univalent in \mathbb{U} . Several well-known subclasses of S plays prominent role in geometric function theory such as S_{γ}^* and C_{γ} which are consisting of starlike and convex functions of order γ ($0 \leq \gamma < 1$), respectively; are usually characterized by the quantities $zf'(z)/f(z)$ or $(1 + zf''(z))/f'(z)$ lying in a given domain in the right half-plane, for more details, see ([15], [19]).

For given two functions $\varphi(z), \psi(z) \in \mathcal{A}$, the function $\varphi(z)$ is said to be *subordinate* to $\psi(z)$, (or $\psi(z)$ is *superordinate* to $\varphi(z)$), if there exists Schwarz function $w(z)$, analytic in \mathbb{U} such that $w(0) = 0$ and $|w(z)| < 1$, satisfying the following condition

$$\varphi(z) = \psi(w(z)), \quad (z \in \mathbb{U}).$$

Moreover, if the function $\psi(z)$ is univalent in \mathbb{U} , then we have the following equivalence(see [11])

$$\varphi(z) \prec \psi(z) \Leftrightarrow \varphi(0) \prec \psi(0) \text{ and } \varphi(\mathbb{U}) \subset \psi(\mathbb{U}).$$

We denote this subordination as $\varphi \prec \psi$. A function $p(z)$ is said to be in the class $\mathcal{P}[A; B]$; if it is analytic in \mathbb{U} with $p(0) = 1$ and

$$p(z) = z + \sum_{n=2}^{\infty} p_n z^n \in \mathcal{P}[A; B] \Leftrightarrow p(z) \prec \frac{1+Az}{1+Bz}, \quad -1 \leq A < B \leq 1.$$

This class was presented by Janowski [7] and explored by a few creators, for example see ([13], [14], [21]). In particular, if $A = 1$ and $B = -1$, we obtain the class $p(z) \prec (1+z)/(1-z)$ of functions with positive real part (see [4]). Kanas and Wisniowska ([8], [9]) presented and examined the class k -ST of k -starlike functions and the relating class k -UCV of k -uniformly convex functions. There were characterized subject to the conic region Ω_k , $k \geq 0$, as

$$\Omega_k = \{u + iv : u > k \sqrt{(u-1)^2 + v^2}\}.$$

For fixed k , Ω_k represents the conic region bounded, successively, by the imaginary axis ($k = 0$), the right branch of hyperbola ($0 < k < 1$), a parabola ($k = 1$) and an ellipse ($k > 1$). The functions which play the role of extremal functions for these conic regions are given as

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} & k = 0, \\ 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 & k = 1, \\ 1 + \frac{2}{1-k^2} \sinh^2 \left\{ \frac{2}{\pi} (\arccos k) \arctan h \sqrt{z} \right\} & 0 < k < 1, \\ 1 + \frac{1}{k^2-1} \sin \left(\frac{\pi}{2R(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{dx}{\sqrt{1-x^2} \sqrt{1-t^2 x^2}} dx \right) + \frac{t^2}{k^2-1} & k > 1, \end{cases} \quad (1.2)$$

where $u(z) = (z - \sqrt{t})/(1 - \sqrt{tz})$, $z \in \mathbb{U}$, and $t \in (0, 1)$ is chosen such that $k = \cosh(\pi R'(t)/(4R(t)))$. Here $R(t)$ is Legendre's complete elliptic integral of first kind and $R'(t) = R(\sqrt{1-t^2})$ and $R'(t)$ is the complementary integral of $R(t)$, see ([6], [8], [9]). If $p_k(z) = 1 + Q_1(k)z + Q_2(k)z^2 + Q_3(k)z^3 + \dots, z \in \mathbb{U}$, then it is shown in [8] that for (1.2) one can have

$$Q_1 := Q_1(k) = \begin{cases} \frac{8(\arccos k)^2}{\pi^2(1-k^2)} & 0 \leq k < 0, \\ \frac{8}{\pi^2} & k = 1, \\ \frac{\pi^2}{4(1-k^2)\sqrt{t}(1+t)R^2(t)} & k > 1. \end{cases} \quad (1.3)$$

In 2011, Noor and Malik [14] use the concept of Janowski functions and conic domain to define the class k - $\mathcal{P}[A, B]$ as follows

DEFINITION 1.1. A function $p(z)$ is said to be in the class $k\mathcal{P}[A, B]$, if and only if,

$$p(z) < \frac{(A+1)p_k(z) - (A-1)}{(B+1)p_k(z) - (B-1)}, \quad k \geq 0, \quad (1.4)$$

where $p_k(z)$ is defined in (1.2) and $-1 \leq B < A \leq 1$. Geometrically, the function $p \in k\mathcal{P}[A, B]$ takes the values from the domain $\Omega_k[A, B]$, $-1 \leq B < A \leq 1$, $k \geq 0$ which is defined as

$$\Omega_k[A, B] = \left\{ w : \operatorname{Re} \left(\frac{(B-1)w - (A-1)}{(B+1)w - (A+1)} \right) > k \left| \frac{(B-1)w - (A-1)}{(B+1)w - (A+1)} - 1 \right| \right\},$$

or equivalently $\Omega_k[A, B]$ is a set of numbers w such that

$$\begin{aligned} & [(B^2 - 1)(u^2 + v^2) - 2(AB - 1)u + (A^2 - 1)]^2 \\ & > k^2 \left[\left\{ -2(B+1)(u^2 + v^2) + 2(A+B+2)u - 2(A+1) \right\}^2 + 4(A-B)^2 v^2 \right]. \end{aligned}$$

This domain represents the conic type domain, for details see ([8], [9], [14]).

DEFINITION 1.2. [14] A function $f \in \mathcal{A}$ is said to be in the class $k - UCV[A, B]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if,

$$\operatorname{Re} \left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)} \right) > k \left| \frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)} - 1 \right|, \quad (1.5)$$

or equivalently $\frac{(zf'(z))'}{f'(z)} \in k\mathcal{P}[A, B]$.

DEFINITION 1.3. [14] A function $f \in \mathcal{A}$ is said to be in the class $k - ST[A, B]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if,

$$\operatorname{Re} \left(\frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)} \right) > k \left| \frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)} - 1 \right|, \quad (1.6)$$

or equivalently $\frac{(zf'(z))'}{f'(z)} \in k\mathcal{P}[A, B]$.

It can be easily seen that

$$f \in k - UCV[A, B] \Leftrightarrow zf' \in k - ST[A, B].$$

REMARK 1.4. From among the many choices of k , A , B which would provide the following well-known subclasses:

- (1) $k - \mathcal{P}[1, -1] = \mathcal{P}(p_k)$, the class introduced by Kanas and Wisniowska [8].
- (2) $0 - \mathcal{P}[A, B] = \mathcal{P}[A, B]$, the class introduced by Janowski [7].
- (3) $k - ST[1, -1] = k - ST$, $k - UCV[1, -1] = k - UCV$, the classes, introduced by Kanas and Wisniowska ([8], [9]).
- (4) $k - ST[1 - 2\alpha, -1] = SD(k, \alpha)$, $k - UCV[1 - 2\alpha, -1] = kD(k, \alpha)$, the classes, introduced by Shams et al. [22].
- (5) $0 - ST[A, B] = S^*[A, B]$, $0 - UCV[A, B] = C[A, B]$, the classes introduced, by Janowski [7].

THEOREM 1.5. [14] A function $f \in \mathcal{A}$ of the form (1.1) in the class $k - ST[A, B]$, if it satisfies the condition

$$\sum_{n=2}^{\infty} \left[2(k+1)(n-1) + |n(B+1) - (A+1)| \right] |a_n| < |B-A|, \quad (1.7)$$

where $k \geq 0, -1 \leq B < A \leq 1$.

THEOREM 1.6. [14] A function $f \in \mathcal{A}$ of the form (1.1) in the class $k - UCV[A, B]$, if it satisfies the condition

$$\sum_{n=2}^{\infty} n \left[2(k+1)(n-1) + |n(B+1) - (A+1)| \right] |a_n| < |B-A|, \quad (1.8)$$

where $k \geq 0, -1 \leq B < A \leq 1$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(C, D)$, ($\tau \in \mathbb{C} \setminus \{0\}, -1 \leq D < C \leq 1$), if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{\tau(C-D) - D(f'(z) - 1)} \right| < 1. \quad (1.9)$$

This class was introduced by Dixit and Pal [2].

LEMMA 1.7. [2] If $f \in \mathcal{R}^\tau(C, D)$ is of the form (1.1) then

$$|a_n| = \frac{|\tau|(C-D)}{n}, \quad n \in N \setminus \{1\}. \quad (1.10)$$

The study of distribution series (theory) plays an important role in the geometric function theory and its related fields. Porwal [16] gives a beautiful application of Poisson distribution series on univalent functions and establish a co-relation between Geometric Function Theory and Probability distribution. Recently, the area of Poisson distribution has attracted the serious attention of researchers. Many researchers have introduced and investigated several interesting distribution series such as Generalized distribution series [5], Pascal distribution series [16], confluent hypergeometric distribution series [18], Binomial distribution series ([12], [23]), Borel distribution series [24], (see also [[1], [10], [20]]) and obtain the analogues results for these distribution series on certain subclasses of univalent functions.

Recently, Porwal [16] introduced a power series whose coefficients are probabilities of Poisson distribution:

$$\psi(m, z) = z + \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{(n-1)!} z^n, \quad z \in \mathbb{U}, \quad (1.11)$$

We note that, by ratio test, the radius of convergence of the above series is infinity. Corresponding to the series $\psi(m, z)$ using the Hadamard product for $f \in \mathcal{A}$, Porwal

and Kumar [17] introduced the linear operator $\mathcal{I}(m)f : \mathcal{A} \rightarrow \mathcal{A}$ defined by using the Hadamard product as

$$\mathcal{I}(m)f = \psi(m, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{(n-1)!} a_n z^n, \quad z \in \mathbb{U}. \quad (1.12)$$

Also, we define the function

$$\begin{aligned} \mathcal{N}(m, v; z) &= (1-v)\psi(m; z) + vz(\psi(m; z))' \\ &= z + \sum_{n=2}^{\infty} [1+v(n-1)] \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad (v \geq 0). \end{aligned} \quad (1.13)$$

In the present investigation, inspired by the works of ([16], [17]), we find necessary and sufficient conditions for the power series with coefficients are the probabilities of Poisson distribution (1.11) and also we obtain inclusion relations for classes $k - ST[A, B]$ and $k - UCV[A, B]$ with $\mathcal{R}^r(C, D)$.

2. Main Results

THEOREM 2.1. *If $m > 0, -1 \leq B < A \leq 1$, then $\psi(m, z) \in k - ST[A, B]$, if and only if*

$$(2k + B + 3)m + (B - A)(1 - e^{-m}) \leq |B - A|. \quad (2.1)$$

PROOF. Since

$$\psi(m, z) = z + \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{(n-1)!} z^n, \quad z \in \mathbb{U}.$$

According to Theorem 1.5, it is enough to show that

$$\sum_{n=2}^{\infty} \left[2(k+1)(n-1) + |n(B+1) - (A+1)| \right] \left| \frac{m^{n-1}}{(n-1)!} \right| e^{-m} < |B - A|. \quad (2.2)$$

It follows from (2.2) that

$$\begin{aligned} &\sum_{n=2}^{\infty} \left[2(k+1)(n-1) + |n(B+1) - (A+1)| \right] \left| \frac{m^{n-1}}{(n-1)!} \right| e^{-m} \\ &= e^{-m} \left[\sum_{n=2}^{\infty} [(n-1)(2k+B+3) + B - A] \right] \left| \frac{m^{n-1}}{(n-1)!} \right| \\ &= e^{-m} \left[(2k+B+3) \sum_{n=2}^{\infty} (n-1) \frac{m^{n-1}}{(n-1)!} + (B-A) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} \left[(2k+B+3) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + (B-A) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} [(2k+B+3)me^m + (B-A)(e^m - 1)] \\ &= (2k+B+3)m + (B-A)(1 - e^{-m}). \end{aligned}$$

But this last expression is bounded by $B - A$ if and only if (2.1) holds.

This completes the proof. \square

COROLLARY 2.2. *If $m > 0$, then $\psi(m, z) \in SD(k, \alpha)$, if and only if*

$$(k+1)m + (\alpha - 1)(1 - e^{-m}) \leq \alpha - 1.$$

COROLLARY 2.3. *If $m > 0, -1 \leq B < A \leq 1$, then $\psi(m, z) \in S^*[A, B]$, if and only if*

$$(B+3)m + (B-A)(1 - e^{-m}) \leq |B - A|.$$

THEOREM 2.4. *If $m > 0, -1 \leq B < A \leq 1$, then $\psi(m, z) \in k - UCV[A, B]$, if and only if*

$$(2k + B + 3)m^2 + (4k + 3B - A + 6)m + (B - A)(1 - e^{-m}) \leq |B - A|. \quad (2.3)$$

PROOF. Since

$$\psi(m, z) = z + \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{(n-1)!} z^n, \quad z \in \mathbb{U}.$$

According to Theorem 1.6, it is enough to show that

$$\sum_{n=2}^{\infty} n \left[2(k+1)(n-1) + |n(B+1) - (A+1)| \right] \left| \frac{m^{n-1}}{(n-1)!} \right| e^{-m} < |B - A|. \quad (2.4)$$

It follows from (2.4) that

$$\begin{aligned} & \sum_{n=2}^{\infty} n \left[2(k+1)(n-1) + |n(B+1) - (A+1)| \right] \left| \frac{m^{n-1}}{(n-1)!} \right| e^{-m} \\ &= e^{-m} \left[\sum_{n=2}^{\infty} n[(n-1)(2k+B+3) + B - A] \right] \left| \frac{m^{n-1}}{(n-1)!} \right| \\ &= e^{-m} \left[(2k+B+3) \sum_{n=2}^{\infty} (n-1)(n-2) \frac{m^{n-1}}{(n-1)!} \right. \\ &\quad \left. + (4k+3B-A+6) \sum_{n=2}^{\infty} (n-1) \frac{m^{n-1}}{(n-1)!} + (B-A) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} \left[(2k+B+3) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-3)!} + (4k+3B-A+6) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} \right. \\ &\quad \left. + (B-A) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} [(2k+B+3)m^2 e^m + (4k+3B-A+6)m e^m + (B-A)(e^m - 1)] \\ &= (2k+B+3)m^2 + (4k+3B-A+6)m + (B-A)(1 - e^{-m}). \end{aligned}$$

But this last expression is bounded by $B - A$ if and only if (2.3) holds.

This completes the proof. \square

COROLLARY 2.5. *If $m > 0$, then $\psi(m, z) \in kD(k, \alpha)$, if and only if*

$$(k+1)m^2 + (2k+\alpha+1)m + (\alpha-1)(1 - e^{-m}) \leq \alpha - 1.$$

COROLLARY 2.6. *If $m > 0, -1 \leq B < A \leq 1$, then $\psi(m, z) \in C[A, B]$, if and only if*

$$(B + 3)m^2 + (3B - A + 6)m + (B - A)(1 - e^{-m}) \leq |B - A|.$$

THEOREM 2.7. *If $m > 0, -1 \leq B < A \leq 1$, then $N(m, v; z) \in k-ST[A, B]$, if and only if*

$$\begin{aligned} v(2k + B + 3)m^2 + (2k(3v + 1) + (B + 3)(v + 1) - vA)m \\ + (4kv + B - A)(1 - e^{-m}) \leq |B - A|. \end{aligned} \quad (2.5)$$

PROOF. Since

$$\psi(m, z) = z + \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{(n-1)!} z^n, \quad z \in \mathbb{U}.$$

According to Theorem 1.5, it is enough to show that

$$\sum_{n=2}^{\infty} [2(k+1)(n-1) + |n(B+1) - (A+1)|] (1 + v(n-1)) \left| \frac{m^{n-1}}{(n-1)!} \right| e^{-m} < |B - A|. \quad (2.6)$$

It follows from (2.6) that

$$\begin{aligned} & \sum_{n=2}^{\infty} [2(k+1)(n-1) + |n(B+1) - (A+1)|] (1 + v(n-1)) \left| \frac{m^{n-1}}{(n-1)!} \right| e^{-m} \\ &= e^{-m} \left[\sum_{n=2}^{\infty} [(n-1)(2k+B+3) + B-A] (1 + v(n-1)) \left| \frac{m^{n-1}}{(n-1)!} \right| \right] \\ &= e^{-m} \left[v(2k+B+3) \sum_{n=2}^{\infty} (n-1)(n-2) \frac{m^{n-1}}{(n-1)!} \right. \\ &\quad \left. + (2k(3v+1) + B(v+1) - vA + 3(v+1)) \sum_{n=2}^{\infty} (n-1) \frac{m^{n-1}}{(n-1)!} \right. \\ &\quad \left. + (4kv + B - A) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} \left[v(2k+B+3) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-3)!} \right. \\ &\quad \left. + (2k(3v+1) + B(v+1) - vA + 3(v+1)) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} \right. \\ &\quad \left. + (4kv + B - A) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} [v(2k+B+3)m^2 e^m + (2k(3v+1) + B(v+1) - vA + 3(v+1))m e^m \\ &\quad + (4kv + B - A)(e^m - 1)] \\ &= v(2k+B+3)m^2 + (2k(3v+1) + (B+3)(v+1) - vA)m \\ &\quad + (4kv + B - A)(1 - e^{-m}). \end{aligned}$$

But this last expression is bounded by $B - A$ if and only if (2.5) holds.

This completes the proof. \square

COROLLARY 2.8. *If $m > 0$, then $N(m, \nu; z) \in SD[k, \alpha]$, if and only if*

$$\begin{aligned} 2\nu(k+1)m^2 + (2k(3\nu+1) + \nu(2\alpha+1) + 2)m \\ + 2(2k\nu+\alpha-1)(1-e^{-m}) \leq 2(1-\alpha). \end{aligned} \quad (2.7)$$

COROLLARY 2.9. *If $m > 0, -1 \leq B < A \leq 1$, then $N(m, \nu; z) \in S^*[A, B]$, if and only if*

$$\nu(B+3)m^2 + ((B+3)(\nu+1) - \nu A)m + (B-A)(1-e^{-m}) \leq |B-A|. \quad (2.8)$$

THEOREM 2.10. *If $m > 0, -1 \leq B < A \leq 1$, then $N(m, \nu; z) \in k-UVC[A, B]$, if and only if*

$$\begin{aligned} \nu(2k+B+3)m^3 + (2k(6\nu+1) + (\nu+1)A + 5B\nu + 3(6\nu+1))m^2 \\ + (20k+3A+7B+30)m + (4k+A+B+6)(1-e^{-m}) \leq |B-A|. \end{aligned} \quad (2.9)$$

PROOF. The proof of Theorem 2.10 is similar to the proof of Theorem 2.7, therefore we omit the details involved. \square

COROLLARY 2.11. *If $m > 0, -1 \leq B < A \leq 1$, then $N(m, \nu; z) \in KD[k, \alpha]$, if and only if*

$$\begin{aligned} \nu(k+1)m^3 + (k(6\nu+1) - \alpha(\nu+1) + 7\nu+2)m^2 \\ + (10k-3\alpha-10)m + (k-\alpha+3)(1-e^{-m}) \leq (\alpha-1). \end{aligned} \quad (2.10)$$

COROLLARY 2.12. *If $m > 0, -1 \leq B < A \leq 1$, then $N(m, \nu; z) \in C[A, B]$, if and only if*

$$\begin{aligned} \nu(B+3)m^3 + ((\nu+1)A + 5B\nu + 3(6\nu+1))m^2 \\ + (3A+7B+30)m + (A+B+6)(1-e^{-m}) \leq |B-A|. \end{aligned} \quad (2.11)$$

3. Inclusion Properties

THEOREM 3.1. *Let $m > 0, -1 \leq B < A \leq 1$ and $-1 \leq D < C \leq 1$. If $f \in \mathcal{R}^\tau(C, D)$, then $I(m)f \in K-ST[A, B]$, if and only if*

$$(C-D)|\tau| \left[(2k+B+3)(1-e^{-m}) + \frac{(B-A)}{m}(1-e^{-m}-me^{-m}) \right] \leq |B-A|. \quad (3.1)$$

PROOF. According to Theorem 1.5, it is enough to show that

$$\sum_{n=2}^{\infty} \left[2(k+1)(n-1) + |n(B+1) - (A+1)| \right] \left| \frac{m^{n-1}}{(n-1)!} \frac{(C-D)\tau}{n} e^{-m} \right| < |B-A|. \quad (3.2)$$

It follows from (3.2) that

$$\begin{aligned}
& \sum_{n=2}^{\infty} [2(k+1)(n-1) + |n(B+1) - (A+1)|] \left| \frac{m^{n-1}(C-D)\tau}{n!} \right| e^{-m} \\
&= e^{-m}(C-D)|\tau| \left[\sum_{n=2}^{\infty} [2(k+1)(n-1) + |n(B+1) - (A+1)|] \right] \left| \frac{m^{n-1}}{n!} \right| \\
&= e^{-m}(C-D)|\tau| \left[(2k+B+3) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} + \frac{(B-A)}{m} \sum_{n=2}^{\infty} \frac{m^n}{n!} \right] \\
&= e^{-m}(C-D)|\tau| \left[(2k+B+3)(e^m - 1) + \frac{(B-A)}{m}(e^m - 1 - m) \right] \\
&= (C-D)|\tau| \left[(2k+B+3)(1 - e^{-m}) + \frac{(B-A)}{m}(1 - e^{-m} - me^{-m}) \right].
\end{aligned}$$

But this last expression is bounded by $B - A$ if and only if (3.1) holds.

This completes the proof. \square

THEOREM 3.2. *Let $m > 0$, $-1 \leq B < A \leq 1$ and $-1 \leq D < C \leq 1$. If $f \in \mathcal{R}^r(C, D)$, then $\mathcal{I}(m)f \in K - UCV[A, B]$, if and only if*

$$(C-D)|\tau|[(2k+B+3)m + (B-A)(1 - e^{-m})] \leq |B - A|. \quad (3.3)$$

PROOF. Since

$$\psi(m, z) = z + \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{(n-1)!} z^n, \quad z \in \mathbb{U}.$$

According to Theorem 1.6, it is enough to show that

$$\sum_{n=2}^{\infty} n [2(k+1)(n-1) + |n(B+1) - (A+1)|] \left| \frac{m^{n-1}}{(n-1)!} \frac{(C-D)\tau}{n} \right| e^{-m} < |B - A|. \quad (3.4)$$

It follows from (2.4) that

$$\begin{aligned}
& \sum_{n=2}^{\infty} n [2(k+1)(n-1) + |n(B+1) - (A+1)|] \left| \frac{m^{n-1}}{(n-1)!} \frac{(C-D)\tau}{n} \right| e^{-m} \\
&= e^{-m}(C-D)|\tau| \left[\sum_{n=2}^{\infty} [(n-1)(2k+B+3) + B - A] \right] \left| \frac{m^{n-1}}{(n-1)!} \right| \\
&= e^{-m}(C-D)|\tau| \left[(2k+B+3) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + (B-A) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\
&= e^{-m}(C-D)|\tau| \left[(2k+B+3)me^m + (B-A)(e^m - 1) \right] \\
&= e^{-m}(C-D)|\tau|[(2k+B+3)me^m + (B-A)(e^m - 1)] \\
&= (C-D)|\tau|[(2k+B+3)m + (B-A)(1 - e^{-m})].
\end{aligned}$$

But this last expression is bounded by $B - A$ if and only if (3.3) holds.

This completes the proof. \square

4. An Integral Operator

In the following theorem, we get similar result in connection with a particular integral operator $I(m, z)$ as follows:

$$I(m, z) = \int_0^z \frac{\psi(m, t)}{t} dt. \quad (4.1)$$

THEOREM 4.1. *If $m > 0$, then $I(m, z)$ defined by (4.1) is in $K - UCV[A, B]$, if and only if*

$$(2k + B + 3)m + (B - A)(1 - e^{-m}) \leq |B - A|. \quad (4.2)$$

PROOF. It is easy to see that

$$I(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{n!} e^{-m} z^n.$$

In view of Theorem 1.6 it is sufficient to show that

$$\sum_{n=2}^{\infty} n \left[2(k+1)(n-1) + |n(B+1) - (A+1)| \right] \left| \frac{m^{n-1}}{n!} e^{-m} \right| < |B - A|. \quad (4.3)$$

It follows from (4.3) that

$$\begin{aligned} & \sum_{n=2}^{\infty} \left[2(k+1)(n-1) + |n(B+1) - (A+1)| \right] \left| \frac{m^{n-1}}{(n-1)!} e^{-m} \right| \\ &= e^{-m} \left[\sum_{n=2}^{\infty} [(n-1)(2k+B+3) + B - A] \right] \left| \frac{m^{n-1}}{(n-1)!} \right| \\ &= e^{-m} \left[(2k+B+3) \sum_{n=2}^{\infty} (n-1) \frac{m^{n-1}}{(n-1)!} + (B-A) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} \left[(2k+B+3) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + (B-A) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} [(2k+B+3)me^m + (B-A)(e^m - 1)] \\ &= (2k+B+3)m + (B-A)(1 - e^{-m}). \end{aligned}$$

But this last expression is bounded by $B - A$ if and only if (4.2) holds.

This completes the proof. \square

THEOREM 4.2. *If $m > 0$, then $I(m, z)$ defined by (4.1) is in $K - ST[A, B]$, if and only if*

$$(2k + B + 3)(1 - e^{-m}) + \frac{(B - A)}{m}(1 - e^{-m} - me^{-m}) \leq |B - A|. \quad (4.4)$$

PROOF. The proof of Theorem 4.2 is similar to the proof of Theorem 4.1, therefore we omit the details involved. \square

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