

GENERALIZED p -FUSION FRAME IN SEPARABLE BANACH SPACE

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Abstract

Concepts of g -fusion frame and gf -Riesz basis in a Hilbert to a Banach space is being presented. Some properties of g -fusion frame and gf -Riesz basis in Banach space have been developed. We discuss perturbation results of g -fusion frame in a Banach space. Finally, we construct g - p -fusion frames in Cartesian product of Banach spaces and tensor product of Banach spaces.

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1. Introduction and Preliminaries

In recent times, several generalization of frame for separable Hilbert space have been introduced. Some of them are K -frame [5], g -frame [12], fusion frame [3] and so on. The combination of g -frame and fusion frame is known as generalized fusion frame or g -fusion frame. Sadri et al. [10] presented g -fusion frame to generalize the theory of fusion frame and g -frame. These frames were further studied by P. Ghosh and T. K. Samanta in [7–9].

A. Aldroubi et al. [1] introduced p -frame in a Banach space and discussed some of its properties. Chistensen and stoeva [4] also developed p -frame in separable Banach space. M. R. Abdollahpour et al. [2] introduced the p g -frames in Banach spaces. The generalization of the g -frame and g -Riesz Basis in a complex Hilbert space to a complex Banach space was also studied by Xiang-Chun Xio et al. [11].

In this paper, we generalize the notion of g -fusion frame in a Hilbert space to a Banach space and establish some of its properties. Generalized Riesz basis in Banach space is also discussed. The relation between g - p -fusion frame and q - gf -Riesz basis is obtained. We describe some perturbation results of g - p -fusion frame in Banach space. At the end, we present g - p -fusion frame in tensor product of Banach spaces.

Throughout this paper, X is considered to be a separable Banach space over the field \mathbb{K} (\mathbb{R} or \mathbb{C}) and X^* , its dual space. I, J denotes the subset of natural numbers \mathbb{N} . $\{X_i\}_{i \in I}$ is a sequence of Banach spaces and $\{V_i\}_{i \in I}$ is a collection of closed subspaces of X . $\mathcal{B}(X, X_i)$ are the collection of all bounded linear operators from X to X_i and in particular, $\mathcal{B}(X)$ denotes the space of all bounded linear operators

on X . It is assumed that $p \in (1, \infty)$ and when p and q are used in a same assertion, they satisfy the relation $1/p + 1/q = 1$.

Now, we recall some necessary definitions and theorems.

THEOREM 1.1. [6] *If $U : X \rightarrow Y$ is a bounded operator from a Banach space X into a Banach space Y then its adjoint $U^* : Y^* \rightarrow X^*$ is surjective if and only if U has a bounded inverse on \mathcal{R}_U (range of U).*

DEFINITION 1.2. [4] Let $1 < p < \infty$. A countable family $\{g_i\}_{i \in I} \subset X^*$ is said to be a p -frame for X if there exist constants $0 < A \leq B < \infty$ such that

$$A \|f\|_X \leq \left(\sum_{i \in I} |g_i(f)|^p \right)^{1/p} \leq B \|f\|_X \quad \forall f \in X.$$

DEFINITION 1.3. [11] A sequence $\{\Lambda_i \in \mathcal{B}(X, X_i) : i \in I\}$ is called a generalized p -frame or g - p -frame for X with respect to $\{X_i\}_{i \in I}$ if there exist two positive constants A and B such that

$$A \|f\|_X \leq \left(\sum_{i \in I} \|\Lambda_i(f)\|^p \right)^{1/p} \leq B \|f\|_X \quad \forall f \in X.$$

A and B are called the lower and upper frame bounds, respectively.

DEFINITION 1.4. [11] Define the linear space

$$l^p(\{X_i\}_{i \in I}) = \left\{ \{f_i\}_{i \in I} : f_i \in X_i, \sum_{i \in I} \|f_i\|^p < \infty \right\}.$$

Then it is a complex Banach space with respect to the norm is defined by

$$\|\{f_i\}_{i \in I}\| = \left(\sum_{i \in I} \|f_i\|^p \right)^{1/p}$$

LEMMA 1.5. [11] *Let $p > 1, q > 1$ be such that $1/p + 1/q = 1$. Then the adjoint space of $l^p(\{X_i\}_{i \in I})$ is $l^q(\{X_i^*\}_{i \in I})$, where X_i^* is the adjoint space of X_i for $i \in I$.*

DEFINITION 1.6. [2, 11] Let $\{\Lambda_i \in \mathcal{B}(X, X_i) : i \in I\}$ be a generalized p -frame or g - p -frame for X . Then the operator defined by

$$U : X \rightarrow l^p(\{X_i\}_{i \in I}), \quad Uf = \{\Lambda_i(f)\}_{i \in I} \quad \forall f \in X,$$

is called the analysis operator and the operator given by

$$T : l^q(\{X_i^*\}_{i \in I}) \rightarrow X^*$$

$$T(\{g_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^* g_i \quad \forall \{g_i\}_{i \in I} \in l^q(\{X_i^*\}_{i \in I})$$

is called synthesis operator.

DEFINITION 1.7. [10] Let $\{v_i\}_{i \in I}$ be a collection of positive weights and $\{H_i\}_{i \in I}$ be a collections of Hilbert spaces and $\{V_i\}_{i \in I}$ be a family of closed subspaces of a Hilbert space H . Then the family $\Lambda = \{(V_i, \Lambda_i, v_i)\}_{i \in I}$ is called a generalized fusion frame or a g-fusion frame for H respect to $\{H_i\}_{i \in I}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \sum_{i \in I} v_i^2 \|\Lambda_i P'_{V_i}(f)\|^2 \leq B \|f\|^2 \quad \forall f \in H, \quad (1.1)$$

where P'_{V_i} is the orthogonal projection of H onto V_i . The constants A and B are called the lower and upper bounds of g-fusion frame, respectively. If Λ satisfies the right inequality of (1.1), it is called a g-fusion Bessel sequence with bound B in H .

2. g-p-fusion frame and it's properties

In this section, we develop the generalized fusion frame and generalized Riesz basis for Banach space.

DEFINITION 2.1. Let $p > 1$ and $\{v_i\}_{i \in I}$ be a collection of positive weights i.e., $v_i > 0$. Let $\Lambda_i \in \mathcal{B}(X, X_i)$ and $\{P_{V_i}\}$ be non-trivial linear projections of X onto V_i such that $P_{V_i}(X) = V_i$, for each $i \in I$. Then the family $\Lambda = \{(V_i, \Lambda_i, v_i)\}_{i \in I}$ is called a generalized p -fusion frame or a g-p-fusion frame for X with respect to $\{X_i\}_{i \in I}$ if there exist $0 < A \leq B < \infty$ such that

$$A \|f\| \leq \left(\sum_{i \in I} v_i^p \|\Lambda_i P_{V_i}(f)\|^p \right)^{1/p} \leq B \|f\| \quad \forall f \in X. \quad (2.1)$$

The constants A and B are called the lower and upper bounds of g-p-fusion frame, respectively. If $A = B$ then Λ is called tight g-p-fusion frame and if $A = B = 1$ then we say Λ is a Parseval g-p-fusion frame. If Λ satisfies only the right inequality of (2.1), it is called a g-p-fusion Bessel sequence with bound B in X .

Suppose that $\Lambda = \{(V_i, \Lambda_i, v_i)\}_{i \in I}$ is a tight g-p-fusion frame for X with bound A . Then for all $f \in X$, we have

$$\begin{aligned} & \left(\sum_{i \in I} v_i^p \|\Lambda_i P_{V_i}(f)\|^p \right)^{1/p} = A \|f\| \\ \Rightarrow & \left(\sum_{i \in I} v_i^p \|A^{-1} \Lambda_i P_{V_i}(f)\|^p \right)^{1/p} = \|f\|. \end{aligned}$$

This verify that $\{(V_i, A^{-1} \Lambda_i, v_i)\}_{i \in I}$ is a Parseval g-p-fusion frame for X .

THEOREM 2.2. Let Λ be a g-p-fusion frame for X with respect to $\{X_i\}_{i \in I}$ having bounds A, B . Suppose $U \in \mathcal{B}(X)$ be an invertible operator on X . Then $\Gamma = \{(U V_i, \Lambda_i P_{V_i} U, v_i)\}_{i \in I}$ is a g-p-fusion frame for X , provided $P_{V_i} U P_{U V_i} = P_{V_i} U$, for $i \in I$.

PROOF. For each $f \in X$, we have

$$\begin{aligned} \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i} U P_{U V_i}(f) \right\|^p \right)^{1/p} &= \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i} U(f) \right\|^p \right)^{1/p} \\ &\leq B \|Uf\| \leq B \|U\| \|f\| \quad [\text{since } \Lambda \text{ is a } g\text{-}p\text{-fusion frame}]. \end{aligned}$$

On the other hand

$$\begin{aligned} \left(\sum_{i \in I} v_i^p \left\| \Lambda_j P_{V_i} U P_{U V_i}(f) \right\|^p \right)^{1/p} &= \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i} U(f) \right\|^p \right)^{1/p} \\ &\geq A \|Uf\| \geq A \|U^{-1}\|^{-1} \|f\| \quad [\text{since } U \text{ is invertible}]. \end{aligned}$$

Hence, Γ is a g - p -fusion frame for X with bounds $B \|U\|$ and $A \|U^{-1}\|^{-1}$. \square

THEOREM 2.3. *Let Λ be a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$ having bounds A, B and $U : X \rightarrow X$ be a bounded linear operator such that for each $i \in I$, $P_{V_i} U P_{U V_i} = P_{V_i} U$. Then the family $\Gamma = \{(U V_i, \Lambda_i P_{V_i} U, v_i)\}_{i \in I}$ is a g - p -fusion frame for X if and only if U is bounded below.*

PROOF. Let Γ be a g - p -fusion frame for X with bounds C and D . Then

$$\begin{aligned} C \|f\| &\leq \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i} U P_{U V_i}(f) \right\|^p \right)^{1/p} \leq D \|f\| \quad \forall f \in X. \\ \Rightarrow C \|f\| &\leq \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i} U(f) \right\|^p \right)^{1/p} \leq D \|f\| \end{aligned} \quad (2.2)$$

Since Λ is a g - p -fusion frame with bounds A and B , in (2.1), replacing f by Uf , we get

$$A \|Uf\| \leq \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i} U(f) \right\|^p \right)^{1/p} \leq B \|Uf\| \quad \forall f \in X. \quad (2.3)$$

Now, from (2.2) and (2.3), for each $f \in X$, we can write

$$C \|f\| \leq B \|Uf\| \Rightarrow \|Uf\| \geq \frac{C}{B} \|f\|.$$

This shows that U is bounded below.

Conversely, suppose that there exists $M > 0$ such that $\|Uf\| \geq M \|f\|$. Now, for each $f \in X$, we have

$$\begin{aligned} \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i} U P_{U V_i}(f) \right\|^p \right)^{1/p} &= \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i} U(f) \right\|^p \right)^{1/p} \\ &\geq A \|Uf\| \geq A M \|f\|. \end{aligned}$$

According to the proof of the Theorem 2.2, the upper frame condition is also satisfied. This completes the proof. \square

We now give a characterization of a g - p -fusion Bessel sequence in X .

THEOREM 2.4. *The family Λ is a g - p -fusion Bessel sequence in X with respect to $\{X_i\}_{i \in I}$ having bound B if and only if the operator given by*

$$T : l^q \left(\{X_i^*\}_{i \in I} \right) \rightarrow X^*, \quad T \left(\{g_i\}_{i \in I} \right) = \sum_{i \in I} v_i P_{V_i} \Lambda_i^* g_i$$

is a well-defined, bounded linear operator with $\|T\| \leq B$.

PROOF. First we suppose that Λ is a g - p -fusion Bessel sequence in X with respect to $\{X_i\}_{i \in I}$ having bound B . Then for any $\{g_i\}_{i \in I} \in l^q \left(\{X_i^*\}_{i \in I} \right)$ and any subset $J \subset I$, we have

$$\begin{aligned} \left\| \sum_{i \in J} v_i P_{V_i} \Lambda_i^* g_i \right\| &= \sup_{f \in X, \|f\|=1} \left| \sum_{i \in J} v_i P_{V_i} \Lambda_i^* g_i(f) \right| \\ &= \sup_{f \in X, \|f\|=1} \left| \sum_{i \in J} g_i v_i \Lambda_i P_{V_i}(f) \right| \leq \sup_{f \in X, \|f\|=1} \sum_{i \in J} \|g_i\| v_i \|\Lambda_i P_{V_i}(f)\| \\ &\leq \sup_{f \in X, \|f\|=1} \left(\sum_{i \in J} \|g_i\|^q \right)^{1/q} \left(\sum_{i \in J} v_i^p \|\Lambda_i P_{V_i}(f)\|^p \right)^{1/p} \\ &\leq B \left(\sum_{i \in J} \|g_i\|^q \right)^{1/q} \quad [\text{since } \Lambda \text{ is a } g\text{-}p\text{-fusion Bessel sequence}]. \end{aligned}$$

This shows that the series $\sum_{i \in I} v_i P_{V_i} \Lambda_i^* g_i$ is unconditionally convergent in X^* . From the above calculation also it follows that

$$\begin{aligned} \left\| \sum_{i \in I} v_i P_{V_i} \Lambda_i^* g_i \right\| &\leq B \left(\sum_{i \in I} \|g_i\|^q \right)^{1/q} \\ \Rightarrow \|T(\{g_i\}_{i \in I})\| &\leq B \left(\sum_{i \in I} \|g_i\|^q \right)^{1/q} = B \|\{g_i\}_{i \in I}\|_q. \end{aligned}$$

Thus T is bounded and $\|T\| \leq B$.

Conversely, suppose that T is well-defined and bounded linear operator. For fixed $f \in X$, consider the mapping $F_f : l^q \left(\{X_i^*\}_{i \in I} \right) \rightarrow \mathbb{C}$ defined by

$$F_f(\{g_i\}_{i \in I}) = T(\{g_i\}_{i \in I})(f) = \sum_{i \in I} v_i g_i \Lambda_i P_{V_i}(f).$$

Then F_f is a bounded linear functional on $l^q \left(\{X_i^*\}_{i \in I} \right)$, so

$$\{v_i \Lambda_i P_{V_i}(f)\} \in l^p \left(\{X_i\}_{i \in I} \right)$$

and

$$\|F_f(\{g_i\}_{i \in I})\| \leq \|T\| \|\{g_i\}_{i \in I}\|_q \|f\|.$$

Now, by the Hahn-Banach Theorem, there exists $\{g_i\}_{i \in I} \in l^q(\{X_i^*\}_{i \in I})$ with $\|\{g_i\}_{i \in I}\|_q \leq 1$ such that

$$\|\{v_i \Lambda_i P_{V_i}(f)\}\|_p = \left| \sum_{i \in I} v_i g_i \Lambda_i P_{V_i}(f) \right|.$$

Thus

$$\begin{aligned} \left(\sum_{i \in I} v_i^p \|\Lambda_i P_{V_i}(f)\|^p \right)^{1/p} &= \|\{v_i \Lambda_i P_{V_i}(f)\}\|_p \\ &\leq \sup_{\|\{g_i\}_{i \in I}\|_q \leq 1} \left| \sum_{i \in I} v_i g_i \Lambda_i P_{V_i}(f) \right| = \|F_f\| \leq \|T\| \|f\|. \end{aligned}$$

This completes the proof. \square

DEFINITION 2.5. Let Λ be a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$. Then the operator given by

$$U : X \rightarrow l^p(\{X_i\}_{i \in I}), \quad Uf = \{v_i \Lambda_i P_{V_i}(f)\}_{i \in I} \quad \forall f \in X.$$

is called the analysis operator and the operator $T : l^q(\{X_i^*\}_{i \in I}) \rightarrow X^*$,

$$T(\{g_i\}_{i \in I}) = \sum_{i \in I} v_i P_{V_i} \Lambda_i^* g_i \quad \forall \{g_i\}_{i \in I} \in l^q(\{X_i^*\}_{i \in I})$$

is called synthesis operator.

LEMMA 2.6. Let Λ be a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$. Then the analysis operator U has closed range.

PROOF. Since Λ be a g - p -fusion frame for X , by the definition of analysis operator U , the inequality (2.1), can be written as $A \|f\| \leq \|Uf\| \leq B \|f\|$. Now, it is easy to verify that U is one-to-one, $X \cong \mathcal{R}_U$ and hence U has closed range. \square

LEMMA 2.7. Let Λ be a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$. If for each $i \in I$, X_i is reflexive then X is reflexive.

PROOF. The proof is follows from the lemma 2.6. \square

THEOREM 2.8. Let Λ be a g - p -fusion Bessel sequence in X with respect to $\{X_i\}_{i \in I}$. Then

- (i) $U^* = T$.
- (ii) If Λ has the lower g - p -fusion frame condition and for each $i \in I$, X_i is reflexive then $T^* = U$.

PROOF. (i) For any $f \in X$ and $\{g_i\}_{i \in I} \in l^q \left(\{X_i^*\}_{i \in I} \right)$, we have

$$\begin{aligned} \langle Uf, \{g_i\}_{i \in I} \rangle &= \left\langle \{v_i \Lambda_i P_{V_i}(f)\}_{i \in I}, \{g_i\}_{i \in I} \right\rangle = \sum_{i \in I} \langle v_i \Lambda_i P_{V_i}(f), g_i \rangle, \\ \langle f, T(\{g_i\}_{i \in I}) \rangle &= \left\langle f, \sum_{i \in I} v_i P_{V_i} \Lambda_i^* g_i \right\rangle = \sum_{i \in I} \langle v_i \Lambda_i P_{V_i}(f), g_i \rangle. \end{aligned}$$

This shows that $U^* = T$.

(ii) The proof is directly follows from the lemma 2.6. \square

The following Theorem gives a characterization of a g - p -fusion frame for X .

THEOREM 2.9. *The family Λ is a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$ if and only if the synthesis operator T is a surjective and bounded linear operator.*

PROOF. First we consider that Λ is a g - p -fusion frame for X . Then by Theorem 2.4, T is well-defined and bounded linear operator. Since U is one-to-one and $U^* = T$, by Theorem 1.1 T is onto.

Conversely, suppose that T is bounded and onto. Then by Theorem 2.4, Λ is a g - p -fusion Bessel sequence in X . Also, by Theorem 1.1, U has a bounded inverse and this gives the lower g - p -fusion frame condition. This completes the proof. \square

We now develop the concept of generalized Riesz basis into the Banach space X .

DEFINITION 2.10. Let $1 < q < \infty$. The family Λ is called a q - gf -Riesz basis for X with respect to $\{X_i\}_{i \in J}$ if

- (i) Λ is gf -complete, i. e., $\{f : \Lambda_i P_{V_i}(f) = 0, i \in I\} = \{0\}$.
- (ii) There exist constants $0 < A \leq B < \infty$ such that for any subset $J \subset I$ and $g_i \in X_i^*, i \in J$,

$$A \left(\sum_{i \in J} \|g_i\|^q \right)^{1/q} \leq \left\| \sum_{i \in J} v_i P_{V_i} \Lambda_i^* g_i \right\| \leq B \left(\sum_{i \in J} \|g_i\|^q \right)^{1/q}.$$

Next theorem establish a relationship between q - gf -Riesz basis and the synthesis operator T .

THEOREM 2.11. *The family Λ is a q - gf -Riesz basis for X with respect to $\{X_i\}_{i \in I}$ having bounds A and B if and only if the synthesis operator T is a bounded linear and invertible such that*

$$A \|g\| \leq \|Tg\| \leq B \|g\| \quad (2.4)$$

for any $g = \{g_i\}_{i \in I} \in l^q \left(\{X_i^*\}_{i \in I} \right)$.

PROOF. Suppose Λ is a q - gf -Riesz basis for X with respect to $\{X_i\}_{i \in J}$ having bounds A and B . Then from the definition of q - gf -Riesz basis, it is easy to verify that $\sum_{i \in I} v_i P_{V_i} \Lambda_i^* g_i$ converges unconditionally for all $\{g_i\}_{i \in I} \in l^q \left(\{X_i^*\}_{i \in I} \right)$,

$$A \left(\sum_{i \in I} \|g_i\|^q \right)^{1/q} \leq \left\| \sum_{i \in I} v_i P_{V_i} \Lambda_i^* g_i \right\| \leq B \left(\sum_{i \in I} \|g_i\|^q \right)^{1/q}$$

and this implies that T is bounded, one-to-one and $A \|g\| \leq \|Tg\| \leq B \|g\|$.

Conversely, suppose that the operator T is a bounded linear and invertible operator from $l^q \left(\{X_i^*\}_{i \in I} \right)$ onto X^* and satisfying (2.4). Then by Theorem 2.9, Λ is a g - p -fusion frame for X with respect to $\{X_i\}_{i \in J}$ having bounds A and B . Now, for $f \in \{f : \Lambda_i P_{V_i}(f) = 0, i \in I\}$, we have

$$A \|f\| \leq \left(\sum_{i \in I} v_i^p \|\Lambda_i P_{V_i}(f)\|^p \right)^{1/p} = 0 \Rightarrow f = 0.$$

Therefore, we obtain that $\{f : \Lambda_i P_{V_i}(f) = 0, i \in I\} = \{0\}$. Hence, Λ is a q - gf -Riesz basis for X with respect to $\{X_i\}_{i \in I}$ having bounds A and B . \square

THEOREM 2.12. *Let Λ be a q - gf -Riesz basis for X with respect to $\{X_i\}_{i \in I}$ having bounds A and B . Then Λ is also a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$ having same bounds.*

PROOF. By Theorem 2.11, T is a bounded linear invertible operator with $\|T\| \leq B$ and $\|T^{-1}\| \leq A^{-1}$. It is easy to verify that $\|(T^*)^{-1}\|^{-1} \geq A$. Then by Theorem 2.9, Λ is a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$ having bounds A and B . \square

THEOREM 2.13. *Let $\{X_i\}_{i \in I}$ be a sequence of reflexive Banach spaces and Λ be a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$. Then the following are equivalent*

- (i) Λ is a q - gf -Riesz basis for X with respect to $\{X_i\}_{i \in I}$.
- (ii) If for any $g = \{g_i\}_{i \in I} \in l^q \left(\{X_i^*\}_{i \in I} \right)$, $\sum_{i \in I} v_i P_{V_i} \Lambda_i^* g_i = 0$, then $g_i = 0 \forall i \in I$.
- (iii) $\mathcal{R}(U) = l^p(\{X_i\}_{i \in I})$

PROOF. From the definition of q - gf -Riesz basis, it is easy to verify (i) \Rightarrow (ii).

(ii) \Rightarrow (i) Suppose that (ii) holds. Since Λ be a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$, by Theorem 2.9, the operator T is bounded linear and surjective. Also by condition (ii), it is easy to verify that T is injective. Hence, by Theorem 2.11, Λ is a q - gf -Riesz basis for X with respect to $\{X_i\}_{i \in I}$.

(i) \Rightarrow (iii) and (iii) \Rightarrow (i) are directly follows from the Theorem 3.16 of [2]. \square

3. Perturbation of g - p -fusion frame

In this section, the stability of g - p -fusion frame in X is presented.

THEOREM 3.1. *Let Λ be a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$ having bounds A and B . Suppose that $\Gamma_i \in \mathcal{B}(X, X_i)$, $i \in I$ such that*

$$\begin{aligned} \left(\sum_{i \in I} v_i^p \left\| (\Lambda_i P_{V_i} - \Gamma_i P_{W_i})(f) \right\|^p \right)^{1/p} &\leq \lambda_1 \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|^p \right)^{1/p} + \\ &+ \lambda_2 \left(\sum_{i \in I} v_i^p \left\| \Gamma_i P_{W_i}(f) \right\|^p \right)^{1/p} + \mu \|f\| \quad \forall f \in X, \end{aligned} \quad (3.1)$$

where $\lambda_1, \lambda_2 \in (-1, 1)$ and $-(1 + \lambda_1)B \leq \mu \leq (1 - \lambda_1)A$. Then $\Gamma = \{(W_i, \Gamma_i, v_i)\}_{i \in I}$ is a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$.

PROOF. For each $f \in X$, by Minkowski inequality, we have

$$\begin{aligned} \left(\sum_{i \in I} v_i^p \left\| \Gamma_i P_{W_i}(f) \right\|^p \right)^{1/p} &\leq \left(\sum_{i \in I} v_i^p \left\| (\Lambda_i P_{V_i} - \Gamma_i P_{W_i})(f) \right\|^p \right)^{1/p} + \\ &+ \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|^p \right)^{1/p} \\ &\leq (1 + \lambda_1) \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|^p \right)^{1/p} + \lambda_2 \left(\sum_{i \in I} v_i^p \left\| \Gamma_i P_{W_i}(f) \right\|^p \right)^{1/p} + \mu \|f\| \end{aligned}$$

Therefore, since Λ is a g - p -fusion frame, we have

$$\begin{aligned} &\left(\sum_{i \in I} v_i^p \left\| \Gamma_i P_{W_i}(f) \right\|^p \right)^{1/p} \\ &\leq \frac{1 + \lambda_1}{1 - \lambda_2} \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|^p \right)^{1/p} + \frac{\mu}{(1 - \lambda_2)} \|f\| \\ &\leq \left[\left(\frac{1 + \lambda_1}{1 - \lambda_2} \right) B + \frac{\mu}{(1 - \lambda_2)} \right] \|f\| = \left[\frac{B(1 + \lambda_1) + \mu}{1 - \lambda_2} \right] \|f\|. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|^p \right)^{1/p} - \left(\sum_{i \in I} v_i^p \left\| \Gamma_i P_{W_i}(f) \right\|^p \right)^{1/p} \\ &\leq \left(\sum_{i \in I} v_i^p \left\| (\Lambda_i P_{V_i} - \Gamma_i P_{W_i})(f) \right\|^p \right)^{1/p}. \end{aligned}$$

Now using (3.1), we obtain

$$\begin{aligned}
 & (1 + \lambda_2) \left(\sum_{i \in I} v_i^p \left\| \Gamma_i P_{W_i}(f) \right\|^p \right)^{1/p} \\
 & \geq (1 - \lambda_1) \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|^p \right)^{1/p} - \mu \|f\| \\
 & \geq [(1 - \lambda_1) A - \mu] \|f\| \quad [\text{since } \Lambda \text{ is a } g\text{-}p\text{-fusion frame}] \\
 & \Rightarrow \left(\sum_{i \in I} v_i^p \left\| \Gamma_i P_{W_i}(f) \right\|^p \right)^{1/p} \geq \left[\frac{A(1 - \lambda_1) - \mu}{1 + \lambda_2} \right] \|f\|.
 \end{aligned}$$

Hence, Γ is a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$ having bounds

$$\left[\frac{A(1 - \lambda_1) - \mu}{1 + \lambda_2} \right] \text{ and } \left[\left(\frac{1 + \lambda_1}{1 - \lambda_2} \right) B + \frac{\mu}{(1 - \lambda_2)} \right].$$

This completes the proof. \square

THEOREM 3.2. Let Λ be a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$ having bounds A and B . Suppose that $\Gamma_i \in \mathcal{B}(X, X_i)$, $i \in I$ such that

$$\left(\sum_{i \in I} v_i^p \left\| (\Lambda_i P_{V_i} - \Gamma_i P_{W_i})(f) \right\|^p \right)^{1/p} \leq R \|f\| \quad \forall f \in X.$$

where $0 < R < A$. Then $\Gamma = \{(W_i, \Gamma_i, v_i)\}_{i \in I}$ is a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$ having bounds $(A - R)$ and $(B + R)$.

PROOF. By Minkowski inequality, for $f \in X$, we get

$$\begin{aligned}
 \left(\sum_{i \in I} v_i^p \left\| \Gamma_i P_{W_i}(f) \right\|^p \right)^{1/p} & \leq \left(\sum_{i \in I} v_i^p \left\| (\Lambda_i P_{V_i} - \Gamma_i P_{W_i})(f) \right\|^p \right)^{1/p} + \\
 & + \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|^p \right)^{1/p} \\
 & \leq (B + R) \|f\| \quad [\text{since } \Lambda \text{ is } g\text{-}p\text{-fusion frame}].
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \left(\sum_{i \in I} v_i^p \left\| \Gamma_i P_{W_i}(f) \right\|^p \right)^{1/p} & \geq \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|^p \right)^{1/p} - \\
 & - \left(\sum_{i \in I} v_i^p \left\| (\Lambda_i P_{V_i} - \Gamma_i P_{W_i})(f) \right\|^p \right)^{1/p} \\
 & \geq (A - R) \|f\|.
 \end{aligned}$$

Hence, Γ is a g - p -fusion frame for X with bounds $(A - R)$ and $(B + R)$.

This completes the proof. \square

We end this section by constructing g - p -fusion frames in Cartesian product of Banach spaces and tensor product of Banach spaces.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces. Then the Cartesian product of X and Y is denoted by $X \oplus Y$ and defined to be an Banach space with respect to the norm

$$\|f \oplus g\|^p = \|f\|_X^p + \|g\|_Y^p, \quad (3.2)$$

for all $f \in X$ and $g \in Y$. Now, if $U \in \mathcal{B}(X, X_i)$ and $V \in \mathcal{B}(Y, Y_i)$, then for all $f \in X$ and $g \in Y$, we define

$$U \oplus V \in \mathcal{B}(X \oplus Y, X_i \oplus Y_i) \text{ by } (U \oplus V)(f \oplus g) = Uf \oplus Vg,$$

$$P_{V_i \oplus W_i}(f \oplus g) = P_{V_i}f \oplus P_{W_i}g,$$

where $\{Y_i\}_{i \in I}$ is a another sequence of Banach spaces and $\{W_i\}_{i \in I}$ is the collection of closed subspaces of Y and P_{W_i} are the linear projections of Y onto W_i such that $P_{W_i}(X) = W_i$, for $i \in I$.

THEOREM 3.3. *Let Λ be a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$ having bounds A, B and $\Gamma = \{(W_i, \Gamma_i, v_i)\}_{i \in I}$ be a g - p -fusion frame for Y with respect to $\{Y_i\}_{i \in I}$ having bounds C, D , where $\Gamma_i \in \mathcal{B}(Y, Y_i)$ for each $i \in I$. Then $\Lambda \oplus \Gamma = \{(V_i \oplus W_i, \Lambda_i \oplus \Gamma_i, v_i)\}_{i \in I}$ is a g - p -fusion frame for $X \oplus Y$ with respect to $\{X_i \oplus Y_i\}_{i \in I}$ having bounds $\min(A^p, C^p)$ and $\max(B^p, D^p)$.*

PROOF. Since Λ and Γ are g - p -fusion frames for X and Y , respectively,

$$A^p \|f\|_X^p \leq \sum_{i \in I} v_i^p \|\Lambda_i P_{V_i}(f)\|_X^p \leq B^p \|f\|_X^p \quad \forall f \in X \quad (3.3)$$

$$C^p \|g\|_Y^p \leq \sum_{i \in I} v_i^p \|\Gamma_i P_{W_i}(g)\|_Y^p \leq D^p \|g\|_Y^p \quad \forall g \in Y. \quad (3.4)$$

Adding (3.3) and (3.4) and then using (3.2), we get

$$\begin{aligned}
 A^p \|f\|_X^p + C \|g\|_Y^p &\leq \sum_{i \in I} v_i^p \|\Lambda_i P_{V_i}(f)\|_X^p + \sum_{i \in I} v_i^p \|\Gamma_i P_{W_i}(g)\|_Y^p \\
 &\leq B^p \|f\|_X^p + D^p \|g\|_Y^p. \\
 \Rightarrow \min(A^p, C^p) \{\|f\|_X^p + \|g\|_Y^p\} &\leq \sum_{i \in I} v_i^p (\|\Lambda_i P_{V_i}(f)\|_X^p + \|\Gamma_i P_{W_i}(g)\|_Y^p) \\
 &\leq \max(B^p, D^p) \{\|f\|_X^p + \|g\|_Y^p\}. \\
 \Rightarrow \min(A^p, C^p) \|f \oplus g\|^p &\leq \sum_{i \in I} v_i^p \|\Lambda_i P_{V_i}(f) \oplus \Gamma_i P_{W_i}(g)\|^p \\
 &\leq \max(B^p, D^p) \|f \oplus g\|^p. \\
 \Rightarrow \min(A^p, C^p) \|f \oplus g\|^p &\leq \sum_{i \in I} v_i^p \|(\Lambda_i \oplus \Gamma_i)(P_{V_i} \oplus P_{W_i})(f \oplus g)\|^p \\
 &\leq \max(B^p, D^p) \|f \oplus g\|^p \quad \forall f \oplus g \in X \oplus Y. \\
 \Rightarrow \min(A^p, C^p) \|f \oplus g\|^p &\leq \sum_{i \in I} v_i^p \|(\Lambda_i \oplus \Gamma_i) P_{V_i \oplus W_i}(f \oplus g)\|^p \\
 &\leq \max(B^p, D^p) \|f \oplus g\|^p \quad \forall f \oplus g \in X \oplus Y.
 \end{aligned}$$

Thus, $\Lambda \oplus \Gamma$ is a g - p -fusion frame for $X \oplus Y$ with respect to $\{X_i \oplus Y_i\}_{i \in I}$ having bounds $\min(A^p, C^p)$ and $\max(B^p, D^p)$. This completes the proof. \square

The tensor product of X and Y is denoted by $X \otimes Y$ and it is defined to be a normed space with respect to the norm

$$\|f \otimes g\|^p = \|f\|_X^p \|g\|_Y^p, \quad (3.5)$$

for all $f \in X$ and $g \in Y$. Then it is easy to verify that $X \otimes Y$ is complete with respect to the above norm. Therefore, $X \otimes Y$ is a Banach space.

Let $U, U' \in \mathcal{B}(X, X_i)$ and $V, V' \in \mathcal{B}(Y, Y_j)$, for $i \in I$ and $j \in J$. Then for $U \otimes V, U' \otimes V' \in \mathcal{B}(X \otimes Y, X_i \otimes Y_j)$, we define

$$(i) \quad (U \otimes V)(f \otimes g) = Uf \otimes Vg \quad \text{for all } f \in X, g \in Y.$$

$$(ii) \quad (U \otimes V)(U' \otimes V') = UU' \otimes VV'.$$

$$(iii) \quad P_{V_i \otimes W_j}(f \otimes g) = P_{V_i}f \otimes P_{W_j}g \quad \text{for all } f \in X, g \in Y.$$

Let $\{v_i\}_{i \in I}, \{w_j\}_{j \in J}$ be two families of positive weights i.e., $v_i > 0 \quad \forall i \in I, w_j > 0 \quad \forall j \in J$ and $\Lambda_i \otimes \Gamma_j \in \mathcal{B}(X \otimes Y, X_i \otimes Y_j)$ for each $i \in I$ and $j \in J$. Then according to the definition (2.1), the family $\Lambda \otimes \Gamma = \{(V_i \otimes W_j, \Lambda_i \otimes \Gamma_j, v_i w_j)\}_{i,j}$ is said to be a g - p -fusion frame for $X \otimes Y$ with respect to $\{X_i \otimes Y_j\}_{i,j}$ if there exist constants $A, B > 0$ such that

$$A \|f \otimes g\| \leq \left(\sum_{i,j} v_i^p w_j^p \left\| (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j}(f \otimes g) \right\|^p \right)^{1/p} \leq B \|f \otimes g\|$$

for all $f \otimes g \in X \otimes Y$. The constants A and B are called the frame bounds.

THEOREM 3.4. *The family $\Lambda \otimes \Gamma$ is a g - p -fusion frame for $X \otimes Y$ with respect to $\{X_i \otimes Y_j\}_{i,j}$ if and only if Λ is a g - p -fusion frames for X with respect to $\{X_i\}_{i \in I}$ and Γ is a g - p -fusion frames for Y with respect to $\{Y_j\}_{j \in J}$.*

PROOF. First we suppose that $\Lambda \otimes \Gamma$ is a g - p -fusion frame for $H \otimes K$ with respect to $\{H_i \otimes K_j\}_{i,j}$. Then there exist constants $A, B > 0$ such that for all $f \otimes g \in H \otimes K - \{\theta \otimes \theta\}$, we have

$$\begin{aligned} A \|f \otimes g\| &\leq \left(\sum_{i,j} v_i^p w_j^p \left\| (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j}(f \otimes g) \right\|^p \right)^{1/p} \leq B \|f \otimes g\| \\ \Rightarrow A \|f \otimes g\| &\leq \left(\sum_{i,j} v_i^p w_j^p \left\| \Lambda_i P_{V_i}(f) \otimes \Gamma_j P_{W_j}(g) \right\|^p \right)^{1/p} \leq B \|f \otimes g\|. \\ \Rightarrow A \|f\|_X \|g\|_Y &\leq \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|_X^p \right)^{1/p} \left(\sum_{j \in J} w_j^p \left\| \Gamma_j P_{W_j}(g) \right\|_Y^p \right)^{1/p} \\ &\leq B \|f\|_X \|g\|_Y \quad [\text{by (3.5)}]. \end{aligned}$$

Since $f \otimes g$ is non-zero vector, f and g are also non-zero vectors and therefore $\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|_X^p$ and $\sum_{j \in J} w_j^p \left\| \Gamma_j P_{W_j}(g) \right\|_Y^p$ are non-zero. Then

$$\begin{aligned} \frac{A \|g\|_Y}{\left(\sum_{j \in J} w_j^p \left\| \Gamma_j P_{W_j}(g) \right\|_Y^p \right)^{1/p}} \|f\|_X &\leq \left(\sum_{i \in I} v_i^p \left\| \Lambda_i P_{V_i}(f) \right\|_X^p \right)^{1/p} \\ &\leq \frac{B \|g\|_Y}{\left(\sum_{j \in J} w_j^p \left\| \Gamma_j P_{W_j}(g) \right\|_Y^p \right)^{1/p}} \|f\|_X \\ \Rightarrow A_1 \|f\|_X &\leq \left(\sum_{j \in J} v_j^p \left\| \Lambda_j P_{V_j}(f) \right\|_X^p \right)^{1/p} \leq B_1 \|f\|_X \quad \forall f \in X, \end{aligned}$$

where

$$A_1 = \min_{g \in Y} \left\{ \frac{A \|g\|_Y}{\left(\sum_{j \in J} w_j^p \left\| \Gamma_j P_{W_j}(g) \right\|_Y^p \right)^{1/p}} \right\}$$

and

$$B_1 = \max_{g \in Y} \left\{ \frac{B \|g\|_Y}{\left(\sum_{j \in J} w_j^p \left\| \Gamma_j P_{W_j}(g) \right\|_Y^p \right)^{1/p}} \right\}.$$

This shows that Λ is a g - p -fusion frame for X with respect to $\{X_i\}_{i \in I}$. Similarly, it can be shown that Γ is g - p -fusion frame for Y with respect to $\{Y_j\}_{j \in J}$.

Conversely, suppose that Λ and Γ are g - p -fusion frames for X and Y . Then there exist positive constants A, B and C, D such that

$$A \|f\|_X \leq \left(\sum_{i \in I} v_i^p \|\Lambda_i P_{V_i}(f)\|_X^p \right)^{1/p} \leq B \|f\|_X \quad \forall f \in X \quad (3.6)$$

$$C \|g\|_Y \leq \left(\sum_{j \in J} w_j^p \|\Gamma_j P_{W_j}(g)\|_Y^p \right)^{1/p} \leq D \|g\|_Y \quad \forall g \in Y. \quad (3.7)$$

Multiplying (3.6) and (3.7), and using (3.5), we get

$$AC \|f \otimes g\| \leq \left(\sum_{i,j} v_i^p w_j^p \|\Lambda_i P_{V_i}(f) \otimes \Gamma_j P_{W_j}(g)\|^p \right)^{1/p} \leq BD \|f \otimes g\|.$$

Therefore, for each $f \otimes g \in H \otimes K$, we get

$$AC \|f \otimes g\| \leq \left(\sum_{i,j} v_i^p w_j^p \left\| (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j}(f \otimes g) \right\|^p \right)^{1/p} \leq BD \|f \otimes g\|.$$

Hence, $\Lambda \otimes \Gamma$ is a g - p -fusion frame for $X \otimes Y$ with respect to $\{X_i \otimes Y_j\}_{i,j}$ with bounds AC and BD . This completes the proof. \square

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References

- [1] A. Aldroubi, Q. Sun and W. Tang, p -frame and shift subspaces of L^p , J. Fourier Anal. Appl., **7** (2001) 1–22.
- [2] M. R. Abdollahpour, M. H. Faroughi and A. Rahimi, PG -frames in Banach spaces, Methods of Functional Analysis and Topology, **13** (2007), no. 3, 201–210.
- [3] P. Casazza and G. Kutyniok, *Frames of subspaces*, Cotemporary Math, AMS **345** (2004), 87–114.
- [4] O. Christensen and Stoeva, p -frames in separable Banach spaces, Adv. Comput. Math., textbf18 (2003), no. 2-4, 117–126.
- [5] Laura Gavrutu, *Frames for operator*, Appl. Comput. Harmon. Anal., **32** (1) (2012), 139–144.
- [6] H. Heuser, Functional Analysis, (Wiley, 1982).
- [7] P. Ghosh and T. K. Samanta, *Stability of dual g -fusion frame in Hilbert spaces*, Methods of Functional Analysis and Topology, **26** (2020) no. 3, 227–240.
- [8] P. Ghosh and T. K. Samanta, *Generalized atomic subspaces for operators in Hilbert spaces*, doi: 10.21136/MB.2021.0130-20.
- [9] P. Ghosh and T. K. Samanta, *Generalized fusion frame in tensor product of Hilbert spaces*, Journal of the Indian Mathematical Society, **89** (1-2) (2022), 58–71.

- [10] V. Sadri, Gh. Rahimlou, R. Ahmadi and Farfar R. Zarghami, *Construction of g -fusion frames in Hilbert Spaces*, Probl. Anal. Issues Anal., **9** (27), no. 1, 110–127.
- [11] Xiang-Chun Xiao, Yu-Can Zhu and Xiao-Ming Zeng, *Generalized p -frame in separable complex Banach spaces*, International Journal of Wavelets, Multiresolution and Information Processing, **8** (2010), no. 1, 133–148.
- [12] W. Sun, *G -frames and G -Riesz bases*, Journal of Mathematical Analysis and Applications, **322** (1) (2006), 437–452.

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