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PRIME $k-$IDEALS AND WEAKLY NOETHERIAN $\Gamma-$SEMIRINGS

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Abstract

The purpose of this paper is to provide some notions like smallest $k-$ ideal containing an ideal of a $\Gamma-$ semiring R, connected ideals, $S-$ inductors, $P-$ primary ideals, weakly noetherian $\Gamma-$ semiring and investigate the structure of a commutative $\Gamma-$ semiring with these concepts. Further, for weakly Noetherian $\Gamma-$ semiring with strong identity, each connected prime ideal of R is equal to $(0 : r)$, which is a $k-$ideal.

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1. Introduction

N. Nobusawa [7] introduced the notion of $\Gamma-$ ring as generalization of ring in 1964. M. K. Sen [9] introduced the notion of $\Gamma-$ semigroup in 1981 and in 1995, the concept of $\Gamma-$ semiring was introduced by Rao [8] as a generalization of $\Gamma-$ ring, ternary semirings, semigroups and semirings. After this T. K. Dutta and S. K. Sardar[1–3], H. Hedayati and K. P. Shum[4], Sharma and Gupta[10–12] and many others obtained interesting results on $\Gamma-$ semirings. Dutta and Sardar [3] studied ideals and prime ideals in $\Gamma-$ semirings and various radicals namely, prime radicals, Jacobson radicals, Levitzki radicals. D.R. La Torre [15] introduced the concept of $k-$ ideals and $h-$ ideals in $\Gamma-$ Semiring. Then, Paul Lescot [6] studied various results on prime and primary ideals in semirings.

The study of prime ideal and prime $k-$ideal has provided important information to the investigation of the structure of commutative $\Gamma-$ Semirings. In this paper we concentrate on a prime ideal and prime $k-$ideal in relation with connected ideals, zero divisors and primary ideals of a $\Gamma-$ Semiring R.

The motivation for this paper is [6], in which Lescot has discussed the concepts of zero divisor, minimal prime ideals, obvious version of primary decomposition and many more. Here our aim is two-fold. First we point out that set of maximal $k-$ ideals is contained in the set of prime $k-$ ideals and for any $k-$ ideal I, a smallest $k-$ ideal denoted by $Sl(I)$, containing I is defined and established some results with radical of this smallest $k-$ ideal. Further the concept of connected ideal, $S-$ inductors, $P-$ primary ideal, radical $k-$ ideal and weakly Noetherian $\Gamma-$ semiring is introduced and for weakly Noetherian $\Gamma-$ semiring with strong identity, each connected prime ideal of R is equal to $(0 : r)$, which is a $k-$ideal.

2. Preliminaries

First we will recall some definitions that will be used in this paper.

Definition 2.1. [8] Let R and Γ be two additive commutative semigroups. Then R is called a $\Gamma-$ semiring if there exists a mapping $R \times \Gamma \times R \to R$ denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying the following conditions:

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$.
(ii) $x(\alpha + \beta)z = x\alpha z + x\beta z$.
(iii) $x\alpha(y + z) = x\alpha y + x\alpha z$.
(iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

Similarly one can define Γ is a $R-$ semiring.

Example 1. Obviousely, every semiring R is a $\Gamma-$ semiring. Let R be a semiring and Γ be a commutative semigroup. Define a mapping $R \times \Gamma \times R \to R$ denoted by $x\alpha y = xy$ for all $x, y, \in R$ and $\alpha \in \Gamma$. Then R is a $\Gamma-$ semiring.

Example 2. Let $R = (\mathbb{Z}^+, +)$ be a semigroup of non negative integers and let $\Gamma = (2\mathbb{Z}^+, +)$ be the semigroup of even non negative integers. Then R is a $\Gamma-$ semiring.

Example 3. Let M be a $\Gamma-$ ring and let R be the set of ideals of M. Define addition in the natural way and if $A, B \in R$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{x\gamma y | x, y \in M\}$. Then R is a $\Gamma-$ semiring

Definition 2.2. [10] A $\Gamma-$ semiring R is said to have a zero element if $0\alpha x = 0 = x\alpha 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\alpha \in \Gamma$.

Definition 2.3. [12] A $\Gamma-$ semiring R is said to have a identity element 1 if for all $x \in R$ there exists $\alpha \in \Gamma$ such that $1\alpha x = x = x\alpha 1$.

Definition 2.4. [12] A $\Gamma-$ semiring R is said to have a strong identity element 1_s if for all $x \in R$, $1_s\alpha x = x = x\alpha 1_s$, for all $\alpha \in \Gamma$.

Definition 2.5. [8] A $\Gamma-$ semiring R is said to be commutative if $x\alpha y = y\alpha x$ for all $x, y \in R$ and for all $\alpha \in \Gamma$.

Example 3. Let R be the set of all even positive integers and Γ be the set of all positive integers divisible by 3. Then with usual addition and multiplication of integers, R is commutative $\Gamma-$ semiring.

Definition 2.6. [8] A $R-$ semiring Γ is said to be commutative if $\alpha x\beta = \beta x\alpha$ for all $x \in R$ and for all $\alpha, \beta \in \Gamma$.

Definition 2.7. [11] A non-zero element x in a $\Gamma-$ semiring R is a left zero divisor if and only if there exists a non-zero element $y \in R$ and $\alpha \in \Gamma$ satisfying $x\alpha y = 0$. Clearly, if $ZD(R)$ is the set of all zero divisors of R then $ZD(R) \cup \{0\} = \bigcap_{r \in R\setminus\{0\}} (0 : r), r \in R$. Similarly we can define right zero divisor.

Definition 2.8. [13] For an ideal I of a commutative $\Gamma-$ semiring R, the prime radical of I given by $r(I)$ and is defined as $r(I) = \{x \in R | (x\alpha)^{n-1}x \in I$ for some positive integer n and for all $\alpha \in \Gamma\}$.

Definition 2.9. [1] An ideal P of a $\Gamma-$ semiring R is said to be a prime ideal if for any two ideals A and B of R, $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$.

Definition 2.10. [12] An ideal P of a $\Gamma-$ semiring R is said be a primary ideal if $P \neq R$ and $x\alpha y \in P$, for all $\alpha \in \Gamma$ and $x, y \in R$, then either $x \in P$ or $(y\beta)^{n-1}y \in P$ for all $\beta \in \Gamma$ and some positive integer n.

OR

Definition 2.11. [14] An ideal P of a $\Gamma-$ semiring R is said be a primary ideal if for any two ideals A and B of R , $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq r(P)$.

Obviously every prime is primary.

Remark 2.12. *Throughout this paper, R will denote a commutative* $\Gamma-$ *semiring with zero element* $'0'$ *and identity element* $'1'$ *unless otherwise stated.*

3. Maximal k- ideal, Prime k-ideal and prime ideals in a $\Gamma-$ Semiring

Let $Prm(R)$ and $Prm(R_k)$ denotes the set of all prime and prime $k-$ideals of a $\Gamma-$ semiring R respectively and $Max(R_k)$ denotes the set of all maximal $k-$ideal in R, which are proper. Min $(Prm(R))$ and Min $(Prm(R_k))$ will denote the sets of minimal elements for inclusion of $Prm(R)$ and $Prm(R_k)$ respectively. So by some classical arguments $(Prm(R), \supseteq)$ and $(Prm(R_k), \supseteq)$ are inductive. Therefore, Zorns lemma implies that each prime (prime $k-$ideal) contains a minimal prime ideal (minimal prime $k-$ideal). We will use these notations through out this paper.

Definition 3.1. Let R be a $\Gamma-$ semiring and I be an ideal of R then I is a $k-$ ideal of R such that whenever $x + y = z$ with $x \in R, y \in I$ and $z \in I$ then $x \in I$.

Definition 3.2. [12] For each ideal I of R, there is a smallest $k-$ ideal, $Sl(I)$ containing I and is given by $Sl(I) = \{x \in R \mid$ there exists $y, z \in I$ such that $x + y = z\}$.

Theorem 3.3. *[1] An ideal P of a commutative* $\Gamma-$*semiring R is prime if and only if* $a\Gamma b \subseteq P$ *implies that either* $a \in P$ *or* $b \in P$.

Theorem 3.4. *Let R be a commutative* $\Gamma-$ *semiring with strong identity. Then* $Max(R_k) \subseteq Prm(R_k)$.

Proof. Let $T \in Max(R_k)$ and let $a \notin T$ and $b \notin T$ such that $a\Gamma b \subseteq T$. Then $a\alpha b \in T$, for all $\alpha \in \Gamma$. Now $T \subseteq Sl(T+R\Gamma a) \subseteq R$ and $T \subseteq Sl(T+R\Gamma b) \subseteq R$, so by maximality of T we have $R = Sl(T+R\Gamma a) = Sl(T+R\Gamma b)$, Therefore one may find $x, y \in T+R\Gamma b$ such that $1_s + x = y$ (as $1_s \in R$). Let $x = t + m\gamma b$ and $y = t' + n\beta b$, where $t, t' \in T, m, n \in R$ and $\gamma, \beta \in \Gamma$, then as $x = t + m\gamma b$ therefore $a\alpha x = a\alpha t + a\alpha m\gamma b = a\alpha t + m\gamma a\alpha b \in T$ (as $a\alpha b \in T$). Similarly, $a\alpha y \in T$. But $a + a\alpha x = a\alpha 1_s + a\alpha x = a\alpha(1_s + x) = a\alpha y$, therefore $a \in Sl(T)$, but $Sl(T) = T$, so $a \in T$, which is a contradiction. Therefore T is prime. □

Theorem 3.5. *Let R be a* $\Gamma-$ *semiring and K, L be the ideals of R. Then* $r(Sl(K \cap L)) = r(Sl(K) \cap Sl(L)) = r(Sl(K)) \cap r(Sl(L))$

Proof. It is simple and straightforward. □

Definition 3.6. Let R be a $\Gamma-$ semiring. Then for $r \in R$, the annihilator of r is defined by $(0 : r) = \{x \in R \mid r\alpha x = 0 \text{ for all } \alpha \in \Gamma\}$

Theorem 3.7. *Let R be a* $\Gamma-$ *semiring. If* $r \in R$ *then* $(0 : r)$ *is a* $k-$*ideal of R.*

Proof. Let $t, t' \in (0 : r)$ and $x + t = t'$ then $r\alpha x = r\alpha x + 0 = r\alpha x + r\alpha t' = r\alpha(x + t) = r\alpha t' = 0$. Thus $x \in (0 : r)$. Hence $(0 : r)$ is a $k-$ideal of R. □

For any subset X of R , we define $(0 : X) = \bigcap_{x \in X} (0 : x)$, then clearly $(0 : X)$ is a $k-$ ideal of R. Again for $r \in R \setminus \{0\}$, let $R'_r = R/(0 : r)$ and let $\theta_r : R \to R'_r$ denote canonical projection.

Definition 3.8. An ideal P of R is connected to $r \in R \setminus \{0\}$, if $\theta_r(P) = Q$ for some minimal prime ideal Q of R'_r. The set of connected ideals of R is denoted by $Ctd(R)$.

The following theorem is proved in [4].

Theorem 3.9. *([4], theorem 3.3) If I is an ideal of a* $\Gamma-$ *semiring R then* $R/I = \{x + I | x \in R\}$ *is a* $\Gamma-$*semiring with the mapping* $* : R/I \times \Gamma \times R/I \to R/I$, *defined by* $(x + I)\alpha(y + I) = x\alpha y + I$, *for all* $x, y \in R$ *and* $\alpha \in \Gamma$.

Theorem 3.10. *Let R be a* $\Gamma-$ *semiring and* $Prm(R)$ *denotes the set of all prime ideals of R then* $Ctd(R) \subseteq Prm(R)$ *and* $Min(Prm(R)) \subseteq Ctd(R)$.

Proof. Let $P \in Ctd(R)$ then there exists $r \in R \setminus \{0\}$ such that $\theta_r(P) = P/(0 : r)$ is minimal prime ideal of R'_r. Let $a\Gamma b \subseteq P$. This implies that $a\alpha b \in P$ for all $\alpha \in \Gamma$. Therefore, $(a\alpha b + (0 : r)) \in P/(0 : r)$ so $(a + (0 : r))\alpha(b + (0 : r) \in P/(0 : r)$. But $P/(0 : r)$ is a prime ideal, so either $(a + (0 : r)) \in P/(0 : r)$ or $(b + (0 : r)) \in P/(0 : r)$. This implies that either $a \in P$ or $b \in P$. Thus, P is a prime. Hence, $Ctd(R) \subseteq Prm(R)$. Furthermore, let $P \in Min(Prm(R))$ then for $1 \in R$, $\theta_1(P) = P/(0 : 1) = P/\{0\} = P$. Hence $P \in Ctd(R)$. □

Definition 3.11. For any ideal S of a $\Gamma-$ semiring R and $a \in R$, let $I_a(S) = \{b \in R \mid a\Gamma b \subseteq S\}$. It is clear that $I_a(S)$ is an ideal of R and a $k-$ideal if S is itself a $k-$ideal of R. Furthermore, $I_a(\{0\}) = (0 : a)$. The ideals of the form $I_a(S)(a \notin S)$ is called $S-$inductors.

Theorem 3.12. *Let R be a commutative* $\Gamma-$ *semiring and S be any ideal of R such that* $b \notin S$ *and let* $I_b(S)$ *be maximal element among* $S-$ *inductors then* $I_b(S)$ *is a prime ideal of R.*

Proof. It is clear that $1 \notin I_b(S)$ (as $b \notin S$), hence $I_b(S) \neq R$. Let us assume that $x\Gamma y \subseteq I_b(S), x, y \in R$ and $x \notin I_b(S)$. Then $x\Gamma b \nsubseteq S$ implies that $x\alpha b \notin S$ for some $\alpha \in \Gamma$. Now we claim that $I_b(S) \subseteq I_{x\alpha b}(S)$. For this, let $m \in I_b(S)$. This implies that $b\Gamma m \subseteq S$. Therefore, $x\alpha(b\Gamma m) \subseteq S$. So, $m \in I_{x\alpha b}(S)$ and $I_b(S) \subseteq I_{x\alpha b}(S)$. But the maximality of $I_b(S)$ among $S-$inductors yields $I_b(S) = I_{x\alpha b}(S)$. Now $y\Gamma(x\alpha b) = (x\Gamma y)\alpha b \subseteq S$. Thus, $y \in I_{x\alpha b}(S) = I_b(S)$. Hence, $I_b(S)$ is prime. □

Theorem 3.13. *Let R be a* $\Gamma-$ *semiring with strong identity and* Γ *be a commutative* $R-$ *semiring. Let* $P \in Min(Prm(R)) \cup Min(Prm(R_k))$*, then* $x \in P \setminus \{0\}$ *implies that x is a zero divisor in R.*

Proof. Let $P \in Min(Prm(R))$ then $x \in P$ such that $x \neq 0$ and assume that x is not a zero-divisor. Then for all $r \in R \setminus \{0\}$, $n \in N$ and $\alpha, \gamma \in \Gamma$, $r\gamma(x\alpha)^{n-1}x \neq 0$. In particular, for all $r \in R \setminus P$, for all $n \in N$ we have $r\gamma(x\alpha)^{n-1}x \neq 0$. Let Ψ be the set of all ideals I of R such that for all $n \in N, r \in R \setminus P$ and $\alpha, \gamma \in \Gamma$, $r\gamma(x\alpha)^{n-1}x \notin I$.
Now it is clear that $\{0\} \in \Psi$, so $\Psi \neq \phi$ and Ψ is inductive w.r.t. $\subseteq$. Hence Ψ contains a maximal element, say I. Now $I \neq R$ and as $1 = 1\gamma x^0 \notin I$, let $a\Gamma b \subseteq I$ such that $a \notin I$ and $b \notin I$. Then $a\gamma b \in I$ for all $\gamma \in \Gamma$. Therefore, $I + R\Gamma a$ and $I + R\Gamma b$ are ideals of R strictly containing I. So $I + R\Gamma a \notin \Psi$ and $I + R\Gamma b \notin \Psi$. Then for any $\alpha \in \Gamma$ we can find $c, d \in R \setminus P$, $i, j \in I$, $u, v \in R$, $\beta_1, \beta_2 \in \Gamma$ and $m, n \in N$ with $c\alpha(x\alpha)^{m-1}x = i + u\beta_1 a$ and $d\alpha(x\alpha)^{n-1}x = j + v\beta_2 b$. Now $c\Gamma d \subseteq R \setminus P$. Therefore, $c\gamma d \in R \setminus P$ for any $\gamma \in \Gamma$ and $(c\gamma d)\alpha(x\alpha)^{m+n-1}x = (c\alpha(x\alpha)^{m-1}x)\gamma(d\alpha(x\alpha)^{n-1}x) = i\gamma(j + v\beta_2 b) + (u\beta_1 a)\gamma j + (u\beta_1 v)\beta_2(a\gamma b) \in I$, which is a contradiction . Thus, I is prime. But by definition for all $r \in R \setminus P, r = r\alpha x^0 \notin I$. Hence, $R \setminus P \subseteq R \setminus I$ and $I \subseteq P$. But the minimality of P implies that $I = P$. Thus, $1\alpha(x\alpha)^{1-1}x = x \in P = I$. But this contradicts the definition of I. Hence, x is a zero divisor.
Further, if $P \in Min(Prm(R_k))$, then same argument can be applied by making a slight change in definition of Ψ by defining Ψ as a set of $k-$ideals I of R satisfying for all $n \in N, r \in R \setminus P$ and $\alpha, \gamma \in \Gamma$, $r\gamma(x\alpha)^{n-1}x \notin I$, we find a maximal element say I of Ψ, so that $I \neq R$. Now assume that $a\Gamma b \subseteq I$ with $a \notin I$ and $b \notin I$, we have $Sl(I + R\Gamma a) \notin \Psi$ and $Sl(I + R\Gamma b) \notin \Psi$. So for any $\alpha \in \Gamma$, we can find $c, d \in R \setminus P, m, n \in N$ such that $c\gamma(x\alpha)^{m-1}x + y = y'$ for some $y, y' \in I + R\Gamma a$ and $d\gamma'(x\alpha)^{n-1}x + z = z'$ for some $z, z' \in I + R\Gamma b$ and $\gamma' \in \Gamma$. Let $y = i + s\beta a$ and $z = j + t\beta' b$ for $i, j \in I, \beta, \beta' \in \Gamma$. Then $y\alpha b = i\alpha b + s\beta(a\alpha b) \in I$. Similarly $y'\alpha a \in I$. Now as $c\gamma(x\alpha)^{m-1}x\alpha b + y\alpha b = (c\gamma(x\alpha)^{m-1}x + y)\alpha b = y'\alpha b$. Thus, $c\gamma(x\alpha)^{m-1}x\alpha b \in I$, since I is $k-$ideal. Again,

$c\gamma(x\alpha)^{m-1}x\alpha z = c\gamma(x\alpha)^{m-1}x\alpha(j + t\beta' b) = c\gamma(x\alpha)^{m-1}x\alpha j + t\beta'(c\gamma(x\alpha)^{m-1}x\alpha b) \in I$, similarly, $c\gamma(x\alpha)^{m-1}x\alpha z' \in I$, since $(c\gamma d)\gamma'[(x\alpha)^{m+n-1}x] + [c\gamma(x\alpha)^{m-1}x]\alpha z = [c\gamma(x\alpha)^{m-1}x]\alpha(d\gamma'(x\alpha)^{n-1}x + z) = c\gamma(x\alpha)^{m-1}x\alpha z'$, Thus, $c\gamma d(\gamma'(x\alpha)^{m+n-1}x) \in I$. But $c\gamma d \in R \setminus P$, which contradicts the definition of I. Hence, each element of P is a zero divisor in R. □

4. Primary Ideals and Weakly Noetherian Γ−Semirings

In this section, we define $P-$ primary ideal and proved that finite intersection of $P-$ primary ideals is again a $P-$ primary ideal. Further we proved that radical of $k-$ ideal is equal to the finite intersection of prime $k-$ ideals. Finally, we prove that for weakly Noetherian $\Gamma-$ semiring with strong identity, each connected prime ideal of R is equal to $(0 : r)$, which is a $k-$ideal.

Theorem 4.1. *Let R be a commutative* $\Gamma-$ *semiring and P be primary ideal of R then* $r(P)$ *is prime.*

Proof. As $P \neq R$, therefore $1 \notin P$, thus $1 \notin r(P)$. Let us assume that $a\Gamma b \subseteq r(P)$ then $a\alpha b \in r(P)$ for any $a, b \in R$ and for any choice of $\alpha \in \Gamma$. Then for some $n \geq 1$, $((a\alpha b)\alpha)^{n-1}(a\alpha b) = ((a\alpha)^{n-1}a)\alpha((b\alpha)^{n-1}b) \in P$. But P is primary, therefore either $((a\alpha)^{n-1}a) \in P$ or there exists $m \geq 1$ with $((b\alpha)^{n-1}b)\alpha)^{m-1}((b\alpha)^{n-1}b) \in P$ or $(b\alpha)^{nm-1}b \in P$. Therefore either $a \in r(P)$ or $b \in r(P)$. So $r(P)$ is prime. □

Definition 4.2. The primary ideal J of a $\Gamma-$ semiring R is $P-$ primary if $P = r(J)$, where P is a prime ideal of R.

Theorem 4.3. *Let R be a* $\Gamma-$ *semiring. Let for a prime ideal P,* $J_1, J_2, ..., J_n$ *are P−primary ideals of R. Then* $\bigcap_{i=1}^{n} J_i$ *is P−primary.*

Proof. Let us assume that $u\Gamma v \subseteq \bigcap_{i=1}^{n} J_i = J$ and $u \notin J$. So there exist $k \in \{1, 2 \ldots, n\}$ such that $u \notin J_k$. But as $u\Gamma v \subseteq J$, we have $u\Gamma v \subseteq J_k$. So there exists $m_k \geq 1$, such that $(v\alpha)^{m_k-1}v \in J_k$, for all $\alpha \in \Gamma$, since $\bigcap_{i=1}^{n} J_i \subset J_k$ and J_k is primary. Thus, $v \in r(J_k) = P$. But all J_i's are $P-$ primary, so for each i, $v \in r(J_i)$. Therefore, for each J_i there exist $m_i \geq 1$, such that $(v\alpha)^{m_i-1}v \in J_i$, for all $\alpha \in \Gamma$. Let $m = \max\{m_i\}$, for $1 \leq i \leq n$, then $(v\alpha)^{m-1}v \in \bigcap_{i=1}^{n} J_i$. Thus $\bigcap_{i=1}^{n} J_i$ is primary. Incidentally we proved that $P \subseteq r(\bigcap_{i=1}^{n} J_i)$. But $r(\bigcap_{i=1}^{n} J_i) \subseteq r(J_k) = P$. So $r(\bigcap_{i=1}^{n} J_i) = P$. Hence, $\bigcap_{i=1}^{n} J_i$ is $P-$ primary. □

Definition 4.4. A $\Gamma-$ semiring R is said to be weakly noetherian if every ascending chain of ideals of R is ultimately stationary.

Theorem 4.5. *Let K denotes a radical k−ideal of a commutative* $\Gamma-$ *semiring R, then K is equal to the finite intersection of prime k− ideals.*

Proof. Let T be a maximal among all radical $k-$ideals which is not equal to any finite intersection of prime $k-$ideals. So, in particular $T \neq R$ and not a prime. Therefore we can find $a \notin T$ and $b \notin T$ such that $a\Gamma b \subseteq T$. Let $M = r(Sl(T + R\Gamma a))$ and $N = r(Sl(T + R\Gamma b))$. Then M and N are $k-$ ideals of R and $T \subset M$ and $T \subset N$. Therefore, $T \subset M \cap N$. Let $x \in M \cap N$, therefore $x \in M$ and $x \in N$. So we can find $y, y' \in T + R\Gamma a$ and $z, z' \in T + R\Gamma b$ such that $(x\alpha)^{m-1}x + y = y'$ and $(x\alpha)^{n-1}x + z = z'$. Let $y = t + c\beta a$ and $z = t' + d\beta' b$, where $c, d \in R$, $t, t' \in T$ and $\beta, \beta' \in \Gamma$. Then $b\alpha(x\alpha)^{m-1}x + b\alpha y = b\alpha y'$. But $b\alpha y = b\alpha(t + c\beta a) = b\alpha t + b\alpha c\beta a = b\alpha t + c\beta a\alpha b \in T$. So $b\alpha y \in T$. Similarly, $b\alpha y' \subseteq T$. Therefore, $b\alpha(x\alpha)^{m-1}x \in Sl(T) = T$. But then $((x\alpha)^{m+n-1}x)\alpha z = (x\alpha)^{m+n-1}x\alpha(t' + d\beta' b) = (x\alpha)^{m+n-1}x\alpha t' + d\beta'(x\alpha)^{n-1}x\alpha((x\alpha)^{m-1}x\alpha b)$, therefore $(x\alpha)^{m+n-1}x\alpha z \in T$. Similarly, $(x\alpha)^{m+n-1}x\alpha z' \in T$. Therefore, it follows that $(x\alpha)^{m+2n-1}x + (x\alpha)^{m+n-1}x\alpha z = (x\alpha)^{m+n-1}x\alpha z'$. So $(x\alpha)^{m+2n-1}x \in Sl(T) = T$. So $x \in r(T) = T$ and $T = M \cap N$. Thus, we get a contradiction. Hence, T is equal to the intersection of finite prime $k-$ideals. □

Corollary 4.6. *Let R be a weakly Noetherian $\Gamma-$ semiring and K is it's $k-$ ideal, then there exist prime $k-$ ideals $Q_1, Q_2, ..., Q_n$ of R such that* $r(K) = \bigcap_{i=1}^{n} Q_i$

Proof. As $r(K)$ is a $k-$ideal and radical, so result follows by theorem 4.5. □

Theorem 4.7. *Let R be a weakly Noetherian $\Gamma-$ semiring then any minimal prime ideal of R is finite.*

Proof. Let P be any minimal prime ideal of R, then by applying corollary 4.6 to $K = \{0\}$, we can find finite family of prime $k-$ideals, say $Q_1, Q_2, ..., Q_n$ such that $r(\{0\}) = Q_1 \cap Q_2 \cap ... \cap Q_n$. Let no Q_i is contained in P. Then for each $i \in \{1, 2, ..., n\}$, we may find $q_i \in Q_i$ such that $q_i \notin P$ for all i. So $q_1\Gamma q_2\Gamma ... \Gamma q_n \subseteq Q_1 \cap Q_2 \cap ... \cap Q_n \subseteq P$ with $q_i \notin P$. But this is a contradiction to the definition of P. So $Q_i \subseteq P$ for some i. Therefore $Q_i = P$. Hence, $\min(Prm(R)) \in \{Q_1, Q_2, ..., Q_n\}$. So it is finite. □

The following remark is followed by [6]

Remark 4.8. *Let R be a $\Gamma-$ semiring. It is obvious that R is weakly noetherian if and only if each k-ideal I is finitely generated as a k-ideal, that is, there is a finite family $(r_1, r_2 \ldots r_n)$ of elements of R such that $I = Sl(< r_1, r_2 \ldots r_n >)$, for that it is enough that each k-ideal be finitly generated as an ideal.*

Finally, we have

Theorem 4.9. *Let R be a weakly noetherian $\Gamma-$ semiring with strong identity 1_s, then each connected prime ideal of R is equal to $(0 : r)$, for some $r \in R \setminus \{0\}$. In particular, it is a k-ideal.*

PROOF. Let P be a prime ideal of R, which is connected to $x \in R \setminus \{0\}$, then $\theta_x(P) = Q$, where $Q \in Min(Prm(R'_x))$. Let us define $M(P) = \{z \in R|$ there exists $\alpha \in \Gamma$ such that $(0 : z\alpha x) \subseteq P\}$. For $y \in M(P)$, let $N_P(y) = \bigcup_{s \in R \setminus P} (0 : s\gamma(x\alpha y))$, for some $\alpha, \gamma \in \Gamma$. Now as for all $s, s' \in R \setminus P$, we have $s\beta s' \in R \setminus P$, for some $\beta \in \Gamma$ and $(0 : s\gamma x\alpha y) \cup (0 : s'\gamma(x\alpha y)) \subseteq (0 : (s\beta s)'\gamma(x\alpha y))$. Now as $N_P(y)$ is equal to the union of family of k-ideals, therefore it is a k-ideal itself. Now let $y \in M(P)$, so $(0 : x\alpha y) \subseteq P$ for any $\alpha \in \Gamma$. Then for $s \in R \setminus P$ and $z \in (0 : s\gamma(x\alpha y))$, for any $\alpha, \beta \in \Gamma$, we have $(s\beta z)\gamma(x\alpha y) = (s\gamma(x\alpha y))\beta z = 0$, for all $\beta \in \Gamma$. This implies that $s\beta z \in (0 : x\alpha y) \subseteq P$, $s\beta z \in P$ and $z \in P$ (as $s \notin P$). Thus, we have shown that $N_P(y) \subseteq P$. As R is weakly noetherian, therefore set $\{N_P(y) \mid y \in M(P)\}$ has a maximal element(the existence of such an element follows from the weak noetherianity hypothesis). Let $J = N_P(y')$ be the maximal element of this set. But from above $J \subseteq P$, hence $J \neq R$. Let $a\Gamma b \subseteq J$ and $a \notin J$, then for all $s \in R \setminus P$, $a \notin (0 : s\gamma(x\alpha y'))$, for all $\alpha, \beta, \gamma \in \Gamma$, we have $s\gamma(x\alpha y'\beta a) = (s\gamma x\alpha y')\beta a \neq 0$. Therefore, $(0 : x\alpha(y'\beta a)) \subseteq P$ or $y'\beta a \in M(P)$. But $N_P(y') \subseteq N_P(y'\beta a)$, for any choice of $\beta \in \Gamma$. Hence, $N_P(y') = N_P(y'\beta a)$ (as $N_P(y')$ is maximal). Now as $a\Gamma b \subseteq J = N_P(y')$, so there exists $s \in R \setminus P$ such that $(s\gamma(x\alpha y'))\beta(a\alpha' b) = 0$, for all $\alpha, \alpha', \beta, \gamma \in \Gamma$. But then $s\gamma(x\alpha y'\beta a)\alpha' b = 0$, where $b \in (0 : (s\gamma)x\alpha(y'\beta a)) \subseteq N_P(y'\beta a) = N_P(y') = J$. Therefore $a\Gamma b \in J$ implies that either $a \in J$ or $b \in J$. Thus, J is prime. Again, as $J \subseteq P$ and $(0 : x) \subseteq (0 : x\alpha y) \subseteq N_P(y') = J$, $\theta_x(J)$ is a prime ideal of R'_x and $\theta_x(J) \subseteq \theta_x(P) = Q$. So minimality of Q gives that $\theta_x(J) = Q$. Now if $u \in P$, then $\theta_x(u) \in \theta_x(P) = Q = \theta_x(J)$. Therefore, $\theta_x(u) = \theta_x(j)$ for some $j \in J$, then we have $t, t' \in (0 : x)$, so that $u + t = j + t'$, hence $u + t \in J$ or $u \in Sl(J) = J$ (as J is a k-ideal). Thus $P \subseteq J$ and $P = J = N_P(y')$. In particular, P is a k-ideal. Now as R is weakly noetherian , therefore there exists a finite family $(p_1 \ldots p_n)$ of elements of P such that $P = Sl(< p_1 \ldots p_n >)$. Now for each $p_i \in P = N_P(y')$, there is an $s_i \in R \setminus P$ such that $p_i \in (0 : s_i\gamma(x\alpha y'))$, for all $\alpha, \beta \in \Gamma$. Let $s_0 \in s_1\Gamma s_2\Gamma \ldots \Gamma s_n$ and $r = s_0\gamma(x\alpha y')$ then each $p_i \in (0 : r)$, where $P = Sl(< p_1 \ldots p_n >) \subseteq Sl((0 : r)) = (0 : r)$. Moreover, $s_0 \in R \setminus P$, therefore $(0 : r) = (0 : s_0\gamma(x\alpha y')0 \subseteq N_P(y') = P$. Hence, $P = (0 : r)$ □

COROLLARY 4.10. *Let R be a weakly noetherian $\Gamma-$ semiring with strong identity, then $Min(Prm(R)) = Min(Prm(R_k))$.*

PROOF. Let $P \in Min(Prm(R))$. Since by theorem 3.10, P is connected and by theorem 4.9, P isa $k-$ideal. So,$P \in Min(Prm(R_k))$. Conversely, if $P \in Min(Prm(R_k))$, then P is prime, hence by Zorn's Lemma it contains some minimal prime ideal, say P_0. Now P_0 is a k-ideal hence (as $P_0 \subseteq P$), $P = P_0 \in Min(Prm(R))$. □

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