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ON GENERALISED HILFER-TYPE FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH TWO POINT AND INTEGRAL BOUNDARY CONDITIONS

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Abstract

This paper is devoted to study the existence and uniqueness results of nonlinear generalized fractional integrodifferential equation with two point and integral boundary condition by using fixed point theorems. Further, an example is discussed to illustrate the theory.

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1. Introduction

In this paper we study the existence, uniqueness results of the class of boundary value problems for the following nonlinear Fractional Integrodifferential Equations

$$ {}_{H}D^{\alpha,\beta;h}u(\tau) = F\left(\tau, u(\tau)\int_{a}^{\tau} h(\tau,s)u(s)\,ds\right), \quad \tau \in J = [a,b] \tag{1.1}$$

$$ d_1u(a) + d_2\int_{a}^{b} h'(s,u(s))\,ds + d_3u(b) = d_4, \tag{1.2}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$. ${}_{H}D^{\alpha,\beta;h}$ is the $h-$ Hilfer fractional derivative of order α and type β, $F : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous $d_i \in \mathbb{R}(i = 1,2,3,4)$, $h'(\tau) \in C^1(J,\mathbb{R})$ be an increasing function with $h'(\tau) \neq 0$ for all $\tau \in J$.

The basic theory of fractional calculus and fractional differential equations has been given in excellent monographs by Kilbas et.al. [17], Podlubny [21] and Samko et.al. [22]. Integrodifferential equations arise in many engineering and scientific disciplines, often as approximation to partial differential equations, which represent much of the continuum phenomena. Many forms of these equations are possible see [1] and the references therein.

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Integral boundary conditions are encountered in population dynamics, blood flow models, chemical engineering, cellular systems, heat transmission, plasma physics, thermoelasticity, etc. Byszewski initiated the nonlocal condition proving the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [5] and Deng [6], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

Many recent papers have dealt with the existence, uniqueness and other properties of solutions of special forms of the equations (1.1) - (1.2), see [2–4, 8–12, 14–16, 18, 19] and some of the references cited therein. Recently, in an interesting paper [13], Kendre et.al. have investigated the existence, uniqueness and boundedness of solutions of special form of (1.1) - (1.2). The aim of the present paper is to prove the existence, uniqueness and boundedness of solution of nonlinear fractional integrodifferential equations (1.1) - (1.2). The main tools employed in our analysis are based on the theory of fractional calculus and fixed point theorems.

The important fact is that with the minimum assumptions on the function f, we have obtained various properties of solutions of the equations (1.1) - (1.2).

2. Preliminaries

We set notations and certain fundamental facts in this part, which will be used in the proofs of the following results.
Let $C(J,\mathbb{R})$ and $L(J,\mathbb{R})$ are the Banach space of continuous functions and Lebesgue integrable functions from J into $\mathbb{R}$ respectively with the norms,

$$\| u \|_\infty = \sup\{|u(\tau)| : \tau \in J\}, \text{ and } \| u \|_L = \int_a^b |u(\tau)|\, d\tau.$$

Definition 2.1. [1] Let $\alpha > 0$, and $g \in L^1(J, R)$ The following expression

$$I^{\alpha;h} g(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^\tau h'(t)(h(\tau) - h(t))^{\alpha-1} g(t)\, dt,$$

is called left sided h-RL fractional integral of order α.

Definition 2.2. [13] The $h-$Hilfer fractional derivative of order α and parameter β is defined by

$$H^{\alpha,\beta;h} g(\tau) = I^{\beta(n-\alpha);h} \left(\frac{1}{h'(\tau)} \frac{d}{d\tau}\right)^n I^{(1-\beta)(n-\alpha);h} g(\tau),$$

where $n - 1 < \alpha < n,\ 0 \le \beta \le 1, \tau > a$.

Lemma 2.3. *([1],[13]) Let α, x and $\delta > 0$. then*

1. $I^{\alpha;h} I^{x;h} g(\tau) = I^{\alpha+x;h} g(\tau)$,
2. $I^{\alpha;h} (h(\tau) - h(a))^{\delta-1} = \dfrac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (h(\tau) - h(a))^{\alpha+\delta-1}$.

Lemma 2.4. *[13] Let $g \in L(a,b)$, $\alpha \in (n-1,n]$, $(n \in N)$, $\beta \in [0,1]$, then*

$$(I^{\alpha;h} H^{D^{\alpha;\beta;h}} g)(\tau) = g(\tau) - \sum_{h=0}^{n} \frac{(h(\tau)-h(a))^{\gamma-h}}{\Gamma(\gamma-h+1)} g_h^{[n-h]} I^{(1-\beta)(n-\alpha);h} g(a)$$

where, $g_h^{[n-h]} = \left(\frac{1}{h'(\tau)}\frac{d}{d\tau}\right)^{[n-h]} g(\tau).$

Lemma 2.5. *Let $u \in C(J,\mathbb{R})$. Then the unique solution of the h-Hilfer type boundary value problem (BVP) (1.1) - (1.2) is given by*

$$u(\tau) = \frac{1}{\Lambda\Gamma(\gamma)} (h(\tau)-h(a))^{\gamma-1}$$
$$\times \left[d_4 - d_2 \int_a^b h'(s) I^{\alpha;h} F\Big(s, u(s), \int_a^s h(s,t)u(t)dt\Big) ds - d_3 I^{\alpha;h} F\Big(b, u(b), \int_a^b h(b,t)u(t)dt\Big)\right]$$
$$+ I^{\alpha;h} F\Big(\tau, u(\tau), \int_a^\tau h(\tau,s)u(s)\,ds\Big), \tag{2.1}$$

$$\text{where,}\quad \Lambda = \left[d_2 + \frac{\gamma d_3}{(h(b)-h(a))}\right] \frac{(h(b)-h(a))^\gamma}{\Gamma(\gamma+1)} \neq 0. \tag{2.2}$$

Proof. Let u be the solution of the first equation of h-Hilfer type BVP (1.1) - (1.2). Applying $I^{\alpha;h}$on the first equation of h-Hilfer type BVP (1.1) - (1.2) with Lemma 2.4 and setting $I^{1-\alpha;h}u(a) = C_0$ We obtain

$$u(\tau) = \frac{C_0}{\Gamma(\gamma)} (h(\tau)-h(a))^{\gamma-1} + I^{\alpha;h} F\Big(\tau, u(\tau), \int_a^\tau h(\tau,s)u(s)\,ds\Big) \tag{2.3}$$

where C_0 is an arbitrary constant. From the condition

$$d_1 u(a) + d_2 \int_a^b h'(s)u(s)\,ds + d_3 u(b) = d_4$$

we have

$$d_4 = d_1 u(a) + d_2 \int_a^b h'(s)u(s)\,ds + d_3 u(b)$$
$$= d_1.0 + d_2 \int_a^b h'(s) \left[\frac{C_0}{\Gamma(\gamma)}(h(s)-h(a))^{\gamma-1} + I^{\alpha;h} F\Big(\tau, u(\tau), \int_a^\tau h(\tau,s)u(s)\,ds\Big)\right]$$
$$+ d_3 \frac{C_0}{\Gamma(\gamma)}(h(b)-h(a))^{\gamma-1} + I^{\alpha;h} F\Big(b, u(b), \int_a^b h(b,s)u(s)\,ds\Big)$$

$$= \frac{C_0}{\Gamma(\gamma)}[d_2 \int_a^b h'(s)(h(s)-h(a))^{\gamma-1}\, ds + d_3(h(b)-h(a))^{\gamma-1}]$$

$$+ d_2 \int_a^b h'(s) I^{\alpha;h} F\Big(s, u(s), \int_a^s h(s,t)u(t)\, dt\Big)$$

$$+ d_3 I^{\alpha;h} F\Big(b, u(b), \int_a^b h(b,t)u(t)\, dt\Big)$$

$$= C_0\left[\frac{d_2}{\Gamma\gamma}\frac{(h(b)-h(a))^{\gamma}}{\gamma} + d_3 \frac{(h(b)-h(a))^{\gamma-1}}{\Gamma\gamma}\right]$$

$$+ d_2 \int_a^b h'(s) I^{\alpha;h} F\Big(s, u(s), \int_a^s h(s,t)u(t)\, dt\Big) ds + d_3 I^{\alpha;h} F\Big(b, u(b), \int_a^b h(b,t)u(t)\, dt\Big)$$

$$= C_0\left[d_2 + \frac{\gamma d_3}{(h(b)-h(a))}\right]\frac{(h(b)-h(a))^{\gamma}}{\Gamma(\gamma+1)}$$

$$+ d_2 \int_a^b h'(s) I^{\alpha;h} F\Big(s, u(s), \int_a^s h(s,t)u(t)\, dt\Big) ds + d_3 I^{\alpha;h} F\Big(b, u(b), \int_a^b h(b,t)u(t)\, dt\Big)$$

$$\therefore d_4 = \frac{1}{\Lambda}\Big[d_4 - d_2 \int_a^b h'(s) I^{\alpha;h} F\Big(s, u(s), \int_a^s h(s,t)u(t)\, dt\Big) ds$$

$$- d_3 I^{\alpha;h} F\Big(b, u(b), \int_a^b h(b,t)u(t)\, dt\Big)\Big].$$

Therefore $u(\tau) = \frac{1}{\Lambda\Gamma\gamma}(h(\tau)-h(a))^{\gamma-1}$

$$\times\left[d_4 - d_2 \int_a^b h'(s) I^{\alpha;h} F\Big(s, u(s), \int_a^s h(s,t)u(t)dt\Big)ds - d_3 I^{\alpha;h} F\Big(b, u(b), \int_a^b h(b,t)u(t)dt\Big)\right]$$

$$+ I^{\alpha;h} F(\tau, u(\tau), \int_a^\tau h(\tau,s)u(s)\, ds)$$

This completes the proof. Here we can suffice to refer to Banach's fixed point theorem [23] and Krasnoselskii's fixed point theorem [24]. □

3. Existence and Uniqueness results

In this part, we demonstrate the results of the Existence and Uniqueness of the h-Hilfer type BVP (1.1) - (1.2) by employing Banach's fixed point theorem and Krasnoselskii's fixed point theorems. To obtain our main results, the following conditions must be satisfied.

(H_1) The function F is continuous and there exists $\lambda > 0$ such that

$$\begin{aligned}
&|F(\tau, u_1(\tau), \int_a^\tau h(\tau,s)u_1(s)ds) - F(\tau, u_2(\tau), \int_a^\tau h(\tau,s)u_2(s)ds)| \\
&\leq \lambda \left[\| u_1(\tau) - u_2(\tau) \| + \| \int_a^\tau h(\tau,s)u_1 ds - \int_a^\tau h(\tau,s)u_2(s)ds \| \right] \\
&\leq \lambda \left[\| u_1(\tau) - u_2(\tau) + \| \int_a^\tau (h(\tau,s)) \left[u_1(s) - u_2(s) \right] \, ds \| \right] \\
&\leq \lambda \left[\| u_1(\tau) - u_2(\tau) \| + \int_a^\tau |(h(\tau,s)| \, \| u_1(s) - u_2(s) \| \, ds \right] \\
&\leq \lambda \left[\| u_1 - u_2 \| + \int_a^\tau h_\tau \| u_1 - u_2 \| \; ds \right] \\
&\leq \lambda \left[\| u_1 - u_2 \| + h_\tau \| u_1 - u_2 \| \int_a^\tau ds \right] \\
&\leq \lambda \left[\| u_1 - u_2 \| + h_\tau (\tau - a) \| u_1 - u_2 \| \right] \\
&\leq \lambda \left[1 + h_\tau (\tau - a) \right] \| u_1 - u_2 \|
\end{aligned}$$

where $h_\tau = sup|h(\tau,s)|$, $\tau \in [a,b]$ and $a < \tau \leq b$ and $u_1, u_2 \in \mathbb{R}$

For convenience purpose,we are setting two constants:

$$\Upsilon_1 = \lambda \left[d_2 \frac{h((b) - h(a))^{\alpha+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\gamma+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)} \frac{h((b) - h(a))^{\alpha}}{\Gamma(\gamma+1)} + \frac{h((b) - h(a))^{\alpha}}{\Gamma(\gamma+1)} \right] (1 + h_b(b-a)) \tag{3.1}$$

$$\Upsilon_1 < 1 \tag{3.2}$$

THEOREM 3.1. *Assume that (H_1)holds.Then the h-Hilfer type BVP (1.1) - (1.2) has atleast one solution on J, provided that $\Upsilon_1 < 1$ Where Υ_1 was defined by (3.1).*

PROOF. In view of Lemma (2.5), we define operator $\pi : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$\begin{aligned}
(\pi u)(\tau) = &\frac{1}{\Lambda\Gamma y} (h(\tau) - h(a))^{\gamma-1} \times \left[d_4 - d_2 \int_a^b h'(s) I^{\alpha;h} F(s, u(s), \int_a^s h(s,t)u(t)dt) ds \right. \\
&\left. - d_3 I^{\alpha;h} F(\tau, u(\tau), \int_a^\tau h(\tau,t)u(t)dt)(b) \right] + I^{\alpha;h} F(\tau, u(\tau), \int_a^\tau h(\tau,t)u(t)\,dt)
\end{aligned}$$

Consider the closed ball by $B_\delta = \{u\varepsilon C(J,\mathbb{R}) :\| u \|\leq \delta\}$ with $\delta \geq \dfrac{\gamma_2}{1-\gamma_1}$ where

$$\gamma_2 = N\left[d_2 \frac{(h((b)-h(a))^{\alpha+\gamma}}{\Lambda\Gamma\gamma\Gamma\gamma+2} + \frac{d_3}{\Lambda\Gamma\gamma}\frac{h((b)-h(a))^{\alpha}}{\Gamma\gamma+1} + \frac{h((b)-h(a))^{\alpha}}{\Gamma\gamma+1}\right]$$
$$+d_4 \frac{(h(b)-h(a))^{\gamma-1}}{\Lambda\Gamma\gamma}$$

and $N = max_{\tau\varepsilon J}|F(\tau,0,0)|$ Now we define the operator π_1, π_2 such that $\pi_1 + \pi_2 = \pi$ on B_δ as

$$\pi_1(u)(\tau) = I^{\alpha;h} F\Big(\tau, u(\tau), \int_{a^\tau} h(\tau,t)u(t)\,dt\Big)$$

$$\pi_2(u)(\tau) = \frac{1}{\Lambda\Gamma y}(h(\tau)-h(a))^{\gamma-1}$$
$$\times\left[d_4 - d_2\int_a^b h'(s) I^{\alpha;h} F\Big(s,u(s),\int_a^s h(s,t)u(t)dt\Big)ds - d_3 I^{\alpha;h} F\Big(\tau,u(\tau),\int_a^\tau h(\tau,t)u(t)dt\Big)(b)\right]$$

By using (H_1) we obtain

$$\begin{aligned}
&\left|F\Big(\tau,u(\tau),\int_a^\tau h(\tau,s)u(s)\,ds\Big)\right| \\
&\leq \left|F\Big(\tau,u(\tau),\int_a^\tau h(\tau,s)u(s)\,ds\Big) - F(\tau,0,0) + F(\tau,0,0)\right| \\
&\leq \lambda\left[\left|F\Big(\tau,u(\tau),\int_a^\tau h(\tau,s)u(s)\Big)ds\right] - F(\tau,0,0)\right|\right] + |F(\tau,0,0)| \\
&\leq \lambda\left[|u(\tau)| + \left|\int_a^\tau h(\tau,s)u(s))\,ds\right|\right] + N \\
&\leq \lambda\left[|u(\tau)| + \int_a^\tau h_\tau |u(s)|\,ds\right] + N \\
&\leq \lambda\left[|u(\tau)| + h_b \| u \| \int_a^\tau ds\right] + N \\
&\leq \lambda[\| u \| + h_b(\tau - a)] + N \\
&\leq \lambda[1 + h_b(b-a)] \| u \| + N \\
&\leq \lambda \| u \| (1 + h_b(b-a)) + N.
\end{aligned}$$

For any $u, u^* \in B_\delta$ we have,

$$\| (\Pi_1 u) + (\Pi_2 u) \|$$

$$\leq \sup_{\tau\in J} \frac{1}{\Lambda\Gamma(\gamma)} (h(\tau) - h(a))^{\gamma-1}$$

$$\times \Big[d_4 + d_2 \int_a^b h'(s) I^{\alpha;h} \Big| F\Big(s, u(s), \int_a^s h(s,t)u(t)dt\Big)\Big| ds + d_3 I^{\alpha;h} \Big| F\Big(\tau, u(\tau), \int_a^\tau h(\tau,t)u(t)dt\Big)(b)\Big|\Big]$$

$$+ I^{\alpha;h} |F\Big(\tau, u(\tau), \int_a^b h(\tau,t)u(t)\,dt\Big)\Big|.$$

$$\leq \sup_{\tau\in J} \frac{1}{\Lambda\Gamma(\gamma)} (h(\tau) - h(a))^{\gamma-1}$$

$$\times \Big[d_4 + d_2 \int_a^b h'(s) \frac{1}{\Gamma(\alpha)} \int_a^s h'(t)((h(t) - h(a))^{\alpha-1}) \Big| F\Big(s, u(s), \int_a^s h(s,t)u(t)dt\Big)\Big| dt\, ds$$

$$+ d_3 \frac{1}{\Gamma(\alpha)} \int_a^\tau h'(\tau)((h(\tau) - h(a))^{\alpha-1}) \Big| F\Big(\tau, u(\tau), \int_a^\tau h(\tau,t)u(t)dt\Big)(b)\Big|\Big]$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^\tau h'(s)((h(s) - h(a))^{\alpha-1}) \Big| F\Big(\tau, u(\tau), \int_a^\tau h(\tau,t)u(t)\,dt\Big)\Big|.$$

$$\leq \sup_{\tau\in J} \frac{1}{\Lambda\Gamma\gamma} (h(\tau) - h(a))^{\gamma-1}$$

$$\times \Big[d_4 + d_2 \int_a^b h'(s) \frac{1}{\Gamma(\alpha)} \frac{(h(s) - h(a))^{\alpha}}{\alpha} (\lambda \| u \| (1 + h_\tau(\tau - a)) + N)$$

$$+ \left(d_3 \frac{(h(b) - h(a))^{\alpha}}{\Gamma(\alpha)\alpha} \right) (\lambda |u(\tau)| (1 + h_\tau(\tau - a)) + N) \Big] + \left(\frac{((h(b) - h(a))^{\alpha}}{\Gamma(\alpha)\alpha} \right)$$

$$\times (\lambda |u(\tau)| (1 + h_\tau(\tau - a)) + N)]$$

$$\leq d_4 \frac{(h(b) - h(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}$$

$$+ \left\{ d_2 \frac{(h(b) - h(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \frac{(h(b) - h(a))^{\alpha+1}}{\Gamma(\alpha+2)} + d_3 \frac{(h(b) - h(a))^{\alpha+\gamma-1}}{\Lambda\Gamma(\gamma)} \right\}$$

$$\times (\lambda \| u \| (1 + h_b(b - a) + N)) + \frac{(h(b) - h(a))^{\alpha}}{\Gamma(\alpha+1)} (\lambda \| u \| (1 + h_b(b - a) + N))$$

$$\leq (\lambda \| u \| (1 + h_b(b - a) + N) \left[d_2 \frac{(h(b) - h(a))^{\alpha+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+2)} + d_3 \frac{(h(b) - h(a))^{\alpha}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+1)} \right]$$

$$+ d_4 \frac{(h(b) - h(a))^{\gamma-1}}{\Lambda\Gamma(\gamma} + \frac{(h(b) - h(a))^{\alpha}}{\Gamma(\alpha+1)} (\lambda \| u \| (1 + h_b(b - a) + N))$$

$$\leq (\lambda \parallel u \parallel (1 + h_b(b-a)) + \left[d_2 \frac{(h(b)-h(a))^{\alpha+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+2)} + d_3 \frac{(h(b)-h(a))^{\alpha}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{(h(b)-h(a))^{\alpha}}{\Gamma(\alpha+1)}\right]$$
$$+ N\left[d_2 \frac{(h(b)-h(a))^{\alpha+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+2)} + d_3 \frac{(h(b)-h(a))^{\alpha}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+1)(h(b)-h(a))^{\alpha}}\Gamma(\alpha+1)\right]$$
$$+ d_4 \frac{(h(b)-h(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}$$
$$\leq (\lambda\delta(1 + h_b(b-a)) + \left[d_2 \frac{(h(b)-h(a))^{\alpha+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+2)} + d_3 \frac{(h(b)-h(a))^{\alpha}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{(h(b)-h(a))^{\alpha}}{\Gamma(\alpha+1)}\right]$$
$$+ N\left[d_2 \frac{(h(b)-h(a))^{\alpha+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+2)} + d_3 \frac{(h(b)-h(a))^{\alpha}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+1)(h(b)-h(a))^{\alpha}}\Gamma(\alpha+1)\right]$$
$$+ d_4 \frac{(h(b)-h(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}.$$

$\delta\Upsilon_1 + \Upsilon_2 \leq \delta$. This shows that $\Pi_1 u + \Pi_2 u \in B_\delta$.
Next, due to continuity of F, we conclude that Π_1 is continuous too. Also Π_1 is uniformly bounded on B_δ as

$\parallel \Pi_1 u \parallel \leq \frac{(h(b)-h(a))^{\alpha}}{\Gamma(\alpha+1)} \lambda\delta(1 + h_b(b-a))$.

In addition we prove the compactness of Π_1 as follows. Let $\tau_1, \tau_2 \in J$ such that $\tau_l < \tau_2$

$$|(\Pi_1 u)(\tau_2) - (\Pi_1 u)(\tau_1)|$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_a^{\tau_1} h'(s)\Big[\Big((h(\tau_2) - h(s))^{\alpha-1} - (h(\tau_1) - h(s))^{\alpha-1}\Big)\left|F\Big(s, u(s), \int_a^s h(s,t)u(t)\Big)dt\right| ds$$
$$+ \int_{\tau_1}^{\tau_2} h'(s)(h(\tau_2) - h(s))^{\alpha-1}\Big]\left|F\Big(s, u(s), \int_a^s h(s,t)u(t)\Big)dt\right| ds$$
$$\leq \frac{1}{\Gamma(\alpha)}\left|\frac{h(\tau_2) - h(\tau_1))^{\alpha}}{\alpha} - \frac{h((\tau_2) - (h(a))^{\alpha}}{\alpha}(\lambda\delta(1 + h_b(b-a) + N)\right.$$
$$+ \frac{-((h(\tau_2) - h(\tau_1))^{\alpha}}{\alpha}(\lambda\delta(1 + h_b(b-a) + N)$$
$$\left. + \frac{((h(\tau_1) - h(a))^{\alpha}}{\alpha}(\lambda\delta(1 + h_b(b-a) + N)\right|$$
$$\leq \frac{(\lambda\delta(1 + h_b(b-a)) + N)}{\Gamma(\alpha+1)}[|h(\tau_2) - h(a))^{\alpha} - h(\tau_1) - h(a))^{\alpha}|].$$

The last inequality with $\tau_2 - \tau_1 \to 0$ gives $|(\Pi_1 u)(\tau_2) - (\Pi_1 u)(\tau_1)|] \to 0$ as $\tau_2 \to \tau_1, u \in B_\delta$. Then Π_1 is relatively compact in B_δ. An application of the Arzel-Ascoli theorem,

$\Pi_1$is compact on B_δ.
Now we show that Π_2 is a contraction. Let $u, u^* \in B_\delta$ then, by (H_1) for $\tau \in J$ we have For any $u, u^* \in B_\delta$ we have,

$$\begin{aligned}
&|(\pi_2 u)(\tau) - (\pi_2)(u^*)(\tau)| \\
&\leq \sup_{\tau \in J} \frac{1}{\Lambda\Gamma\gamma}(h(\tau) - h(a))^{\gamma-1} \\
&\times \Bigg[d_4 + d_2 \int_a^b h'(s) I^{\alpha;h} \left| F\Big(s, u(s), \int_a^s h(s,t)u(t)dt\Big)) - F\Big(s, u^*(s), \int_a^s h(s,t)u^*(t)dt\Big) \right| ds \\
&+ d_3 I^{\alpha;h} \left| F\Big(\tau, u(\tau), \int_a^\tau h(\tau,t)u(t)dt\Big)(b) - F\Big(\tau, u^*(\tau), \int_a^\tau h(\tau,t)u^*(t)dt\Big) \right| ds \\
&+ I^{\alpha;h} \left| F\Big(\tau, u(\tau), \int_a^\tau h(\tau,t)u(t)dt\Big) - F\Big(\tau, u^*(\tau), \int_a^\tau h(\tau,t)u^*(t)dt\Big) \right| \Bigg] \\
&\leq \lambda \parallel u - u^* \parallel (1 + h_b(b-a)) \left[d_2 \frac{(h(b) - h(a))^{\alpha+\gamma}}{\lambda\Gamma(\gamma)(\Gamma(\alpha+2)} + d_3 \frac{((h(b) - h(a))^{\alpha}}{\lambda\Gamma(\gamma)\Gamma(\alpha+1)} \right].
\end{aligned}$$

Thus, Π_2 is a contraction in B_δ.Thus, all the conditions in Krasnoselskii's fixed point theorem are satisfied. So the h-Hilfer type boundary value problem (1.1)-(1.2) has atleast one solution in on J, provided that $\Upsilon_1 < 1$.

□

THEOREM 3.2. *Assume that (H_1) holds. Then the h-Hilfer type BVP (1.1)-(1.2) has unique solution on J, provided that $\Upsilon_1 < 1$ Where Υ_1 was defined by (3.1).*

PROOF. We shall show that Π has a unique fixed point by using Banach theorem in ([24]). By Theorem 3.1 we have $\parallel \Pi u \parallel \leq \parallel \Pi_1 u \parallel + \parallel \Pi_2 u \parallel \leq \delta$.

Thus $\Pi(B_\delta) \subset B_\delta$. Now we show that Π has a contraction. For $u, u^* \in B_\delta$ and $\tau \in J$ we have

$$\begin{aligned}
&\parallel (\Pi u) - (\Pi u^*) \parallel \\
\leq\ &\parallel (\Pi_1 u) - (\Pi_1 u^*) \parallel + \parallel (\Pi_2 u) - (\Pi_2 u^*) \parallel
\end{aligned}$$

$$\leq \sup_{\tau\in J} \frac{(h(\tau)-h(a))^{\gamma-1}}{\Lambda\Gamma y}$$

$$\times\Big\{\Big[d_4 + d_2\int_a^b h'(s)I^{\alpha;h}|F\Big(s,u(s),\int_a^s h(s,t)u(t)dt\Big) - F\Big(s,u^*(s),\int_a^s h(s,t)u^*(t)dt\Big)|\,ds$$

$$+ d_3 I^{\alpha;h}|F\Big(\tau,u(\tau),\int_a^\tau h(\tau,t)u(t)dt\Big) - F\Big(\tau,u^*(\tau),\int_a^\tau h(\tau,t)u^*(t)dt\Big)|\Big]$$

$$+ I^{\alpha;h}|F\Big(\tau,u(\tau),\int_a^\tau h(\tau,t)u(t)\,dt\Big) - F\Big(\tau,u^*(\tau),\int_a^\tau h(\tau,t)u^*(t)\,dt\Big)|\Big\}$$

$$\leq\| u-u^* \|\,(1+h_b(b-a))\left[d_2\frac{(h((b)-h(a))^{\alpha+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+2)} + \frac{d_3}{\Lambda\Gamma\gamma}\frac{h((b)-h(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{h((b)-h(a))^{\alpha}}{\Gamma(\alpha+1)}\right]$$

$$\leq \Upsilon_1 \| u-u^* \|,$$

which implies that $\| \Pi u - \Pi u^* \| \leq \Upsilon_1 \| u-u^* \|$ by (3.2) we realize that Π is a contraction. Then by Krasnoselskii theorem, the h Hilfer type BVP (1.1) - (1.2) has a unique solution on J.

□

3.1. Sepecial Cases According to our previous results,in this subsetion we present several special cases.

Case (1): If $h(\tau)=\tau$, then the h-Hilfertype BVP (1.1) - (1.2) is reduced to the following Hilfer type problem

$$H^{D^{\alpha,\beta}}u(\tau) = F\Big(\tau,u(\tau),\int_a^b h(\tau,s)u(s)\,ds\Big), \tau\in J := [a,b] \tag{3.3}$$

$$d_1u(a) + d_2\int_a^b u(s)\,ds + d_3u(b) = d_4. \tag{3.4}$$

where ${}_HD^{\alpha,\beta,\tau}$ is the Hilfer fractional derivaitve of order α, $F : J\times\mathbb{R}\to\mathbb{R}$ is continuous function, $d_i\in\mathbb{R}(i=1,2,3,4)$. The solution of the Hilfer type BVP (3.3) - (3.4) is given by

$$(u)(\tau) = \frac{1}{\Lambda\Gamma y}(\tau - a)^{\gamma-1}$$
$$\times\left[d_4 - d_2\int_a^b I^{\alpha}F\Big(s, u(s), \int_a^s h(s,t)u(t)dt\Big)ds - d_3 I^{\alpha;h}F\Big(b, u(b), \int_a^b h(b,t)u(t)dt\Big)\right]$$
$$+ I^{\alpha}F\Big(\tau, u(\tau), \int_a^{\tau} h(\tau,t)u(t)\,dt\Big).$$

where $\Lambda = \left[d_2 + \frac{\gamma d_3}{(b-a)}\right]\frac{(b-a)^{\gamma}}{\Gamma(\gamma+1)}$.
Then the following corollary is extracted from Theorem 3.2.

Corollary 3.3. *Assume that* (H_1) *is satisfied. Then the BVP (3.3)-(3.4) has a unique solution on J,provided that* $\Upsilon_1^* < 1$ *where*

$$\Upsilon_1^* = \lambda[1 + h_b(b-a)]\left[d_2\frac{(b-a)^{\alpha+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a))^{\alpha}}{\Gamma(\alpha+1)}\right].$$

Case 2: If $h(\tau) = \log\tau$, then the h-Hilfer type BVP (1.1) - (1.2) is reduced to the following Hilfer Hadamard type problem

$$_{H}D^{\alpha,\beta,\log(\tau)}u(\tau) = F\Big(\tau, u(\tau), \int_a^b h(\tau,s)u(s)\,ds\Big), \tau \in J := [a,b] \tag{3.5}$$

$$d_1u(a) + d_2\int_a^b u(s)\,ds + d_3u(b) = d_4 \tag{3.6}$$

where $_{H}D^{\alpha,\beta,\log(\tau)}$ is the Hilfer Hadamard fractional derivaitve of order α, $F : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous function, $d_i \in \mathbb{R}$ $(i = 1,2,3,4)$. The solution of the Hilfer type BVP (3.5)-(3.6) is given by

$$u(\tau) = \frac{\log(\frac{b}{a})^{\gamma-1}}{\Lambda\Gamma y}$$
$$\times\left[d_4 - d_2\int_a^b I^{\alpha\,\log\sigma}F\Big(s, u(s), \int_a^s h(s,t)u(t)dt\Big)ds - d_3 I^{\alpha;h}F\Big(b, u(b), \int_a^b h(b,t)u(t)dt\Big)\right]$$
$$+ I^{\alpha;\log\sigma}F\Big(\tau, u(\tau), \int_a^{\tau} h(\tau,t)u(t)\,dt\Big).$$

where $\Lambda = \left[d_2 + \frac{\gamma d_3}{\log(\frac{b}{a})}\right]\frac{\log(\frac{b}{a})^{\gamma}}{\Gamma(\gamma+1)} \neq 0.$

Then the following corollary is deduced from the Theorem 3.2.

Corollary 3.4. *Assume that* (H_1) *is satisfied. Then the BVP (3.5)-(3.6) has a unique solution on J,provided that* $\Upsilon_1^{**} < 1$ *where*

$$\Upsilon_1^{**} = \lambda[1 + h_b(b-a)]\left[d_2 \frac{\log(\frac{b}{a})^{\alpha+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+2)} + \frac{d_3 \log(\frac{b}{a})^{\alpha}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{\log(\frac{b}{a})^{\alpha}}{\Gamma(\alpha+1)}\right].$$

Case 3: If $h(\tau) = \tau^\rho, \rho > 0$ then the h-Hilfer type BVP (1.1) - (1.2) is reduced to the following Hilfer-Katugumpolel type problem

$$_HD^{\alpha,\beta,\tau^\rho}u(\tau) = F\Big(\tau, u(\tau), \int_a^\tau h(\tau,t)u(t)\,dt\Big), \tau \in J := [a,b] \tag{3.7}$$

$$d_1u(a) + d_2\int_a^b u(s)\,ds + d_3u(b) = d_4, \tag{3.8}$$

where $_HD^{\alpha,\beta,\tau^\rho}$ is the Hilfer -Katugumpolel fractional derivaitve of order $\alpha, \rho > 0$, $F : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function, $d_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$. The solution of the Hilfer Katugumpolel type BVP (3.7)-(3.8) is given by

$$\begin{aligned} u(\tau) &= \frac{(\tau^\rho - a^\rho)^\gamma - 1}{\Lambda\Gamma(\gamma)} \\ &\times \left[d_4 - d_2\int_a^b I^{\alpha:\tau^\rho}F\Big(s, u(s), \int_a^s h(s,t)u(t)dt\Big)ds - d_3I^{\alpha;\tau^\rho}F\Big(b, u(b), \int_a^b h(b,t)u(t)dt\Big)\right] \\ &+ I^{\alpha,\tau^\rho}F\Big(\tau, u(\tau), \int_a^\tau h(\tau,t)u(t)\,dt\Big). \end{aligned}$$

where $\Lambda = \left[d_2 + \frac{\gamma d_3}{(b^\rho - a^\rho)}\right]\frac{(b^\rho - a^\rho)^\gamma}{\Gamma(\gamma+1)} \neq 0.$

Corollary 3.5. *Assume that* (H_1) *is satisfied. Then the BVP (3.7)-(3.8) has a unique solution on J, provided that* $\Upsilon_1^{***} < 1$ *where*

$$\Upsilon_1^{***} = \lambda[1 + h_b(b-a)]\left[d_2 \frac{(b^\rho - a^\rho)^{\alpha+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\alpha+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(b^\rho - a^\rho)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b^\rho - a^\rho))}{\Gamma(\alpha+1)}\right].$$

Example 3.1. *Consider the following problem*

$$_HD^{\frac{1}{3},\frac{5}{6};\sqrt{\tau+1}}u(\tau) = \left(\frac{\tau}{9}\right)\frac{|u(\tau)|}{1+|u(\tau)|} + \frac{1}{\sqrt{2}} + \frac{1}{6}\int_a^\tau \frac{1}{(1+\tau)^3}u(s)ds, \tau \in J := [a,b] \tag{3.9}$$

$$\frac{1}{10}u(1) + \frac{2}{5}\int_{1}^{\frac{3}{2}} \frac{1}{2\sqrt{\tau+1}}u(\tau)\,d\tau + \frac{5}{7}u\left(\frac{3}{2}\right) = 1 \tag{3.10}$$

where $\alpha = \frac{1}{3}, \beta = \frac{5}{6}, h(\tau) = \sqrt{\tau+1}, j = [1, \frac{3}{2}], d_1 = \frac{1}{10}, d_2 = \frac{2}{5}, d_3 = \frac{5}{7}, d_4 = 1$
From these settings ,we compute constants as $u = \frac{8}{9}, \Lambda = 0.89 \neq 0$. *For* $u_1, u_2 \in \mathbb{R}^+$, *we have*

$$\left|F\left(\tau, u_1, \int_a^\tau h(\tau, s)ds\right) - F\left(\tau, u_2, \int_a^\tau h(\tau, s)ds\right)\right|$$

$$= \left(\frac{\tau}{9}\right)\left|\frac{u_1}{1+u_1} - \frac{u_2}{1+u_2}\right| + \frac{1}{6}\frac{1}{(1+\tau)^3}\left|\frac{u_1}{1+u_1} - \frac{u_2}{1+u_2}\right|\int_a^\tau ds$$

$$\leq \left(\frac{\tau}{9}\right)|u_1 - u_2|$$

$$\leq \left(\frac{1}{6}\right)|u_1 - u_2|$$

Hence (H_1) *holds with* $\lambda = \frac{1}{6} > 0$. *Also, the condition (3.2) is fulfilled, i. e.* $\Upsilon_1 = 0.19125 < 1$ *Therefore, by applying Banach's fixed point theorem, we conclude that the problem (3.9)-(3.10) has a unique solution u.*

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