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SOLITONS ON PARA-SASAKIAN MANIFOLD WITH RESPECT TO THE SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract

The purpose of the present paper is to study some soliton types on a Para-Sasakian manifold for which the Ricci soliton is steady, shrinking, or expanding. We also discussed the geometrical properties of the Ricci solitons and Yamabe solitons. Also, we have studied the curvature properties of the Schouten-van Kampen connection on the n-dimensional Para Sasakian manifold.

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1. Introduction

In [23], Sato introduced the concept of the almost para-contact manifold. Further, in [24], Sato and K. Matsumoto obtained some special cases of an almost para-contact structure which were recognized as Para-Sasakian and Special Para-Sasakian manifolds. Adati and Matsumoto also gave some interesting results on Para-Sasakian and SP-Sasakian manifolds in [1]. The geometrical properties of the Para-Sasakian manifold have been extensively studied by several authors such as Sasaki et al [25], Shukla et al [26], K. Mondal and U.C. De [14], Yildiz et al[29], Matsumoto, Ianus, and Mihai[15], Ozgur[17], Adati and Miyazawa[2], and many others.

One of the most natural connections, the Schouten-van Kampen connection, is the pair of complementary distributions on a differential manifold bearing an affine connection [21]. This connection has been studied by several geometers on several almost contact(para) structures [4],[13],[28],[8],[16],[20], [22].

Hamilton familiarized the concept of the Ricci flow in [9][10], to get the canonical metric on a smooth manifold. The study of manifolds with positive curvature became easier after the introduction of Ricci flow. With the help of Ricci flow, Perelman proved the Poincare conjecture in [18][19]. The Ricci soliton is the limit of the solutions of the Ricci flow. In general, an almost Ricci soliton is a simplification of an Einstein metric. A Riemannian metric g on a Riemannian manifold M of dimension n is called

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an almost Ricci soliton if one can find a complete vector field Y on M, that satisfies the following equation,

$$L_Y g + 2S + 2\alpha g = 0 \tag{1.1}$$

where L refers the Lie derivative, S denotes the Ricci tensor and α is a smooth function. If α is a constant then the metric g satisfying (1.1), is said to be a Ricci soliton. A Ricci soliton is expanding, steady, or shrinking if $\alpha > 0$, $\alpha = 0$ and $\alpha < 0$ respectively. Further, in [5] Cho and Kimura introduced the idea of η-Ricci soliton. A Riemannian manifold M with Riemannian metric g is called an almost η-Ricci soliton if there occures a smooth vector field Y such that,

$$L_Y g + 2S + 2\alpha g + 2\beta\eta \otimes \eta = 0 \tag{1.2}$$

where α and β, both are smooth functions. On other side, if both the terms, α and β are constants, then the metric g is called a η-Ricci soliton. For a compact manifold of dimension 2 or 3, a Ricci soliton has constant curvature [9],[12]. Many geometers studied the Ricci solitons. For a detailed study, we refer to Hui et al[11], Derdzinski[7], and Chow and Knoof[6].

In [9], Hamilton introduced the concept of Yamabe flow. The Yamabe solitons are self-similar results of the Yamabe flow, which are moved by a one-parameter family of diffeomorphisms and generated by a static vector field Y on Riemannian manifold M. A triplet (g, Y, γ) is called an almost Yamabe soliton on a Riemannian manifold (M, g), if [3]

$$\frac{1}{2}(L_Y g) = (r - \gamma) \tag{1.3}$$

where, r refers to the scalar curvature of the manifold (M, g) and γ is a smooth function. If the value of γ is constant, then the almost Yamabe soliton converts to a Yamabe soliton. A Yamabe soliton is expanding, steady or shrinking if $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$ respectively. For manifold of dimension 2, Yamabe soliton and Ricci soliton coincide, but for dimensions greater than 2, they have distinct behaviors. Moreover, Einstein manifolds are always almost Yamabe solitons. Also, the Riemannian metric g is a Yamabe metric if the Riemannian manifold M is of constant scalar curvature.

We got the inspiration for the present paper from the above studies. In the present paper, we studied some types of Ricci solitons and Yamabe solitons on an n-dimensional Para-Sasakian manifold M, admitting the Schouten-van Kampen connection. This research paper is organized as follows:

Section 2 is keen to a brief account of the Para-Sasakian manifold and the Schouten-van Kampen connection. In section 3, we introduce the Schouten-van Kampen connection on the n-dim Para-Sasakian manifold and attain the terms for curvature tensor, Ricci tensor, Ricci operator, and scalar curvature. In section 4, we study the Ricci Solitons and η-Ricci solitons for n-dimensional Para-Sasakian manifold M admitting the Schouten-van Kampen connection, and in the end, section 5, deals with the study of Yamabe solitons on n-dim Para-Sasakian manifold M admitting the Schouten-van Kampen connection.

2. Preliminaries

An n-dimensional differentiable manifold M, is said to be an almost para-contact manifold if it acknowledges an almost para-contact structure (ϕ, ξ, η) containing a $(1, 1)$- tensor field ϕ, a vector field ξ, and a 1-form η satisfying[24]

$$\phi^2 X = X - \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta \circ \phi = 0 \tag{2.1}$$

Let g be a compatible Riemannian metric with (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.2}$$

Or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \xi) = \eta(X) \tag{2.3}$$

The fundamental 2-form F of the manifold M is defined by

$$F(X, Y) = g(X, \phi Y)$$

for all$X, Y \in TM$. Then M converts to an almost paracontact Riemannian manifold equipped with an almost paracontact Riemannian structure (ϕ, ξ, η, g).

An n-dimensional almost paracontact Riemannian manifold M is called a P-Sasakian manifold if it satisfies

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X \tag{2.4}$$

for all$X, Y \in TM$, where ∇ is the Levi-Civita connection of the Riemannain metric. Also,

$$(\nabla_X \eta)Y = g(X, \phi Y), \nabla_X \xi = -\phi X \tag{2.5}$$

In an n-dimensional Para-Sasakian manifold M, the Ricci tensor S and the Ricci operator Q satisfy

$$S(X, \xi) = -(n - 1)\eta(X), \tag{2.6}$$

$$Q\xi = -(n - 1)\xi, \tag{2.7}$$

An n-dimensional almost para-contact Riemannian manifold M is said to be an η-Einstein manifold if the Ricci tensor S satisfied the condition[14],

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then M is an Einstein manifold.

On the other hand, we have two naturally defined distributions in the tangent bundle TM of manifold M as follows,

$$H = ker\eta, V = span\xi.$$

Then we have $TM = H \oplus V$, $H \cap V = \{0\}$ and $H \perp V$. This decomposition allows one to define the Schouten Van Kampen connection $\tilde{\nabla}$ over an almost (para) contact metric structure. The Schouten-van Kampen connection $\tilde{\nabla}$ on an almost (para) contact metric manifold with respect to the Levi-Civita connection ∇ is defined by[27]

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi \tag{2.8}$$

Thus, with the help of the Schouten Van Kampen connection, many properties of some geometric objects connected with the distributions H, V can be characterized [27]. For example, g, ξ and η are parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla}\xi = 0$, $\tilde{\nabla}g = 0$, $\tilde{\nabla}\eta = 0$. Also, the torsion $\tilde{T}$ for $\tilde{\nabla}$ is defined by,

$$\tilde{T}(X, Y) = \eta(X)\nabla_Y \xi - \eta(Y)\nabla_X \xi + 2d\eta(X, Y)\xi$$

3. Para-Sasakian manifold with respect to the Schouten-van Kampen connection

Let M be a Para-Sasakian manifold, then by using (2.5) and (2.8), we can define M with respect to the Schouten-van Kampen connection such that,

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X + g(X, \phi Y)\xi \tag{3.1}$$

Also,

$$\tilde{\nabla}_X \xi = 0 \tag{3.2}$$

Let R and $\tilde{R}$ be the curvature tensors of Levi-Civita connection ∇ and Schoten-van Kampen connection $\tilde{\nabla}$ respectively, then

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{3.3}$$

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z \tag{3.4}$$

Then, by using the equations (3.1), (3.3) and (3.4), the curvature tensor of the Para-Sasakian manifold with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ is given by,

$$\begin{aligned} \tilde{R}(X, Y)Z = R(X, Y)Z &+ \eta(Z)(\eta(Y)X - \eta(X)Y) \\ &+ (g(Y, Z)\eta(X) - g(X, Z)\eta(Y))\xi \\ &- g(Z, \phi X)\phi Y + g(Z, \phi Y)\phi X \end{aligned} \tag{3.5}$$

From (3.5), we obtain the Ricci curvature with respect to the Schouten-van Kampen connection,

$$\tilde{S}(Y, Z) = S(Y, Z) + (n - 1)\eta(Y)\eta(Z) + g(Z, \phi Y)trace\phi \tag{3.6}$$

The Ricci operator $\tilde{Q}$ for Para-Sasakian manifold with respect to Schouten-van Kampen connection $\tilde{\nabla}$ is given by,

$$\tilde{S}(X,Y) = g(\tilde{Q}X,Y) \tag{3.7}$$

Now, from (3.6) and (3.7), we get

$$\tilde{Q}X = QX + (n-1)\eta(X)\xi + (\phi X)trace\phi \tag{3.8}$$

also, the constant curvature $\tilde{r}$ is given by,

$$\tilde{r} = r + n - 1 + (trace\phi)^2 \tag{3.9}$$

4. Ricci soliton types on an n-dimensional Para-Sasakian manifold with respect to the Schouten-van Kampen connection

In this section, we study Ricci soliton types on an n-dimensional Para-Sasakian manifold M with respect to the Schouten-van Kampen connection.

In an n-dimensional Para-Sasakian manifold M admits the Schouten-van Kampen connection is metric, i.e, $\tilde{\nabla}g = 0$ and the torsion tensor $\tilde{T} \neq 0$. Using (3.1), we get

$$(\tilde{L}_Y g)(X,Y) = g(\nabla_X Y, Z) + g(X, \nabla_Z Y) = (L_Y g)(X,Y) \tag{4.1}$$

Where L and $\tilde{L}$ are Lie derivatives on n-dimensional Para-Sasakian manifold M with respect to the Levi-Civita connection and Schouten-van Kampen connection respectively.

Now, we consider an almost Ricci soliton on an n-dimensional Para-Sasakian manifold M with respect to the Schouten-van Kampen connection. Now, from (1.1), we have

$$\tilde{L}_Y g + 2\tilde{S} + 2\alpha g = 0 \tag{4.2}$$

From (4.1) and (4.2),

$$g(\nabla_X Y, Z) + g(X, \nabla_Z Y) + 2\tilde{S}(X,Y) + 2\alpha g(X,Y) = 0$$

That is,

$$2\tilde{S}(X,Y) = -g(\nabla_X Y, Z) - g(X, \nabla_Z Y) - 2\alpha g(X,Y) \tag{4.3}$$

Now, putting $Y = \xi$ in (4.3) and using (2.3) and (2.5), we have

$$\tilde{S}(X,Z) = -\alpha g(X,Z) \tag{4.4}$$

Now, put the value from (3.6) in (4.4), we get

$$S(X,Z) = -(n-1)\eta(X)\eta(Z) - g(Z,\phi X)trace\phi - \alpha g(X,Z) \tag{4.5}$$

Conversely, let an n-dimensional Para-Sasakian manifold M be an Einstein manifold with respect to the Schouten-van Kampen connection. Then for $Y = \xi$ we have

$$\tilde{S}(X,Z) = -\lambda g(X,Z)$$

where λ is a constant. We have $(\tilde{L}_{\xi}g)(X, Z) = 0$, from (4.1) and (2.5). So we have,

$$(\tilde{L}_{\xi}g)(X, Z) + 2\tilde{S}(X, Z) + \alpha g(X, Z) = 2(\alpha + \lambda)g(X, Z) \tag{4.6}$$

Hence, from (4.6), it is clear that the manifold M admits the Ricci soliton if $\alpha + \lambda = 0$. Hence, we can state the following theorem:

THEOREM 4.1. *An n-dimensional Para-Sasakian manifold M is an Einstein manifold with respect to the Schouten-van Kampen connection if and only if M admits a Ricci soliton (ξ, α, g) with respect to the Schouten-van Kampen connection.*

Now, put $Z = \xi$ in (4.5), we get

$$S(X, \xi) = -(n + \alpha - 1)\eta(X) \tag{4.7}$$

From (2.6), we have,

$$\alpha\eta(X) = 0$$

As, $\eta(X) \neq 0$, so, $\alpha = 0$. Hence, we can state the following theorem:

THEOREM 4.2. *On an n-dim Para-Sasakian manifold, an almost Ricci soliton (ξ, α, g) is always steady with respect to the Schouten-van Kampen connection.*

From (4.4), we have,

$$\tilde{r} = -\alpha n$$

Hence, we have the theorem:

THEOREM 4.3. *The scalar curvature of an n-dimensional Para-Sasakian manifold M having an almost Ricci soliton (ξ, α, g) with respect to the Schouten-van Kampen connection is $\tilde{r} = -\alpha n$.*

Now, from (1.2) the η-Ricci soliton on an n-dim Para-Sasakian manifold with respect to the Schouten-van Kampen connection is given by

$$(\tilde{L}_Y g + 2\tilde{S} + 2\alpha g + 2\beta\eta \otimes \eta)(X, Z) = 0 \tag{4.8}$$

From (4.1) and (4.8),

$$g(\nabla_X Y, Z) + g(X, \nabla_Z Y) + 2\tilde{S}(X, Z) + 2\alpha g(X, Z) + 2\beta\eta(X)\eta(Z) = 0 \tag{4.9}$$

Putting the values from (2.3) and (2.5) in (4.9), we get

$$\tilde{S}(X, Z) = -\alpha g(X, Z) - \beta\eta(X)\eta(Z) \tag{4.10}$$

Thus, from (4.10) we have the following theorem:

THEOREM 4.4. *Let M be an n-dim Para-Sasakian manifold admitting an η-Ricci soliton, then M will be an η-Einstein manifold with respect to the Schouten-van Kampen connection.*

5. Yamabe soliton on para-sasakian manifold with respect to the schouten-van kampen connection

In this section, we studied the almost Yamabe soliton on an n-dimensional Para-Sasakian manifold admitting Schouten-van Kampen connection.
Now, we consider an n-dimensional Para-Sasakian manifold admitting an almost Yamabe soliton explained in (1.3) with respect to the Schouten-van Kampen connection. Then we have,

$$\frac{1}{2}(\tilde{L}_Y g)(X,Z) = (\tilde{r} - \gamma)g(X,Z) \tag{5.1}$$

From (4.1), we have,

$$\frac{1}{2}(L_Y g)(X,Z) = (\tilde{r} - \gamma)g(X,Z) \tag{5.2}$$

Now, from (3.9), we get

$$\frac{1}{2}(L_Y g)(X,Z) = (r + n - 1 + (trace\phi)^2 - \gamma)g(X,Z) \tag{5.3}$$

From (5.3), we can state the following theorem:

Theorem 5.1. *An almost Yamabe soliton (M, Y, γ, g) on an n-dim Para-Sasakian manifold is invariant with respect to the Schouten-van Kampen connection iff $n - 1 + (trace\phi)^2 = 0$.*

Now, from (5.1) and (4.1), we have

$$\frac{1}{2}(g(\nabla_X Y, Z) + g(X, \nabla_Z Y)) = (\tilde{r} - \gamma)g(X,Z) \tag{5.4}$$

Now, put $Y = \xi$ in (5.4), we get

$$\frac{1}{2}(g(\nabla_X \xi, Z) + g(X, \nabla_Z \xi)) = (\tilde{r} - \gamma)g(X,Z) \tag{5.5}$$

now, put the values from (2.3) and (2.5) in (5.5), we obtain,

$$\tilde{r} = \gamma \tag{5.6}$$

Hence, from (5.6), we can state the following theorem:

Theorem 5.2. *Let M be an n-dim Para-Sasakian manifold admitting an almost Yamabe soliton (M, Y, γ, g) with respect to the Schouten-van Kampen connection. The scalar curvature $\tilde{r}$ of M is equal to γ if and only if Y and ξ are pairwise colinear in TM.*

From the above theorem, we state the following corollaries:

Corollary 5.3. *An n-dim Para-Sasakian manifold M admitting a Yamabe soliton with respect to the Schouten-van Kampen connection is of constant scalar curvature with respect to the Schouten-van Kampen connection if Y is pointwise colinear with ξ.*

Corollary 5.4. *If an n-dim Para-Sasakian manifold M admits a Yamabe soliton with respect to the Schouten-van Kampen connection and Y is pointwise colinear with ξ, then the Riemannian metric g is Yamabe metric for manifold M.*

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