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A NEW KIND OF HERMITE INTERPOLATION USING NON-UNIFORM NODES ON THE UNIT CIRCLE

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Abstract

The aim of this research article is to investigate an interpolation problem that lies between Lagrange and Hermite .We have taken problem on the nodes obtained by projecting vertically the zeroes of the $(1-x^2)P_n^{(\alpha,\beta)}(x)$ onto the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial of degree n. We prove the regularity of the problem, give explicit forms and establish a convergence theorem for the same.

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1. Introduction

In recent decades, Lagrange and Hermite interpolation have drawn the attention of plenty of researchers and been the focus of numerous studies (see [1], [4], [5], [6], [7], [12] and [13]). The process of finding a polynomial which coincides with the continuous function at specific predetermined places, known as nodes and whose derivative corresponds with some but not all of the nodes of the nodal system is known as Lagrange-Hermite interpolation.

Convergence of sequence of interpolating polynomials to the interpolated function is discussed by many researchers in their reseach article. There are various ways to determine convergence such as mean square convergence, in terms of integral or uniform norm. G. Freud [8] defined L_p modulus of continuity for meseauring convergence of interpolatory polynomial. The set nodes plays an important role to determine the convergence of the interpolatory polynomial. S. Bahadur and Varun [2] have studied Hermite-Fejér interpolation and discussed their convegence. Various types of interpolatory polynomials and their convergence are discussed in [9], [11], [14] and [16].

The main concern of this paper is intermediate problem between the Lagrange interpolation and the Hermite interpolation. In the present paper, we investigate a Lagrange-Hermite interpolation on unit circle problem on the nodal system made up of projection of zeros of $P_n^{(\alpha,\beta)}(x)$ vertically onto the unit circle along with the unit circle's end points on the real line, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial of degree n.

The paper is set up as follows. Some important results are given in section (2) which

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will also be used in this research article later on. The interpolation problem and its regularity are introduced in section (3) and section (4) respectively. Explicit representation of the interpolatory polynomial is covered in section (5). Finding estimates is covered in section (6) , while the convergence theorem and its proof are covered in section (7). Conclusion of this research article is given in section (8).

2. PRELIMINARIES

This section includes the following well known results, which we shall use.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is,

$$(1-x^2)P_n^{(\alpha,\beta)''}(x)+[\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)'}(x)+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0, \quad (2.1)$$

where $x = \frac{1+z^2}{2z}$.

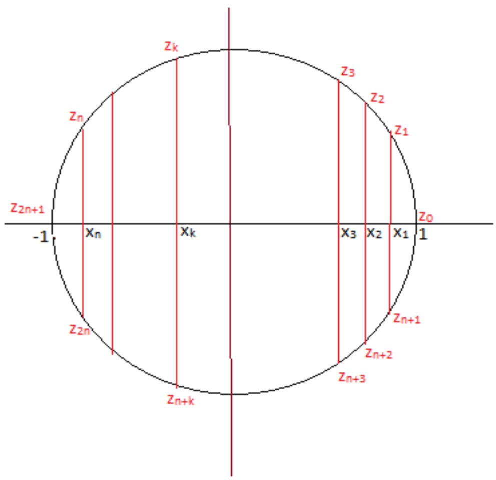

FIGURE 1. Nodal System Z_{2n}

Consider the set of nodes $\mathbb{Z}_{2n}$, which is produced by vertically projecting the zeros of $P_n^{(\alpha,\beta)}(x)$ on the unit circle.

$$\mathbb{Z}_{2n} = \{z_k = x_k + iy_k = \cos\theta_k + i\sin\theta_k\, ;\, z_{n+k} = \bar{z}_k\, ;\, k = 1, 2, ..., n;\ x_k,\, y_k \in R\}, \quad (2.2)$$

where, R is the set of real numbers.

$$\mathscr{M}(z) = \prod_{k=1}^{2n}(z - z_k) = K_n P_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n, \quad (2.3)$$

$$\mathbb{L}_k(z) = \frac{\mathscr{M}(z)}{(z-z_k)\mathscr{M}'(z_k)} \qquad ,k = 1,2,...,2n \tag{2.4}$$

For $-1 \le x \le 1$,

$$(1-x^2)^{1/2} \mid P_n^{(\alpha,\beta)}(x) \mid = O(n^{\alpha-1}), \tag{2.5}$$

$$\mid P_n^{(\alpha,\beta)}(x) \mid = O(n^{\alpha}). \tag{2.6}$$

Let $x_k = cos\theta_k\, , k = 1(1)n$ be the zeros of $P_n^{(\alpha,\beta)}(x)$, then

$$(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{2.7}$$

$$\mid P_n^{(\alpha,\beta)'}(x_k) \mid \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}. \tag{2.8}$$

We can write $z = x + iy$, where $x, y \in R$. If $|z| = 1$, then

$$|z^2 - 1| = 2\sqrt{1-x^2}, \tag{2.9}$$

$$\frac{1}{|z-z_k|} \le \frac{1-xx_k}{x-x_k}. \tag{2.10}$$

For more information see [15].

3. THE PROBLEM

Let $\{z_k\}_{k=0}^{2n+1}$ be set of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial of degree n (suppose, $z_o = 1$ and $z_{2n+1} = -1$). Here, we are interested in determining the convergence of interpolatory polynomial $\mathbb{LH}_n(z)$ of degree $\le 2n+1$ on the above said nodes and satisfying the conditions.

$$\begin{cases} \mathbb{LH}_n(f, z_k) = \nu_k & ; k = 0, 2n+1 \\ \mathbb{LH}_n'(f, z_k) = \mu_k & ; k = 1,2,...,2n, \end{cases} \tag{3.1}$$

where ν_k and μ_k are arbitrary complex constants.

4. REGULARITY

Theorem 4.1. *$\mathbb{LH}_n(z)$ is regular on $\{z_k\}_{k=0}^{2n+1}$.*

Proof. We only need to demonstrate that $\mathbb{LH}_n(z) \equiv 0$ is the only possible solution to the problem (3.1).

For $k = 1(1)2n$, let us consider

$$\mathbb{LH}_n'(z_k) = 0. \tag{4.1}$$

Since z_k's are the zeros of $\mathscr{M}(z)$, so we can write

$$\mathbb{LH}_n'(z) = a\mathscr{M}(z), \tag{4.2}$$

where a is a constant independent of n and z. On integrating (4.2), we get

$$\mathbb{LH}_n(z) = a\int_{-1}^{z} \mathscr{M}(u)du. \tag{4.3}$$

Now, since $\mathbb{LH}_n(z)$ satisfies the conditions given in (3.1), we have

$$\mathbb{LH}_n(1) = a\int_{-1}^{1} \mathscr{M}(u)du. \tag{4.4}$$

Since, $\int_{-1}^{1} \mathscr{M}(u)du \neq 0$. This implies $a = 0$.
Hence, the theorem follows. □

5. Explicit Representation of Interpolatory Polynomials

We shall write $\mathbb{LH}_n(z)$ satisfying (3.1),

$$\mathbb{LH}_n(z) = \sum_{k=0,2n+1} \nu_k\, \mathcal{X}_k(z) + \sum_{k=1}^{2n} \mu_k \mathcal{Y}_k(z), \tag{5.1}$$

where $\mathcal{X}_k(z)$ and $\mathcal{Y}_k(z)$ are unique polynomials, each of degree atmost $2n+1$ satisfying the conditions

$$\begin{cases} \mathcal{X}_k(z_j) = \delta_{kj} & ; j, k = 0, 2n+1, \\ \mathcal{X}_k'(z_j) = 0 & ; k = 0, 2n+1,\ \ j = 1, 2, ..., 2n \end{cases} \tag{5.2}$$

and

$$\begin{cases} \mathcal{Y}_k(z_j) = 0 & ; j = 0, 2n+1,\ \ k = 1, ..., 2n \\ \mathcal{Y}_k'(z_j) = \delta_{kj} & ; j, k = 1, 2, .., 2n. \end{cases} \tag{5.3}$$

THEOREM 5.1. *Fundamental polynomial $\mathcal{Y}_k(z)$ for $k = 1, 2, ..., 2n$ is given as*

$$\begin{aligned} \mathcal{Y}_k(z) \;&=\; (z^2-1)\mathbb{L}_k(z) - \int_{-1}^{z} (u^2-1)\mathbb{L}_k'(u)du \\ &+\; (1-2z_k)\int_{-1}^{z} \mathbb{L}_k(u)du - \frac{H_k(1)}{\int_{-1}^{1} \mathscr{M}(u)du} \int_{-1}^{z} \mathscr{M}(u)du. \end{aligned} \tag{5.4}$$

Proof. Let

$$\mathcal{Y}_k(z) = (z^2 - 1)\mathbb{L}_k(z) + H_k(z) + b\int_{-1}^{z} \mathcal{M}(u)du, \tag{5.5}$$

where $\mathcal{Y}_k(z)$ is atmost of degree $2n + 1$ satisfying the conditions given in (5.3), then

$$\begin{aligned}
\mathcal{Y}_k(1) &= H_k(1) + b\int_{-1}^{1} \mathcal{M}(u)du, \\
0 &= H_k(1) + b\int_{-1}^{1} \mathcal{M}(u)du, \\
b &= -\frac{H_k(1)}{\int_{-1}^{1} \mathcal{M}(u)du}.
\end{aligned} \tag{5.6}$$

On differentiating (5.5) with respect to z, we get

$$\mathcal{Y}'_k(z) = 2z\mathbb{L}_k(z) + (z^2 - 1)\mathbb{L}'_k(z) + H'_k(z) + b\mathcal{M}(z). \tag{5.7}$$

At $z = z_j$, we have

$$\mathcal{Y}'_k(z_j) = 2z_j\mathbb{L}_k(z_j) + (z_j^2 - 1)\mathbb{L}'_k(z_j) + H'_k(z_j). \tag{5.8}$$

On satisfying the second set of conditions given in (5.3) for $j \neq k$, we have

$$\begin{aligned}
0 &= (z_j^2 - 1)\mathbb{L}'_k(z_j) + H'_k(z_j), \\
H'_k(z_j) &= -(z_j^2 - 1)\mathbb{L}'_k(z_j).
\end{aligned}$$

Since, $\mathbb{L}_k(z_j) = 0$ for $j \neq k$, we can write

$$H'_k(z) = -(z^2 - 1)\mathbb{L}'_k(z) + c_k\mathbb{L}_k(z), \tag{5.9}$$

where c_k is a constant independent of n and z. Integrating above equation, we get

$$H_k(z) = -\int_{-1}^{z} (u^2 - 1)\mathbb{L}'_k(u)du + c_k\int_{-1}^{z} \mathbb{L}_k(u)du. \tag{5.10}$$

Now, to determine the constant c_k, we use the second conditions given in (5.3) for $j = k$,

$$\mathcal{Y}'_k(z_k) = 2z_k\mathbb{L}_k(z_k) + (z_k^2 - 1)\mathbb{L}'_k(z_k) + H'_k(z_k). \tag{5.11}$$

Substituting the value of $H'_k(z)$ from (5.9) at $z = z_k$ in (5.11), we get

$$c_k = 1 - 2z_k. \tag{5.12}$$

Hence, from (5.5), (5.6), (5.10) and (5.12), we have theorem 5.1 . □

Theorem 5.2. *For $k = 0, 2n + 1$*

$$\mathcal{X}_k(z) = (-1)^k \frac{\int_{-z_k}^{z} \mathcal{M}(u)du}{\int_{-1}^{1} \mathcal{M}(u)du}. \tag{5.13}$$

Proof. Consider

$$\mathcal{X}_k(z) = (-1)^k d_k \int_{-z_k}^{z} \mathcal{M}(u)du, \tag{5.14}$$

where $\mathcal{X}_k(z)$ is atmost of degree $2n + 1$. At $z = z_j$, we get

$$\mathcal{X}_k(z_j) = (-1)^k d_k \int_{-z_k}^{z_j} \mathcal{M}(u)du. \tag{5.15}$$

On satisfying the conditions in (5.2), we get

$$d_k = \frac{1}{\int_{-1}^{1} \mathcal{M}(u)du}.$$

Hence, the theorem follows. □

6. ESTIMATES OF FUNDAMENTAL POLYNOMIALS

We need to calculate estimates in order to obtain the convergence of interpolatory polynomials.

Lemma 6.1. *Let $\mathcal{Y}_k(z)$ be given by theorem 5.1, then*

$$\sum_{k=1}^{2n} |\ \mathcal{Y}_k(z)\ | = \begin{cases} O(\log\ n) & -1 < \alpha \le \frac{-1}{2}, \\ O(n^{\alpha+\frac{1}{2}} \log\ n) & \frac{-1}{2} < \alpha < 0, \\ O(n^{\alpha+\frac{1}{2}}) & \alpha \ge 0. \end{cases}$$

Proof. From (5.4), we have

$$\begin{aligned} \mathcal{Y}_k(z) &= (z^2-1)\mathbb{L}_k(z) - \int_{-1}^{z} (u^2-1)\mathbb{L}_k'(u)du \\ &+ (1-2z_k)\int_{-1}^{z} \mathbb{L}_k(u)du - \frac{H_k(1)}{\int_{-1}^{1} \mathcal{M}(u)du} \int_{-1}^{z} \mathcal{M}(u)du. \end{aligned} \tag{6.1}$$

On taking modulus both the sides, we get

$$\begin{aligned} |\mathcal{Y}_k(z)| &\le \underbrace{|(z^2-1)\mathbb{L}_k(z)|}_{M_1} + \underbrace{\left| \int_{-1}^{z} (1-u^2)\mathbb{L}_k'(u)du + (1-2z_k)\int_{-1}^{z} \mathbb{L}_k(u)du \right|}_{M_2} \\ &+ \underbrace{\left| \frac{H_k(1)}{\int_{-1}^{1} \mathcal{M}(u)du} \int_{-1}^{z} \mathcal{M}(u)du \right|}_{M_3}. \end{aligned} \tag{6.2}$$

Taking summation on both the sides, we get

$$\sum_{k=1}^{2n} |\mathcal{Y}_k(z)| \le \sum_{k=1}^{2n} M_1 + \sum_{k=1}^{2n} M_2 + \sum_{k=1}^{2n} M_3. \tag{6.3}$$

From (6.2), we have

$$M_1 = |(z^2-1)\mathbb{L}_k(z)|.$$

Since, $|z_k| = 1$ and using (2.3) and (2.4), we get

$$M_1 = \frac{|(z^2-1)||P_n^{(\alpha,\beta)}(x)||z^n|}{|(z-z_k)||P_n^{(\alpha,\beta)'}(x_k)|}.$$

Using (2.9) and (2.10), we get

$$M_1 \leq \frac{2\sqrt{1-x^2}\,|P_n^{(\alpha,\beta)}(x)|\,\sqrt{1-xx_k}}{|x-x_k|\,|P_n^{(\alpha,\beta)'}(x_k)|},$$

For $| x - x_k | \geq \frac{1}{2} | 1 - x_k^2 |$, we have

$$\begin{aligned} M_1 &\leq \frac{4\sqrt{1-x^2}\,|P_n^{(\alpha,\beta)}(x)|\,\sqrt{1-xx_k}}{|1-x_k^2|\,|P_n^{(\alpha,\beta)'}(x_k)|}, \\ &\leq \frac{8\sqrt{1-x^2}\,|P_n^{(\alpha,\beta)}(x)|}{|1-x_k^2|\,|P_n^{(\alpha,\beta)'}(x_k)|}. \end{aligned}$$

Using (2.5), (2.7) and (2.8), we get

$$M_1 = O\left(\frac{n^{\alpha-1}}{(k^2n^{-2})(k^{-\alpha-3/2}n^{\alpha+2})}\right).$$

$$M_1 = O(n^{-1}k^{\alpha-\frac{1}{2}}). \tag{6.4}$$

The estimate remains the same in the case, where $| x - x_k | < \frac{1}{2} | 1 - x_k^2 |$. On taking summation , we get

$$\sum_{k=1}^{2n} M_1 = \sum_{k=1}^{2n} O(n^{-1}k^{\alpha-\frac{1}{2}}). \tag{6.5}$$

Now, to determine M_2, let us first determine $\mathbb{L}_k'(z)$. From (2.4), we have

$$\mathbb{L}_k(z) = \frac{\mathscr{M}(z)}{(z-z_k)\mathscr{M}'(z_k)}.$$

$$\mathbb{L}_k(z)(z-z_k)\mathscr{M}'(z_k) = \mathscr{M}(z).$$

Differentiating above equation with respect to z, we get

$$\mathbb{L}_k'(z)(z-z_k)\mathscr{M}'(z_k) + \mathbb{L}_k(z)\mathscr{M}'(z_k) = \mathscr{M}'(z).$$

$$\mathbb{L}_k'(z) = \frac{\mathscr{M}'(z) - \mathbb{L}_k(z)\mathscr{M}'(z_k)}{(z-z_k)\mathscr{M}'(z_k)}.$$

Taking Modulus on both the sides, we get

$$|\mathbb{L}_k'(z)| \leq \left|\frac{\mathscr{M}'(z)}{(z-z_k)\mathscr{M}'(z_k)}\right| + \left|\frac{\mathbb{L}_k(z)}{(z-z_k)}\right|, \tag{6.6}$$

$$|\mathbb{L}_k'(z)| \leq \frac{|2nP_n^{(\alpha,\beta)}(x)z^{n-1} + P_n^{(\alpha,\beta)'}(x)(z^2-1)z^{n-2}|}{|(z-z_k)(z_k^2-1)z_k^{n-2}P_n^{(\alpha,\beta)'}(x_k)|} + \frac{2|P_n^{(\alpha,\beta)}(x)z^n|}{|(z-z_k)^2(z_k^2-1)P_n^{(\alpha,\beta)'}(x_k)\, z_k^{n-2}|}. \tag{6.7}$$

From (6.2), we have

$$M_2 = \left|\int_{-1}^{z}(1-u^2)\mathbb{L}_k'(u)du + (1-2z_k)\int_{-1}^{z}\mathbb{L}_k(u)du\right|,$$

$$M_2 \leq \underbrace{\int_{-1}^{z} |\,(u^2-1)\,||\,\mathbb{L}_k'(u)\,|\,du}_{I_1} + (1+2\,|\,z_k\,|)\underbrace{\int_{-1}^{z}|\,\mathbb{L}_k(u)\,|\,du}_{I_2}. \tag{6.8}$$

$$I_1 \leq \frac{|1-z^2|}{|z-z_k||P_n^{(\alpha,\beta)}(x_k)|\,|z_k^2-1||z_k^{n-2}|}\left\{|z^2-1|\,|P_n^{(\alpha,\beta)'}(x)|\left|\int_{-1}^{z}u^{n-2}du\right| + n|p_n^{(\alpha,\beta)}(x)|\left|\int_{-1}^{z}u^{n-1}du\right|\right\} + \frac{|1-z^2||P_n^{(\alpha,\beta)}(x)|}{|(z-z_k)^2|\,|P_n^{(\alpha,\beta)'}(x_k)|\,|z_k^2-1|\,|z_k^{n-2}}\left|\int_{-1}^{z}u^n du\right|.$$

Since, $|z| = 1$ and $|z_k| = 1$. Using (2.9) and (2.10), we get

$$I_1 \leq \frac{\sqrt{1-x^2}\,\sqrt{1-xx_k}}{|x-x_k|\,|P_n^{(\alpha,\beta)'}(x_k)|\,\sqrt{1-x_k^2}}\left\{\frac{4\sqrt{1-x^2}|P_n^{(\alpha,\beta)'}(x)|}{n-1} + 2n^{\alpha}\right\} + \frac{\sqrt{1-x^2}|P_n^{(\alpha,\beta)}(x)|(1-xx_k)}{|x-x_k|^2|P_n^{(\alpha,\beta)'}(x_k)|\,\sqrt{1-x_k^2}}.\frac{2}{n+1}.$$

For $|\,x-x_k\,| \geq \frac{1}{2}\,|\,1-x_k^2\,|$ and using (2.5), (2.8) and (2.10), we have

$$I_1 \leq \frac{8n}{(n-1)k^{-\alpha+\frac{1}{2}}} + \frac{2}{k^{-\alpha+\frac{1}{2}}} + \frac{n}{n+1}\frac{2}{k^{-\alpha+\frac{3}{2}}} \tag{6.9}$$

From (6.8)

$$I_2 \leq \frac{2|P_n^{(\alpha,\beta)}(x)|}{|z_k^2-1|\,|P_n^{(\alpha,\beta)'}(x_k)|\,|z_k^{n-2}|}\left|\int_{-1}^{z}u^n du\right|.$$

For $| x - x_k | \geq \dfrac{1}{2} | 1 - x_k^2 |$ and using (2.6), (2.8)

$$I_2 \leq \frac{4}{k^{-\alpha-\frac{1}{2}}\, n(n+1)}. \tag{6.10}$$

Combining (6.9) and (6.10) we can write (6.8) as

$$M_2 \leq \frac{8n}{(n-1)k^{-\alpha+\frac{1}{2}}} + \frac{2}{k^{-\alpha+\frac{1}{2}}} + \frac{n}{n+1}\frac{2}{k^{-\alpha+\frac{3}{2}}} + \frac{4}{k^{-\alpha-\frac{1}{2}}\, n(n+1)},$$

$$\sum_{k=1}^{2n} M_2 = \sum_{k=1}^{2n} O\Big(\frac{1}{k^{-\alpha+\frac{1}{2}}}\Big). \tag{6.11}$$

Also

$$M_3 = \left| H_k(1) \frac{\int_{-1}^{z} \mathcal{M}(u)du}{\int_{-1}^{1} \mathcal{M}(u)du} \right|.$$

Using (2.3), (2.4) and (5.10), we have

$$\sum_{k=1}^{2n} M_3 = \sum_{k=1}^{2n} O\Big(\frac{1}{k^{-\alpha+\frac{1}{2}}}\Big). \tag{6.12}$$

Using (6.5), (6.11) and (6.12) in (6.3), lemma follows. □

LEMMA 6.2. *Let $\mathcal{X}_k(z)$ be given by theorem 5.2, then*

$$\sum_{k=0,2n+1} | \mathcal{X}_k(z) | = O(1).$$

PROOF.

$$\sum_{k=0,2n+1} | \mathcal{X}_k(z) | = | \mathcal{X}_0(z) | + | \mathcal{X}_{2n+1}(z) | .$$

$$| \mathcal{X}_0(z) | \leq \frac{1}{| \int_{-1}^{1} \mathcal{M}(u)du |} \Big| \int_{-1}^{z} \mathcal{M}(u)du \Big| .$$

Using (2.3), we have

$$| \mathcal{X}_0(z) | = O(1). \tag{6.13}$$

Similarly, we have

$$| \mathcal{X}_{2n+1}(z) | = O(1). \tag{6.14}$$

Combining (6.13) and (7.1), we get lemma 6.2. □

7. CONVERGENCE

THEOREM 7.1. *Let $f(z)$ be a function continuous on closed unit disk and analytic on open unit disk. Let the arbitrary numbers μ_k's be such that*

$$|\mu_k| = O(n\,\omega_r(f, n^{-1})) \qquad ; k = 1, 2, ..., 2n. \tag{7.1}$$

Then, the sequence of interpolatory polynomial $\{\mathbb{LH}_n(z)\}$ defined by

$$\mathbb{LH}_n(z) = \sum_{k=0,2n+1} f(z_k)\mathcal{X}_k(z) + \sum_{k=1}^{2n} \mu_k \mathcal{Y}_k(z), \tag{7.2}$$

satisfies the relation

$$|\mathbb{LH}_n(z) - f(z)| \leq \begin{cases} O(\omega_r(f, n^{-1}) \log n) & -1 < \alpha \leq \frac{-1}{2} \\ O(\omega_r(f, n^{-1})\, n^{\alpha+\frac{3}{2}} \log n) & \frac{-1}{2} < \alpha < 0 \\ O(\omega_r(f, n^{-1})\, n^{\alpha+\frac{3}{2}}) & \alpha \geq 0 \end{cases} \tag{7.3}$$

where $\omega_r(f, n^{-1})$ be the r^{th} modulus of continuity of $f(z)$.

Remark : Let $f(z)$ be a function continuous on closed unit disk and analytic on open unit disk and $f' \epsilon\, Lip\, q$, $q > 0$, then the sequence $\{\mathbb{LH}_n(z)\}$ converges uniformly to $f(z)$ on closed unit disk , which follows from (7.3) as

$$\omega_r(f, n^{-1}) = O(n^{-r-q+1}) \qquad \{q > \alpha - r + \frac{5}{2}\} \tag{7.4}$$

where $\omega_r(f, n^{-1})$ be the r^{th} modulus of continuity of $f(z)$, To prove the Theorem 7.1, we shall need following.

Let $f(z)$ be a function continuous on closed unit disk and analytic on open unit disk. Then, there exists a polynomial $H_n(z)$ of degree $\leq 2n + 1$ satisfying **Jackson's** inequality.

$$|f(z) - H_n(z)| \leq C\,\omega_r(f, n^{-1}), \qquad z = e^{i\theta}(0 \leq \theta < 2\pi) \tag{7.5}$$

and also an inequality by **O.Kiŝ**

$$|H_n^{(m)}(z)| \leq C\, n^m\, \omega_r(f, n^{-1}) \qquad , m\, \varepsilon\, \mathbb{Z}^+ \tag{7.6}$$

where C is a constant independent of n and z.

PROOF. Since, $\mathbb{LH}_n(z)$ be the uniquely determined polynomial of degree $\leq 2n + 1$ and the polynomial $H_n(z)$ satisfying (7.5) and (7.6) can be expressed as

$$H_n(z) = \sum_{k=0,2n+1} H_n(z_k)\mathcal{X}_k(z) + \sum_{k=1}^{2n} H_n'(z_k)\mathcal{Y}_k(z). \tag{7.7}$$

Then, we can write

$$|\mathbb{LH}_n(z) - f(z)| \leq |\mathbb{LH}_n(z) - H_n(z)| + |H_n(z) - f(z)|\,.$$

Using (7.2) and (7.7), we have

$$\begin{aligned} |\,\mathbb{LH}_n(z) - f(z)\,| &\leq \sum_{k=0,2n+1} |\,f(z_k) - H_n(z_k)\,||\,\mathcal{X}_k(z)\,| \\ &+ \sum_{k=1}^{2n} |\,\mu_k - H_n'(z_k)\,||\,\mathcal{Y}_k(z)\,| + |\,H_n(z) - f(z)\,|, \\ |\,\mathbb{LH}_n(z) - f(z)\,| &\leq \underbrace{\sum_{k=0,2n+1} |\,f(z_k) - H_n(z_k)\,||\,\mathcal{X}_k(z)\,|}_{A_1} + \underbrace{\sum_{k=1}^{2n} |\,\mu_k\,||\,\mathcal{Y}_k(z)\,|}_{A_2} \\ &+ \underbrace{\sum_{k=1}^{2n} |\,H_n'(z_k)\,||\,\mathcal{Y}_k(z)\,|}_{A_3} + \underbrace{|\,H_n(z) - f(z)\,|}_{A_4}. \end{aligned} \tag{7.8}$$

Using (7.5) and lemma 6.2 , we get

$$A_1 = O(\omega_r(f, n^{-1})). \tag{7.9}$$

Using (7.1) and lemma 6.1, we get

$$A_2 = \begin{cases} O(\omega_r(f, n^{-1}) \log n) & -1 < \alpha \leq \frac{-1}{2} \\ O(\omega_r(f, n^{-1})\, n^{\alpha+\frac{3}{2}} \log n) & \frac{-1}{2} < \alpha < 0 \\ O(\omega_r(f, n^{-1})\, n^{\alpha+\frac{3}{2}}) & \alpha \geq 0 \end{cases} \tag{7.10}$$

Using (7.6) and lemma 6.1., we get

$$A_3 = \begin{cases} O(\omega_r(f, n^{-1}) \log n) & -1 < \alpha \leq \frac{-1}{2} \\ O(\omega_r(f, n^{-1})\, n^{\alpha+\frac{3}{2}} \log n) & \frac{-1}{2} < \alpha < 0 \\ O(\omega_r(f, n^{-1})\, n^{\alpha+\frac{3}{2}}) & \alpha \geq 0 \end{cases}. \tag{7.11}$$

Using (7.6), we have

$$A_4 = O(\omega_r(f, n^{-1})). \tag{7.12}$$

Using (7.9), (7.10), (7.11), (7.12) in (7.8), we get

$$|\,\mathbb{LH}_n(z) - f(z)\,| \leq \begin{cases} O(\omega_r(f, n^{-1}) \log n) & -1 < \alpha \leq \frac{-1}{2} \\ O(\omega_r(f, n^{-1})\, n^{\alpha+\frac{3}{2}} \log n) & \frac{-1}{2} < \alpha < 0 \\ O(\omega_r(f, n^{-1})\, n^{\alpha+\frac{3}{2}}) & \alpha \geq 0 \end{cases} \tag{7.13}$$

Hence, theorem 7.1 follows. □

8. CONCLUSION

Bahadur and Varun [3] considered a Lagrange-Hermite interpolation problem making use of zeros of Legendre polynomial for nodal system whereas this research paper poses a problem, which is an extension to the same problem, since it involves the more general Jacobi polynomial zeros for the costruction of nodal system. By putting the value of α equal to zero in our main result given in section (6) and comparing it with the convergence theorem of the paper published by Bahadur and Varun [3], we can conclude that when α equals to zero, results are comparable. Since, we are not restricted to use different value of α, we get a good approximation of a function, which is continuous on closed unit disk and analytic on open unit disk.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this paper.
Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] Bahadur S. and Mathur K. K., *Quasi-Hermite interpolation on the unit circle*, GANITA **49(1)** (1998) 1–6.

[2] Bahadur S. and Varun, *A note on Hermite-Fejér interpolation on the non-uniform nodes of unit circle*, South East Asian J. of Mathematics and Mathematical Sciences **17(2)** (2021) 83–92.

[3] Bahadur S. and Varun, *Convergence of Interpolatory Polynomial Between Lagrange and Hermite*, Annals of Pure and Applied Mathematics **17(1)** (2018) 23–28.

[4] Berriochoa E. and Cachafeiro A., *Algorithms for solving Hermite interpolation problems using the fast Fourier transform*, Journal of Computational and Applied Mathematics **235** (2010) 882–894.

[5] Berriochoa E., Cachafeiro A. and García-Amor J. M., *Lagrange interpolation in non-uniformly distributed nodes on unit circle*, Analysis **6** (2012) 651—664.

[6] Brück R, *Lagrange interpolation in non – uniformly distributed zeros on unit circle*, Analysis **16(3)** (1996) 273–282.

[7] Eneduanya S. A. N., *On interpolation polynomials using the roots of Ultraspherical polynomials*, Demonstratio Math. **17** (1984) 253–261.

[8] Freud G., *On polynomial approximation with the weight* $e^{-\frac{1}{2}x^{2k}}$, Acta Mathematica Academiae Scientiarum Hungaricae Tomus **24** (1973) 363–371.

[9] Goodman T.N.T., Ivanov K.G. and Sharma A., *Hermite interpolation in the roots of unity*, J .Approx. Theory **84** (1996) 41–60.

[10] Kiš O., *Remarks on interpolation*, Acta Math. Acad. Sci. Hungar **11** (1960) 117–128.

[11] Mastroianni G. and Vértesi P., *Mean convergence of Lagrange interpolation on arbitrary system of nodes*, Acta Math. Hung. Szeged **57** (1993) 429–441.

[12] Mathur K. K. and Sharma A., *Some interpolatory properties of Hermite Polynomials*, Acta. Math. Acad. Sci. Hung. **12** (1961) 193–207.

[13] Mathur P. and Srivastava Ashwini Kr., *Hermite interpoltion on unit circle*, J. Ultra. Phys. Sci. **21** (2010).

[14] Prasad J., *On a uniform convergence of interpolatory polynomials*, J. Austral. Math. Soc. Ser. A. **27(1)** (1979) 7–16.

[15] Szegö G., *Orthogonal polynomials*, Amer. Math. Soc., coll, **23**.

[16] Vertesi P. and Xu Yuan, *Weighted* L^p *convergence of Hermite interpolation of higher order*, Acta Math. Hungar., **59** (1992) 423–438.

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