GANITA,Vol.73(1), 2023, 65-71

ON A SUBCLASS OF MEROMORPHIC STARLIKE FUNCTIONS WITH POSITIVE COEFFICIENTS

SAYALI S. JOSHI and SANTOSH B. JOSHI

Abstract

In the present paper we introduce a subclass of meromorphic univalent functions with positive coefficients. Further, we investigate properties like coefficient inequalities, radius of convexity and the closure theorem.

2010 *Mathematics subject classification:* primary 30C45; secondary 30A20.
Keywords and phrases: Meromorphic fuctions, univalent functions, distortion theorems.

1. Introduction

Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in $U^* = \{z : 0 < |z| < 1\}$ having simple pole at z=0 and residue 1 there. Also, let $\sum_s$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_n| z^n$$

which are analytic and univalent in U^*.

A function $f \in \sum_s$ satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha\ ,\ |z| < 1\ , \tag{1.2}$$

is said to be meromorphically starlike of order α $(0 \leq \alpha < 1)$. Also a function $f \in \sum_s$ satisfying the condition

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < -\alpha\ ,\ |z| < 1\ , \tag{1.3}$$

The work is supported by NBHM research project(NBHM/(R.P)/R & D II/14977) sanctioned to first author.

is said to be meromorphically convex of order α $(0 \leq \alpha < 1)$. Let these classes be denoted respectively by $\sum_s^*(\alpha)$ and $\sum_s^k(\alpha)$.

The classes $\sum_s^*(\alpha)$ and similar other classes of meromorphically univalent functions have been extensively studied by Pommerenke [6], Aouf and Joshi [2], Clunie [3], Miller [4] and Aouf [1].

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n , \quad a_n \geq 0 \tag{1.4}$$

that are analytic and univalent in U^*.

Definition 1.1. . A function $f(z) \in \sum$ is said to be in the class $\sum^*(\alpha, \beta, A, B)$ if it satisfies the condition

$$\left| \frac{z^2 f'(z) + 1}{[Bz^2 f'(z)] + [B + (A - B)(1 - \alpha)]} \right| < \beta , \quad z \in U^*$$

where $0 \leq \alpha < 1,\ 0 < \beta \leq 1, -1 \leq A < B \leq 1$ and $0 < B \leq 1$.

Also, we denote $\sum_p^*(\alpha, \beta, A, B) = \sum_p \cap \sum^*(\alpha, \beta, A, B)$. Such type of classes were introduced and studied by Mogra et al. [5], Uralegaddi [8], Joshi [7], Uralegaddi and Ganigi [9].

2. Main Results

The following theorem gives a sufficient condition for a function to be in $\sum^*(\alpha, \beta, A, B)$.

Theorem 2.1. *Let the function f(z) defined by* (1.1) *and regular in* U^* *and if*

$$\sum_{n=1}^{\infty} n(1 - B\beta)|a_n| \leq \beta(A - B)(1 - \alpha) \tag{2.1}$$

Then $f(z) \in \sum^*(\alpha, \beta, A, B)$.

Proof. Let, (2.1) holds true. Consider the expression

$$H(f, f') = \left|z^2 f'(z) + 1\right| - \beta \left|\left[Bz^2 f'(z)\right] + [B + (A - B)(1 - \alpha)]\right|$$

Substituting f and f' by their expansion, we have for $0 < |z| = r < 1$.

$$H(f, f') = \left|\sum_{n=1}^{\infty} n\, a_n\, z^{n+1}\right| - \beta \left|B \sum_{n=1}^{\infty} n\, a_n\, z^{n+1} + [(A - B)(1 - \alpha)]\right|$$

or

$$\begin{aligned} r\,H(f,f') &\le \sum_{n=1}^{\infty} n\,a_n\,r^{n+2} - \beta\,[(A-B)(1-\alpha)] - \sum_{n=1}^{\infty} n\,B\beta\,a_n\,r^{n+2} \\ &= \sum_{n=1}^{\infty} n\,(1-B\beta)\,a_n\,r^{n+2} - \beta(A-B)(1-\alpha) \end{aligned} \tag{2.2}$$

As above inequality holds for all r $(0 < r < 1)$, letting $r \to 1$ we get

$$H(f,f') \le \sum_{n=1}^{\infty} n\,(1-B\beta) - \beta(A-B)(1-\alpha),$$

Hence by (2.1) we get

$$\left|z^2 f'(z) + 1\right| < \beta\left|\left[Bz^2 f'(z)\right] + [B + (A-B)(1-\alpha)]\right|.$$

which implies that $f \in \sum^*(\alpha,\beta,A,B)$. Hence Theorem 2.1 is established □

Theorem 2.2. *Let the function f(z) be given by* (1.4) *is analytic and univalent in* U^*. *Then* $f \in \sum_p^*(\alpha,\beta,A,B)$ *if and only if* (2.1) *is satisfied.*

Proof. In view of Theorem 2.1, it is sufficient to show that the "only if" part. Assume that $f(z)$ given by (1.4) is in $\sum_p^*(\alpha,\beta,A,B)$. Then

$$\begin{aligned} &\left|\frac{z^2\,f'(z) + 1}{B\,z^2\,f'(z) + [B + (A-B)(1-\alpha)]}\right| \\ &= \left|\frac{\sum_{n=1}^{\infty} n\,a_n\,z^{n+1}}{[(A-B)(1-\alpha)] + \sum_{n=1}^{\infty} n\,B\,a_n\,z^{n+1}}\right| < \beta \end{aligned}$$

for all $z \in U^*$. Using the fact that $Re(z) \le |z|$ for all z; it follows that

$$Re\left\{\frac{\sum_{n=1}^{\infty} n\,a_n\,z^{n+1}}{[(A-B)(1-\alpha)] + \sum_{n=1}^{\infty} n\,B\,a_n\,z^{n+1}}\right\} < \beta \quad (z \in U^*) \tag{2.3}$$

Upon clearing denominator in (2.3) and letting $z \to 1$, through positive values, we get

$$\sum_{n=1}^{\infty} n(1-B\beta)|a_n| \le \beta(A-B)(1-\alpha)$$

Hence the result follows.

□

THEOREM 2.3. *Let $f(z)$ defined by (1.4) is in the class $\sum_p^*(\alpha,\beta,A,B)$ then for* $0<|z|=r<1$.

$$\frac{1}{r}-\frac{\beta(A-B)(1-\alpha)r}{(1-B\beta)}\leq f(z)\leq\frac{1}{r}+\frac{\beta(A-B)(1-\alpha)r}{(1-B\beta)} \tag{2.4}$$

Equality holds for the function

$$f(z)=\frac{1}{z}+\frac{\beta(A-B)(1-\alpha)}{(1-B\beta)}\,z. \tag{2.5}$$

PROOF. Suppose $f(z)$ given by (1.4) is in $\sum_p^*(\alpha,\beta,A,B)$. In view of Theorem 2.2, we have

$$\sum_{n=1}^{\infty}a_n\leq\frac{\beta(A-B)(1-\alpha)}{(1-B\beta)}$$

Then, for $0<|z|=r<1$.

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z}+\sum_{n=1}^{\infty}a_n\,z^n\right|\\ &\leq \left|\frac{1}{z}\right|+\sum_{n=1}^{\infty}a_n\,|z|^n\\ &\leq \frac{1}{r}+r\sum_{n=1}^{\infty}a_n\\ &\leq \frac{1}{r}+\frac{\beta(A-B)(1-\alpha)r}{(1-B\beta)}\end{aligned} \tag{2.6}$$

which gives right hand side of inequality (2.4). Also,

$$\begin{aligned}|f(z)| &\geq \left|\frac{1}{z}\right|-\sum_{n=1}^{\infty}a_n\,|z|^n\\ &\geq \frac{1}{r}-r\sum_{n=1}^{\infty}a_n\\ &\geq \frac{1}{r}-\frac{\beta(A-B)(1-\alpha)r}{(1-B\beta)}\end{aligned} \tag{2.7}$$

which gives left hand side of inequality (2.4). It can be easily seen that the function $f(z)$ given by (2.5) is extremal. □

THEOREM 2.4. *Let $f(z)$ is in the class $\sum_p^*(\alpha,\beta,A,B)$, then $f(z)$ is meromorphically convex of order $\delta(0<|z|=r<1)$ in $|z|=r=r(\alpha,\beta,A,B)$ where*

$$r(\alpha,\beta,A,B)=\inf_n\left\{\frac{(1-\delta)(1-B\beta)}{\beta(A-B)(1-\alpha)(n+2-\delta)}\right\}^{\frac{1}{(n+1)}},\quad n=1,\ 2,\ \cdots.$$

Proof. Let $f(z) \in \sum_p^*(\alpha,\beta,A,B)$ then by Theorem 2.2, we have

$$\sum_{n=1}^{\infty} \frac{n(1-B\beta)}{\beta(A-B)(1-\alpha)}\, a_n \leq 1 \tag{2.8}$$

It is sufficient to show that

$$\left|2+\frac{zf''(z)}{f'(z)}\right| \leq (1-\delta) \quad \textit{for } |z| = r(\alpha,\beta,A,B)\,,$$

that is to show that

$$\left|\frac{f'(z)+[zf'(z)]'}{f'(z)}\right| \leq (1-\delta)\,.$$

Substituting the series of expansion of $f'(z)$ and $[zf'(z)]'$ in left hand side of expression we get

$$\left|\frac{\sum_{n=1}^{\infty} n(n+1)\, a_n\, z^{n-1}}{-\frac{1}{z^2}+\sum_{n=1}^{\infty} n\, a_n\, z^{n-1}}\right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)\, a_n\, |z|^{n+1}}{1-\sum_{n=1}^{\infty} n\, a_n\, |z|^{n+1}}$$

This will be bounded above by $(1-\delta)$ if

$$\sum_{n=1}^{\infty} \frac{n(n+2-\delta)}{1-\delta})\, a_n\, |z|^{n+1} \leq 1 \tag{2.9}$$

In view of (2.8), it follows that (2.9) is true if

$$\frac{n(n+2-\delta)}{1-\delta}\, |z|^{n+1} \leq \frac{n(1-B\beta)}{\beta(A-B)(1-\alpha)}\,, \quad n = 1,\ 2,\ \cdots\,,$$

or

$$|z| \leq \left\{\frac{(1-\delta)(1-B\beta)}{\beta(A-B)(1-\alpha)(n+2-\delta)}\right\}^{\frac{1}{(n+1)}}\,, \quad n = 1,\ 2,\ \cdots\,. \tag{2.10}$$

Setting $|z| = r = r(\alpha,\beta,A,B)$, in (2.10), result follows. Sharpness can be verified easily. □

3. Convex linear combination

In this section we will provide that the class $\sum_p^*(\alpha,\beta,A,B)$ is closed under convex linear combination.

Theorem 3.1. *Let* $f_0(z) = \frac{1}{z}$ *and*

$$f_n(z) = \frac{1}{z} + \frac{\beta(A-B)(1-\alpha)}{n(1-B\beta)} z^n\,, \quad n = 1,\ 2,\ \cdots\,.$$

Then $f(z)$ *in* $\sum_p^*(\alpha,\beta,A,B)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \lambda_n\, f_n(z)\,, \quad \textit{where } \lambda_n \geq 0 \textit{ and } \sum_{n=0}^{\infty} \lambda_n = 1.$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$ with $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$. Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \lambda_n f_n(z) \\ &= \lambda_0 f_0(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ &= \left[1 - \sum_{n=1}^{\infty} \lambda_n\right] \frac{1}{z} + \sum_{n=1}^{\infty} \lambda_n \left[\frac{1}{z} + \frac{\beta(A-B)(1-\alpha)}{n(1-B\beta)} z^n\right] \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \lambda_n \frac{\beta(A-B)(1-\alpha)}{n(1-B\beta)} z^n \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} \frac{n(1-B\beta)}{\beta(A-B)(1-\alpha)} \lambda_n \frac{\beta(A-B)(1-\alpha)}{n(1-B\beta)} = \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1 .$$

Hence by Theorem 2.1, $f \in \sum_p^*(\alpha,\beta,A,B)$.
Conversely suppose that $f(z)$ in $\sum_p^*(\alpha,\beta,A,B)$, since

$$a_n \leq \frac{\beta(A-B)(1-\alpha)}{n(1-B\beta)} , \quad n = 1,\ 2,\ \cdots .$$

Setting

$$\lambda_n = \frac{n(1-B\beta)}{\beta(A-B)(1-\alpha)} a_n , \quad n = 1,\ 2,\ \cdots .$$

and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$, it follows that

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z).$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *The class $\sum_p^*(\alpha,\beta,A,B)$ is closed under convex linear combination.*

Proof. Proof of Theorem 3.2 is straightforward, hence omitted. □

4. Acknowledgements

The authors would like to thank the reviewers for their comments and valuable suggestions.

Author contributions:
Conceptualisation: Sayali S. Joshi ; *Calculation:* Sayali S. Joshi ; *Writing-Original Draft:* Sayali S. Joshi, Santosh B. Joshi

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] M.K. Aouf, *A certain subclass of meromorphically starlike functions with positive coefficients*, Rend. di Math. Ser. VII. **9** (1989) 225–235.

[2] M.K. Aouf and S.B.Joshi, *On certain subclasses of meromorphically starlike functions with positive coefficients*, Soochow J. Math. **24(2)** (1998) 79–90.

[3] J. Clunie, *On meromorphic schlict functions*, J. London Math. Soc. **34** (1959) 215-216.

[4] J.E. Miller, *Convex meromorphic mapping and related functions*, Proc. Amer. Math. Soc. **25** (1970) 220–228.

[5] M.L. Mogra, T.R. Reddy, O.P. Juneja, *Meromorphic univalent functions with positive coefficients*, Bull. Austral. Math. Soc. **32** (1985) 161–176.

[6] C. Pommerenke, *On meromorphic starlike functions*, Pacific J. Math. Soc. **13** (1963) 221–235.

[7] S.B.Joshi, *Certain classes of meromorphic univalent functions with positive coefficients*, Bull.Pure Appl. Math. **2(2)** (2008) 178–184.

[8] B.A. Uralegaddi, *Meromorphically starlike functions with positive and fixed second coefficients*, Kyungpook. Math. J. **29(1)** (1989) 64–68.

[9] B.A. Uralegaddi and M.D. Ganigi, *A certain class of meromorphically starlike functions with positive coefficients*, Pure. Appl. Math. Sci. **26** (1987) 75–81.

Sayali S. Joshi, Department of Mathematics, Sanjay Bhokare Group of Institutes, Miraj 416410, India
e-mail: joshiss@sbgimiraj.org

Santosh B. Joshi, Department of Mathematics, Walchand College of Engineering, Sangli 416415, India
e-mail: joshisb@hotmail.com, santosh.joshi@walchandsangli.ac.in