GANITA, Vol.73(1), 2023, 73-85

STUDY OF τ CURVATURE TENSOR ON PARA-SASAKIAN MANIFOLD

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Abstract

In this paper, we investigate the τ-curvature tensor in para-Sasakian manifold. We analyze τ-Flat, φ-τ-Flat, ζ-τ-Flat and quasi-τ-Flat in the para-Sasakian manifold. We study φ-τ-Ricci recurrent and τ-φ-recurrent in para-Sasakian manifold and also obtained some results satisfying the τ-semisymmetry criteria on para-Sasakian manifold.

2010 *Mathematics subject classification:* primary 53C25; secondary 53C50, 53D15..
Keywords and phrases: τ-curvature tensor, φ-τ-Ricci recurrent, τ-φ-recurrent, η-Einstein manifold, para-sasakian manifold .

1. Introduction

Adati and Matsumoto defined the para-Sasakian and special para-Sasakian manifolds [1], which are subclasses of the almost-para contact manifold described by I. Satõ [11]. Numerous authors [3] [4] [6] [7] [8] [9] analysed various conditions of the curvature tensor on P-Sasakian manifolds with some connection and solitons.
Several varieties of curvature tensors exist, with the exception of the Riemannian curvature tensor. It is possible to study the Riemannian geometry of manifolds through the use of curvature tensors that have algebraic and differential properties. The Weyl conformal curvature tensor, the conharmonic curvature tensor, and the concircular curvature tensor are some illustrations of the curvature tensors. With this concept in mind, Tripathi and Gupta [13] introduce a new curvature tensor that they termed as τ-curvature tensor. As a result, the τ-curvature tensor can be considered of as a refinement of special curvature tensors.

$$\begin{aligned} \tau &= a_0Cur(M_1, M_2)M_3 + a_1Ric(M_2, M_3)M_1 + a_2Ric(M_1, M_3)M_2 \\ &+ a_3Ric(M_1, M_2)M_3 + a_4g(M_2, M_3)QM_1 + a_5g(M_1, M_3)QM_2 \\ &+ a_6g(M_1, M_2)QM_3 + a_7r(g(M_2, M_3)M_1 - g(M_1, M_3)M_2), \end{aligned} \tag{1.1}$$

where Cur symbolises curvature tensor, Ric symbolises Ricci tensor, Q symbolises Ricci operator, and r symbolises scalar curvature and a_0, $a_1...a_7$ are smooth function on M and M_1, M_2, M_3 are vectors fields on $\chi(M)$. Additionally, the same authors

[14] investigated the τ-curvature tensor on K-contact and Sasakian manifolds. Under specific conditions, Tripathi and Gupta [14] have developed a classification system for K-contact and Sasakian manifolds based on the τ-curvature tensor. A outcome on (K, μ)-contact manifolds with a τ-curvature tensor was obtained by Nagaraja and Somashekhara [10]. Gupta [5] studied φ-τ-symmetric condition on (ϵ)- para-Sasakian manifolds.

In light of the above research, we investigate the τ-curvature tensor in para-Sasakian manifolds in this paper. Following a brief introduction in section 2, we examine φ-τ-Flat, τ-Flat, quasi-τ-Flat and ζ-τ-Flat in the para-Sasakian manifold in section 3. In section 4 and 5, we investigate φ-τ-Ricci recurrent and τ-φ- recurrent in para-Sasakian manifolds and establish conditions for their Einstein manifold. The next segment 6 presents results satisfying the τ-semisymmetry criterion on para-Sasakian manifolds. Some application related to τ curvature is given in section 7. The last two section 8 and 9 followed by example and conclusion.

2. Preliminaries

An m-dimensional differentiable manifold M^{m} is a P-Sasakian manifold if it admit a (1, 1) tensor field φ, a covariant vector field η, a contravariant vector field ζ and a Riemannian metric g , which provide

$$\varphi^2(M_1) = M_1 - \eta(M_1)\zeta, \tag{2.1}$$

$$g(M_1, \zeta) = \eta(M_1), \varphi\zeta = 0, \tag{2.2}$$

$$g(\varphi M_1, \varphi M_2) = g(M_1, M_2) - \eta(M_1)\eta(M_2), \tag{2.3}$$

$$((\nabla_{M_1}\varphi)M_2) = -g(M_1, M_2)\zeta - \eta(M_2)M_1 + 2\eta(M_1)\eta(M_2)\zeta, \tag{2.4}$$

$$\nabla_{M_1}\zeta = \varphi M_1, \tag{2.5}$$

$$(\nabla_{M_1}\eta)(M_2) = g(\varphi M_1, M_2) = g(\varphi M_2, M_1). \tag{2.6}$$

for any vector field M_1 and M_2, where ∇ denotes covariant differentiation with respect to g [2].
The following condition holds in a P-Sasakian manifold M^m with the structure (φ, ζ, η, g) observed,

$$\eta \circ \varphi = 0; \quad \eta(\zeta) = 1, \tag{2.7}$$

$$rank(\varphi) = m - 1. \tag{2.8}$$

The following expression [12], [1] is known to possess on a P-Sasakian manifold,

$$\eta(Cur(M_1, M_2)M_3) = g(M_1, M_3)\eta(M_2) - g(M_2, M_3)\eta(M_1), \tag{2.9}$$

$$Cur(M_1, M_2)\zeta = \eta(M_1)M_2 - \eta(M_2)M_1, \tag{2.10}$$

$$Cur(\zeta, M_1)M_2 = \eta(M_2)M_1 - g(M_1, M_2)\zeta, \tag{2.11}$$

$$Cur(\zeta, M_1)\zeta = M_1 - \eta(M_1)\zeta, \tag{2.12}$$

$$Ric(M_1, \zeta) = -(m-1)\eta(M_1), \tag{2.13}$$

$$Ric(\zeta, \zeta) = -(m-1), \tag{2.14}$$

$$Ric(\varphi M_1, \varphi M_2) = Ric(M_1, M_2) + (m-1)\eta(M_1)\eta(M_2), \tag{2.15}$$

$$QM_1 = -(m-1)M_1, \tag{2.16}$$

$$Q\zeta = -(m-1)\zeta. \tag{2.17}$$

for any vector field M_1, M_2, M_3 on M^m.

It is claimed that a P-Sasakian manifold M^m is η-Einstein if its Ricci tensor (Ric) is of type

$$Ric(M_1, M_2) = \alpha_1 g(M_1, M_2) + \alpha_2\eta(M_1)\eta(M_2). \tag{2.18}$$

for any vector field M_1 and M_2, where α_1 and α_2 are smooth function on M^m. In particular if α_2=0 in the above equation the η-Einstein becomes an Einstein manifold. Also in a m-dimensional P-Sasakian manifold M, If the expression (e_1, $e_2...e_{m-1}$, ζ) is a local orthonormal basis of vector fields in M then the expression (φe_1, $\varphi e_2...\varphi e_{m-1}$, ζ) must likewise be local orthonormal basis. It is simple to demonstrate

$$\sum_{j=1}^{m-1} g(e_j, e_j) = \sum_{j=1}^{m-1} g(\varphi e_j, \varphi e_j) = m-1, \tag{2.19}$$

$$\begin{aligned} \sum_{j=1}^{m-1} g(e_j, M_3)Ric(M_2, e_j) &= \sum_{j=1}^{m-1} g(\varphi e_j, M_3)Ric(M_2, \varphi e_j) \\ &= Ric(M_2, M_3) + (m-1)\eta(M_2)\eta(M_3). \end{aligned} \tag{2.20}$$

from (2.15) we have

$$\sum_{j=1}^{m-1} Ric(e_j, e_j) = \sum_{j=1}^{m-1} Ric(\varphi e_j, \varphi e_j) = r + (m-1), \tag{2.21}$$

where r is a scalar curvature. In a P-Sasakian manifold from equation (2.11) we have

$$Cur(\zeta, M_2, M_3, \zeta) = -g(\varphi M_2, \varphi M_3). \tag{2.22}$$

thus,

$$\begin{aligned} \sum_{j=1}^{m-1} Cur(e_i, M_2, M_3, e_j) &= \sum_{j=1}^{m-1} Cur(\varphi e_j, M_2, M_3, \varphi e_j) \\ &= Ric(M_2, M_3) + g(\varphi M_2, \varphi M_3). \end{aligned} \tag{2.23}$$

$$\sum_{j=1}^{m-1} Cur(\varphi e_j, \varphi M_3)g(\varphi M_2, \varphi e_j) = Ric(\varphi M_2, \varphi M_3). \tag{2.24}$$

$$\sum_{j=1}^{m-1} g(Cur(\varphi e_j, \varphi M_2)\varphi M_3, \varphi e_j) = Ric(\varphi M_2, \varphi M_3) + g(\varphi M_2, \varphi M_3). \tag{2.25}$$

3. SOME STRUCTURE THEOREMS

DEFINITION 3.1. A P-Sasakian manifold (M^m, g) is said to be
1. τ-Flat if

$$\tau(M_1, M_2)M_3 = 0. \tag{3.1}$$

2. quasi τ-Flat if

$$g(\tau(M_1, M_2)M_3, \varphi M_4) = 0. \tag{3.2}$$

3. ζ-τ-Flat if

$$\tau(M_1, M_2)\zeta = 0. \tag{3.3}$$

4. φ-τ if

$$g(\tau(\varphi M_1, \varphi M_2)\varphi M_3, \varphi M_4) = 0. \tag{3.4}$$

for any vector fields M_1, M_2, M_3 and M_4.

THEOREM 3.2. *Let M be m-dimensional quasi τ-Flat P-Sasakian manifold,*
1. If $(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6) \neq 0$, then the manifold is η-Einstein. In fact, M becomes an Einstein manifold provided

$$a_0 - (m-1)[a_2 + a_3 + a_5 + a_6] - a_7 r = 0.$$

2. If $a_0 + ma_1 + a_2 + a_3 + a_5 + a_6 = 0$ and $m(a_4 + ma_7 - 2a_7) + a_7 \neq 0$, then

$$r = \frac{-(m-1)[a_0 + a_2 + a_3 + na_4 + a_5 + a_6]}{m(a_4 + ma_7 - 2a_7) + a_7}. \tag{3.5}$$

PROOF. Let us suppose M be an m-dimensional P-Sasakian manifold. For quasi τ-Flat P-Sasakian manifold, from equation (1.1) and (3.2) we have

$$\begin{aligned} 0 &= a_0 g(Cur(M_1, M_2)M_3, \varphi M_4) + a_1 Ric(M_2, M_3)g(M_1, \varphi M_4) \\ &+ a_2 Ric(M_1, M_3)g(M_2, \varphi M_4) + a_3 Ric(M_1, M_2)g(M_3, \varphi M_4) \\ &+ a_4 g(M_2, M_3)Ric(M_1, \varphi M_4) + a_5 g(M_1, M_3)Ric(M_2, \varphi M_4) \\ &+ a_6 g(M_1, M_2)Ric(M_3, \varphi M_4) + a_7 r(g(M_2, M_3)g(M_1, \varphi M_4) \\ &- g(M_1, M_3)g(M_2, \varphi M_4)). \end{aligned} \tag{3.6}$$

Replacing M_1 by φM_1 in (3.15), we get

$$\begin{aligned} 0 &= a_0 g(Cur(\varphi M_1, M_2)M_3, \varphi M_4) + a_1 Ric(M_2, M_3)g(\varphi M_1, \varphi M_4) \\ &+ a_2 Ric(\varphi M_1, M_3)g(M_2, \varphi M_4) + a_3 Ric(\varphi M_1, M_2)g(M_3, \varphi M_4) \\ &+ a_4 g(M_2, M_3)Ric(\varphi M_1, \varphi M_4) + a_5 g(\varphi M_1, M_3)Ric(M_2, \varphi M_4) \\ &+ a_6 g(\varphi M_1, M_2)Ric(M_3, \varphi M_4) + a_7 r(g(M_2, M_3)g(\varphi M_1, \varphi M_4) \\ &- g(\varphi M_1, M_3)g(M_2, \varphi M_4)). \end{aligned} \tag{3.7}$$

If the expression $(e_1, e_2...e_{m-1}, \zeta)$ is a basis for local orthonormal vector fields in M, then the expression $(\varphi e_1, \varphi e_2...\varphi e_{m-1}, \zeta)$ must likewise be local orthonormal basis. Now from (3.16), we attained

$$\begin{aligned} 0 &= a_0 \sum_{j=1}^{m-1} g(Cur(\varphi e_j, M_2)M_3, \varphi e_j) + a_1 \sum_{j=1}^{m-1} Ric(M_2, M_3)g(\varphi e_j, \\ \varphi e_j) & +a_2 \sum_{j=1}^{m-1} Ric(\varphi e_j, M_3)g(M_2, \varphi e_j) + a_3 \sum_{j=1}^{m-1} Ric(\varphi e_j, M_2)g(M_3, \varphi e_j) \\ &+ a_4 \sum_{j=1}^{m-1} Ric(\varphi e_j, \varphi e_j)g(M_2, M_3) + a_5 \sum_{j=1}^{m-1} Ric(M_2, \varphi e_j)g(\varphi e_j, M_3) \\ &+ a_6 \sum_{j=1}^{m-1} Ric(M_3, \varphi e_j)g(\varphi e_j, M_2) + a_7 r(\sum_{j=1}^{m-1} g(M_2, M_3)g(\varphi e_j, \varphi e_j) \\ &- \sum_{j=1}^{m-1} g(\varphi e_j, M_3)g(M_2, \varphi e_j)). \end{aligned} \tag{3.8}$$

In view of (2.3), (2.19) and (2.23), (3.17) becomes

$$\begin{aligned} (a_0 + ma_1 + a_2 + a_3 + a_5 + a_6)Ric(M_2, M_3) &= (2a_7 r - a_0 - a_4 r - a_4(m-1) \\ &- na_7 r)g(M_2, M_3) + (a_0 - (m-1)(a_2 \\ &+ a_3 + a_5 + a_6) - a_7 r)\eta(M_2)\eta(M_3). \end{aligned} \tag{3.9}$$

which implies that,

$$Ric(M_2, M_3) = P_1 g(M_2, M_3) + P_2 \eta(M_2)\eta(M_3), \tag{3.10}$$
$$(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6) \neq 0,$$

where,

$$P_1 = \frac{2a_7 r - a_0 - a_4 r - a_4(m-1) - ma_7 r}{a_0 + ma_1 + a_2 + a_3 + a_5 + a_6} \tag{3.11}$$

and

$$P_2 = \frac{a_0 - a_7 r - (m-1)[a_2 + a_3 + a_5 + a_6]}{(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6)}. \tag{3.12}$$

1. If $(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6) \neq 0$ and $a_0 - a_7 r - (m-1)(a_2 + a_3 + a_5 + a_6) = 0$, then (3.19) reduces to

$$Ric(M_2, M_3) = P_1 g(M_2, M_3). \tag{3.13}$$

This implies that manifold is Einstein.
2. If $(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6) = 0$, we obtain the following result (3.14) by contracting (3.18). Hence the theorem proved. □

Theorem 3.3. *Let M be m-dimensional quasi τ-Flat P-Sasakian manifold,*
1. If $(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6) \neq 0$, then the manifold is η-Einstein. In fact, M becomes an Einstein manifold provided

$$a_0 - (m-1)[a_2 + a_3 + a_5 + a_6] - a_7 r = 0.$$

2. If $a_0 + ma_1 + a_2 + a_3 + a_5 + a_6 = 0$ and $m(a_4 + ma_7 - 2a_7) + a_7 \neq 0$, then

$$r = \frac{-(m-1)[a_0 + a_2 + a_3 + na_4 + a_5 + a_6]}{m(a_4 + ma_7 - 2a_7) + a_7}. \tag{3.14}$$

Proof. Let us suppose M be an m-dimensional P-Sasakian manifold. For quasi τ-Flat P-Sasakian manifold, from equation (1.1) and (3.2) we have

$$\begin{aligned} 0 &= a_0 g(Cur(M_1, M_2)M_3, \varphi M_4) + a_1 Ric(M_2, M_3)g(M_1, \varphi M_4) \\ &+ a_2 Ric(M_1, M_3)g(M_2, \varphi M_4) + a_3 Ric(M_1, M_2)g(M_3, \varphi M_4) \\ &+ a_4 g(M_2, M_3)Ric(M_1, \varphi M_4) + a_5 g(M_1, M_3)Ric(M_2, \varphi M_4) \\ &+ a_6 g(M_1, M_2)Ric(M_3, \varphi M_4) + a_7 r(g(M_2, M_3)g(M_1, \varphi M_4) \\ &- g(M_1, M_3)g(M_2, \varphi M_4)). \end{aligned} \tag{3.15}$$

Replacing M_1 by φM_1 in (3.15), we get

$$\begin{aligned} 0 &= a_0 g(Cur(\varphi M_1, M_2)M_3, \varphi M_4) + a_1 Ric(M_2, M_3)g(\varphi M_1, \varphi M_4) \\ &+ a_2 Ric(\varphi M_1, M_3)g(M_2, \varphi M_4) + a_3 Ric(\varphi M_1, M_2)g(M_3, \varphi M_4) \\ &+ a_4 g(M_2, M_3)Ric(\varphi M_1, \varphi M_4) + a_5 g(\varphi M_1, M_3)Ric(M_2, \varphi M_4) \\ &+ a_6 g(\varphi M_1, M_2)Ric(M_3, \varphi M_4) + a_7 r(g(M_2, M_3)g(\varphi M_1, \varphi M_4) \\ &- g(\varphi M_1, M_3)g(M_2, \varphi M_4)). \end{aligned} \tag{3.16}$$

If the expression $(e_1, e_2 ... e_{m-1}, \zeta)$ is a basis for local orthonormal vector fields in M, then the expression $(\varphi e_1, \varphi e_2 ... \varphi e_{m-1}, \zeta)$ must likewise be local orthonormal basis. Now from (3.16), we attained

$$\begin{aligned} 0 &= a_0 \sum_{j=1}^{m-1} g(Cur(\varphi e_j, M_2)M_3, \varphi e_j) + a_1 \sum_{j=1}^{m-1} Ric(M_2, M_3)g(\varphi e_j, \\ \varphi e_j) &+ a_2 \sum_{j=1}^{m-1} Ric(\varphi e_j, M_3)g(M_2, \varphi e_j) + a_3 \sum_{j=1}^{m-1} Ric(\varphi e_j, M_2)g(M_3, \varphi e_j) \\ &+ a_4 \sum_{j=1}^{m-1} Ric(\varphi e_j, \varphi e_j)g(M_2, M_3) + a_5 \sum_{j=1}^{m-1} Ric(M_2, \varphi e_j)g(\varphi e_j, M_3) \\ &+ a_6 \sum_{j=1}^{m-1} Ric(M_3, \varphi e_j)g(\varphi e_j, M_2) + a_7 r(\sum_{j=1}^{m-1} g(M_2, M_3)g(\varphi e_j, \varphi e_j) \\ &- \sum_{j=1}^{m-1} g(\varphi e_j, M_3)g(M_2, \varphi e_j)). \end{aligned} \tag{3.17}$$

In view of (2.3), (2.19) and (2.23), (3.17) becomes

$$\begin{aligned}(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6)Ric(M_2, M_3) &= (2a_7r - a_0 - a_4r - a_4(m-1) \\ &- na_7r)g(M_2, M_3) + (a_0 - (m-1)(a_2 \\ &+ a_3 + a_5 + a_6) - a_7r)\eta(M_2)\eta(M_3).\end{aligned} \tag{3.18}$$

which implies that,

$$Ric(M_2, M_3) = P_1 g(M_2, M_3) + P_2\eta(M_2)\eta(M_3), \quad (a_0 + ma_1 + a_2 + a_3 + a_5 + a_6) \neq 0, \tag{3.19}$$

where,

$$P_1 = \frac{2a_7r - a_0 - a_4r - a_4(m-1) - ma_7r}{a_0 + ma_1 + a_2 + a_3 + a_5 + a_6} \tag{3.20}$$

and

$$P_2 = \frac{a_0 - a_7r - (m-1)[a_2 + a_3 + a_5 + a_6]}{(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6)}. \tag{3.21}$$

1. If $(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6) \neq 0$ and $a_0 - a_7r - (m-1)(a_2 + a_3 + a_5 + a_6) = 0$, then (3.19) reduces to

$$Ric(M_2, M_3) = P_1 g(M_2, M_3). \tag{3.22}$$

This implies that manifold is Einstein.
2. If $(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6) = 0$, we obtain the following result (3.14) by contracting (3.18). Hence the theorem proved. □

4. φ-τ Ricci Recurrent

Now from (1.1),

$$\begin{aligned}Ric_\tau(M_1, M_2) &= (a_0 + ma_1 + a_2 + a_3 + a_5 + a_6)Ric(M_1, M_2) \\ &+ (a_4 + (m-1)a_7)rg(M_1, M_2).\end{aligned} \tag{4.1}$$

From (4.1), it follows that

$$Q_\tau M_1 = (a_0 + ma_1 + a_2 + a_3 + a_5 + a_6)QM_1 + (a_4 + (m-1)a_7)rM_1, \tag{4.2}$$

where $Ric_\tau(M_1, M_2)$ is the τ-Ricci tensor of type (0, 2) and Q_τ is the Ricci operator. In view of (2.17) and (4.2), we obtain

$$Q_\tau\zeta = -(m-1)(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6)\zeta + (a_4 + (m-1)a_7)r\zeta. \tag{4.3}$$

Definition 4.1. A P-Sasakian manifold (M^m, g) is called φ-τ-Ricci recurrent if its τ-Ricci operator Q_τ satisfies the condition

$$\varphi^2((\nabla_{M_4}Q_\tau)M_1) = A(M_4)Q_\tau(M_1). \tag{4.4}$$

where A is non zero 1-form.

Now using (2.1) in (4.4), we get

$$(\nabla_{M_4} Q_\tau)M_1 - \eta((\nabla_{M_4} Q_\tau)M_1)\zeta = A(M_4)Q_\tau(M_1). \tag{4.5}$$

Putting $M_1 = \zeta$

$$(\nabla_{M_4} Q_\tau)\zeta - \eta((\nabla_{M_4} Q_\tau)\zeta)\zeta = A(M_4)Q_\tau(\zeta). \tag{4.6}$$

Taking inner product of above by M_3, we have

$$\begin{aligned} A(M_4)g(Q_\tau\zeta, M_3) &= g(\nabla_{M_4} Q_\tau\zeta, M_3) - g(Q_\tau(\nabla_{M_4}\zeta)M_3) \\ &- \eta(\nabla_{M_4} Q_\tau)\zeta)\eta(M_3). \end{aligned} \tag{4.7}$$

Replacing M_3 by φM_3 and using (2.1), (2.5) in (4.7), we obtain

$$\begin{aligned} Ric_\tau(\varphi M_4, \varphi M_3) &= (-(m-1)(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6 \\ &+ (a_4 + (m-1)a_7)r)g(\varphi M_4, \varphi M_3). \end{aligned} \tag{4.8}$$

By virtue of (2.3), (2.25), (4.1), we get

$$Ric(M_4, M_3) = -(m-1)g(M_4, M_3). \tag{4.9}$$

Thus, we have

Theorem 4.2. *A φ-τ-Ricci recurrent P-Sasakian manifold (M^m, g) is an Einstein Manifold.*

5. τ-φ-Ricci Recurrent

Definition 5.1. A P-Sasakian manifold (M^m, g) is called τ-φ recurrent manifold if $\exists$ a non zero 1-form A such that

$$\varphi^2((\nabla_{M_4}\tau)(M_1, M_2)M_3) = A(M_4)\tau(M_1, M_2)M_3. \tag{5.1}$$

Now from (2.1), we have

$$(\nabla_{M_4}\tau)(M_1, M_2)M_3 - \eta((\nabla_{M_4}\tau)(M_1, M_2)M_3)\zeta = A(M_4)T(M_1, M_2)M_3. \tag{5.2}$$

From which it follows that

$$\begin{aligned} A(M_4)g(T(M_1, M_2)M_3, M_5) &= g((\nabla_{M_4}\tau)(M_1, M_2)M_3, M_5) \\ &- \eta((\nabla_{M_4}\tau)(M_1, M_2)M_3)\eta(M_5) \end{aligned} \tag{5.3}$$

Let $\{e_j\}$; $j = 1, 2, 3...m$ represent the orthonormal basis of the tangent spaces at any and every point on the manifold. Then by inputting $M_1 = M_5 = e_j$ in (5.3) and taking summation over j, $1 \leq j \leq m$, we get

$$\begin{aligned} \sum_{j=1}^{m} A(M_4)g(\tau(e_j, M_2)M_3, e_j) &= \sum_{j=1}^{m} g((\nabla_{M_4}\tau)(e_j, M_2)M_3, e_j) \\ &- \sum_{j=1}^{m} \eta((\nabla_{M_4}\tau)(e_j, M_2)M_3)\eta(e_j) \end{aligned} \tag{5.4}$$

In view of (2.13), (2.14) and (5.4)

$$\begin{aligned}(\; a_0 &+ma_1 + a_2 + a_3)(\nabla_{M_4}Ric)(M_2, M_3) + (a_4 + (m-1)a_7)(\nabla_{M_4}r)g(M_2, M_3) \\ + &\; a_5 g((\nabla_{M_4}Q)M_2, M_3) + a_6((\nabla_{M_4}Q)M_3, M_2) - a_0\eta((\nabla_{M_4}Cur)(\zeta, M_2)M_3) \\ - &\; a_1(\nabla_{M_4}Ric)(M_2, M_3) - a_2(\nabla_{M_4}Ric)(\zeta, M_3)\eta(M_2) - a_3(\nabla_{M_4}Ric)(M_2,\zeta)\eta(M_3) \\ - &\; a_4 g(M_2, M_3)\eta((\nabla_{M_4}Q)\zeta) - a_5\eta(M_3)\eta((\nabla_{M_4}Q)M_2) - a_6\eta(M_2)\eta((\nabla_{M_4}Q)M_3) \\ - &\; a_7(\nabla_{M_4}r)[g(M_2, M_3) - \eta(M_2)\eta(M_3)] = A(M_4)[(a_0 + ma_1 + a_2 + a_3 + a_5 \\ + &\; a_6)Ric(M_2, M_3) + (a_4 + (m-1)a_7)rg(M_2, M_3)]\end{aligned} \tag{5.5}$$

Putting M_3=ζ and using (2.1), (2.7) and (2.14), we get

$$\begin{aligned}A &\;(M_4)[(-(m-1)(a_0 + ma_1 + a_2 + a_3 + a_5 + a_6))\eta(M_2) \\ + &\;(a_4 + (m-1)a_7)r\eta(M_2)] = (a_0 + ma_1 + a_2 + a_5 + a_3)(\nabla_{M_4}Ric)(Y,\zeta) \\ + &\;(a_4 + (m-1)a_7)(\nabla_{M_4}r)\eta(M_2) + a_6((\nabla_{M_4}Q)\zeta, M_2) - a_3(\nabla_{M_4}Ric)(M_2,\zeta)\end{aligned} \tag{5.6}$$

Now we have ,

$$(\nabla_{M_4}Ric)(M_2,\zeta) = \nabla_{M_4}Ric(M_2,\zeta) - Ric(\nabla_{M_4}M_2,\zeta) - Ric(M_2, \nabla_{M_4}\zeta).$$

Using (2.5), (2.6) and (2.14) in above relation we obtain,

$$(\nabla_{M_4}Ric)(M_2,\zeta) = -(m-1)g(M_2,\varphi M_4) - Ric(M_2,\varphi M_4). \tag{5.7}$$

and,

$$(\nabla_{M_4}Ric)(\zeta,\zeta) = 0. \tag{5.8}$$

Replacing M_2 by φM_2 in (5.6) and using (2.3), (2.7), (5.7) and (5.8), we get

$$Ric(M_2, M_4) = -(m-1)g(M_2, M_4), \tag{5.9}$$

and

$$a_0 + na_1 + a_2 + a_6 \neq 0.$$

Thus we can state that

THEOREM 5.2. *A τ-φ recurrent P-Sasakian manifold of m-dimension is an Einstein manifold provided $a_0 + ma_1 + a_2 + a_5 + a_6 \neq 0$.*

6. τ-semi symmetric conditions on para-Sasakian manifolds

In this segment, we examine the τ-curvature tensor for P-Sasakian manifolds satisfying certain symmetries.

THEOREM 6.1. *A τ-semi symmetric P-Sasakian manifold M^m, g is an η-Einstein manifold provided $a_0 + 2(m-1)a_1 + 2a_2 + 2a_3 + a_5 + a_6 \neq 0$.*

Proof. Let M be m-dimensional P-Sasakian manifold satisfying the symmetry criterion. Now,

$$\begin{aligned}(Cur(M_1,M_2)\tau)(M_3,M_4)M_5 &= Cur(M_1,M_2)\tau(M_3,M_4)M_5 \\ &- \tau(Cur(M_1,M_2)M_3,M_4)M_5 \\ &- \tau(M_3,Cur(M_1,M_2)M_4)M_5 \\ &- \tau(M_3,M_4)Cur(M_1,M_2)M_5\end{aligned} \tag{6.1}$$

from $Cur(M_1,M_2)\tau=0$, we have

$$\begin{aligned}0 &= g(Cur(\zeta,M_2)\tau(M_3,M_4)M_5,\zeta) - g(\tau(Cur(\zeta,M_2)M_3,M_4)M_5,\zeta) \\ &- g((\tau(M_3,Cur(\zeta,M_2)M_4)M_5,\zeta) - g(\tau(M_3,M_4)(\zeta,M_2)M_5,\zeta).\end{aligned} \tag{6.2}$$

Putting M_2=M_3 in (6.2) and using (2.7) and (2.10), we get

$$\begin{aligned}0 &= g(\tau(M_3,M_4)M_5,M_3) + g(M_3,U_3)\eta(\tau(\zeta,M_4)M_5) \\ &+ g(M_3,M_4)\eta(\tau(M_3,\zeta)M_5) - \eta(M_3)\eta(\tau(M_3,M_3)M_5) \\ &+ g(M_3,M_5)\eta(\tau(M_3,M_4)\zeta) - \eta(M_5)\eta(\tau(M_3,M_3)U_3).\end{aligned} \tag{6.3}$$

If (e_1, e_2...e_{m-1}, ζ) is a basis for local orthonormal vector fields in M then equation (6.3) can be written as

$$\begin{aligned}0 &= \sum_{j=1}^{m-1} g(\tau(e_j,M_4)M_5,e_j) + \sum_{j=1}^{m-1} g(e_j,e_j)\eta(\tau(\zeta,M_4)M_5) \\ &+ \sum_{j=1}^{m-1} g(e_j,M_4)\eta(\tau(e_j,\zeta)M_5) - \sum_{j=1}^{m-1} \eta(M_4)\eta(\tau(e_j,e_j)M_5) \\ &+ \sum_{j=1}^{m-1} g(e_j,M_5)\eta(\tau(e_j,M_4)\zeta) - \sum_{j=1}^{m-1} \eta(M_5)\eta(T(e_j,M_4)e_j).\end{aligned} \tag{6.4}$$

In view of (2.1), (2.7), (2.17), (2.19), (2.20), (2.23), we obtain

$$\begin{gathered}Ric(M_4,M_5) = E_1 g(M_4,M_5) + E_2\eta(M_4)\eta(M_5), \\ a_0 + 2(m-1)a_1 + 2a_2 + 2a_3 + a_5 + a_6 \neq 0\end{gathered} \tag{6.5}$$

where,

$$E_1 = \frac{-2a_0 + (m-1)[a_0 - (n-2)a_4 + a_5 + a_6] - ra_4}{a_0 + 2(m-1)a_1 + 2a_2 + 2a_3 + a_5 + a_6} \tag{6.6}$$

and

$$E_2 = \frac{a_0 + (m-1)[a_0 + (m-1)a_2 + na_3 - (n-2)a_6 - ra_7]}{a_0 + 2(m-1)a_1 + 2a_2 + 2a_3 + a_5 + a_6} \frac{-r(a_2 - a_3)}{a_0 + 2(m-1)a_1 + 2a_2 + 2a_3 + a_5 + a_6}. \tag{6.7}$$

□

THEOREM 6.2. *A P-Sasakian manifold of dimension m fulfilling the condition $\tau{\cdot}Ric$=0 is an η-Einstein manifold, only if and when $a_3 \neq 0$.*

PROOF. Let (M^m, g) be a P-Sasakian manifold. Suppose (M^m, g) satisfies the condition $\tau{\cdot}Ric$=0 then we have,

$$(\tau(M_1, M_2)Ric)(M_3, \zeta) = 0. \tag{6.8}$$

$$Ric(\tau(M_1, M_2)M_3, \zeta) + Ric(M_3, \tau(M_1, M_2)\zeta) = 0. \tag{6.9}$$

taking $M_3 = \zeta$ in (6.9), we obtain

$$Ric(\tau(M_1, M_2)\zeta, \zeta) = 0. \tag{6.10}$$

Using (1.1) and (2.14) in (6.10), we get

$$Ric(M_1, M_2) = F_1 g(M_1, M_2) + F_2\eta(M_1)\eta(M_2), a_3 \neq 0 \tag{6.11}$$

where,

$$F_1 = \frac{(m-1)a_6}{a_3} \tag{6.12}$$

and

$$F_2 = \frac{(m-1)[a_1 + a_2 + a_4 + a_5]}{a_3}. \tag{6.13}$$

□

7. Application

In differential geometry, the Riemannian curvature tensor is used to characterise the curvature of n-dimensional spaces such as Riemannian manifolds. The Riemannian tensor serves a crucial role in the theories of general relativity, gravity, and spacetime curvature. Curvature defines the deviation of a geometric object, such as a curve or surface from a straight line or flat plane. This difference can be expressed as a simple scalar that represents its magnitude.

The τ curvature tensor has numerous applications in the spacetimes spectrum. Einstein's Field equation is satisfied by a viscous fluid spacetime that admits a vanishing τ-curvature tensor under general relativity. It has extensive applications in cosmology as well.

8. Example

Here, we would like to provide an example of a 3-dimensional para-Sasakian manifold with the tau curvature tensor.

We considered 3-dimension manifold $M^3 = \{(u, v, w)\} \,\varepsilon\, \mathbf{R}^3$, where (u, v, w) are the standard co-ordinates in $\mathbf{R}^3$. We select vector fields that are linearly independent of one another:

$$e_1 = \frac{\partial}{\partial u}, \quad e_2 = 2e^u \frac{\partial}{\partial v}, \quad e_3 = 2e^u \frac{\partial}{\partial w}$$

Let g denote the Riemannian metric defined by the expression: g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1 and g(e_i, e_j) = 0, for i ≠ j

Let η be 1-form defined by $\eta(M_1)$ = g(M_1, e_1) for any $M_1 \varepsilon M^3$, let φ be the (1, 1) tensor field defined by:

$$\varphi(e_1) = 0, \quad \varphi(e_2) = e_2, \quad \varphi(e_3) = e_3$$

Using the linearity of g and φ, we have:$\varphi^2(M_1) = M_1 - \eta(M_1)e_1$ and g$(\varphi M_1, \varphi M_2)$ = -g$(M_1, M_2) + \eta(M_1)\eta(M_2)$ where $M_1, M_2 \varepsilon M^3$

For Levi-Civita connection ∇ we have the following:

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_2, e_3] = 0$$

Now using the Koszul formula for metric g, we obtain the following:

$$\begin{aligned} \nabla_{e_1} e_3 &= 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = 0, \\ \nabla_{e_2} e_3 &= 0, \quad \nabla_{e_2} e_2 = e_1, \quad \nabla_{e_2} e_1 = -e_2, \\ \nabla_{e_3} e_3 &= e_1, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = -e_3. \end{aligned}$$

As a result of the preceding findings, we can deduce that the structures $(\varphi, \zeta, \eta, g)$ satisfies: $(\nabla_{M_1}\varphi)M_2$ = -g$(M_1, M_2)\zeta + \eta(M_2)M_1$; M_1, $M_2 \varepsilon M^3$, where $\eta(\zeta) = \eta(e_1) = 1$. Hence $(\varphi, \zeta, \eta, g)$ is a para-Sasakian structure and $M^3(\varphi, \zeta, \eta, g)$ is a 3-dimensional para-Sasakian manifold.

The components of Riemannian curvature (Cur) in terms of the Levi-Civita connection ∇ are as follows:

$$\begin{aligned} Cur(e_1, e_2)e_2 &= -e_1, \quad Cur(e_1, e_3)e_3 = -e_1, \quad Cur(e_1, e_2)e_3 = 0, \\ Cur(e_2, e_3)e_3 &= -e_2, \quad Cur(e_2, e_1)e_1 = -e_2, \quad Cur(e_3, e_2)e_2 = -e_3, \\ Cur(e_3, e_1)e_1 &= -e_3, \quad Cur(e_3, e_2)e_3 = e_2, \quad Cur(e_3, e_1)e_2 = 0. \end{aligned}$$

The Ricci tensor with regards to Levi-Civita connection and by using above curvature tensor are as follows:

$$Ric(e_1, e_1) = -2, \quad Ric(e_2, e_2) = -2, \quad Ric(e_3, e_3) = -2.$$

From the above value of Ricci tensor and g metric the theorem from section 4 is verified. Hence the manifold is $\varphi - \tau$ Ricci recurrent.

9. Discussions

In this paper, we glanced into the fact that the τ- curvature tensor in the para-Sasakian manifold is τ-flat, φ-τ Flat ζ-τ-flat and quasi τ-flat. And it also satisfies the τ-semi symmetry criteria on a para-Sasakain manifold. In this subsequence, we will study τ-curvature tensors on different differentiable manifolds concerning other connections.

Conflicts of Interest: The authors declare no conflict of interest.

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