GANITA,Vol.73(1), 2023, 87-97

NON-INTERPOLATORY COMPLEX TRIGONOMETRIC SPLINE ON THE UNIT CIRCLE

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Abstract

Considering a mesh of equally spaced points z_j; $j = 1, 2, \ldots, n$ on the unit circle K, we have constructed a non interpolatory complex trigonometric spline on each arc in which the points z_j divides the unit circle K. Further, a quantitative estimate corresponding to a function, analytic on K has also been obtained.

2010 *Mathematics subject classification:* 41A15; 41A25.
Keywords and phrases: Splines, Non-Interpolatory Splines, Trigonometric Spline Interpolation, Rate of Convergence..

1. Introduction

Let K denote the unit circle $|z| = 1$ of the complex plane and $\Delta = \{z_1, z_2, \ldots, z_n\}$ be a mesh of n distinct points of K arranged in cyclic counter-clockwise order. A complex valued function $\mathbb{S}_\Delta(z)$ defined on K is called a spline function of degree $m - 1$, if it satisfies the conditions

1. $\mathbb{S}_\Delta(z) \in C^{m-2}(K)$,

2. $\mathbb{S}_\Delta(z)$ agrees in values with a polynomial of degree at most $m - 1$, on each arc in which the points z_j divide the circle K.

If $\mathbb{S}_1(z), \mathbb{S}_2(z), \ldots, \mathbb{S}_n(z)$ denote the polynomial components of $\mathbb{S}_\Delta(z)$ on the arcs $K_j = \{e^{i\theta} : \arg z_j \leq \theta \leq \arg z_{j+1},\ j = 1, 2, \ldots, n\}$ respectively, where $z_{n+1} = z_1$, then the condition (1) or more explicitly $\mathbb{S}_\Delta(e^{i\theta}) \in C^{m-2}(K)$, is equivalent to the conditions

$$\mathbb{S}_j^{(\nu)}(z_{j+1}) = \mathbb{S}_{j+1}^{(\nu)}(z_{j+1}), \quad \nu = 0, 1, 2, \ldots, m-2, \quad j = 1, 2, \ldots, n \tag{1.1}$$

where $\mathbb{S}_{n+1}(z) = \mathbb{S}_1(z)$.

The first author thankfully acknowledges the financial suppport from University Grants Commission, New Delhi, India in the form of a **UGC JRF** (UGC-Ref. No.: 1251/(OBC)(CSIR-UGC NET DEC.2016), Government of India w.e.f. March 18, 2019.
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In 1971, the problem of complex spline interpolation was initiated by Schoenberg [13] and Ahlberg, Nilson and Walsh in a sequence of papers [1–3]. The solutions were completely different. A related problem on the trigonometric spline interpolation was beautifully studied by Schoenberg [12], connecting the study to the differential operators $\Delta_m = D(D^2 + 1^2)\dots(D^2 + m^2)$, $(D = d/dx)$. Micchelli [6] exploiting Schoenberg's idea and using the cardinal $\mathcal{L}$-splines related to the differential operator $\mathcal{L} = \prod_{j=0}^{n}(D-\gamma_j)$ with γ_j as real numbers, gave a complete and systematic treatment to the interpolation problem. Schoenberg [14] revisited Micchelli's theory and extended it to the operator $\mathcal{L}$ with imaginary γ_j's. Sharma and Tzimbalario [7] and Tzimbalario [20] further extended the study for cardinal splines related to the operators Δ_m and $\mathcal{L} = \prod_{j=0}^{n}(D - i(j + \ell)\eta)$ for some $\eta > 0$ and ℓ real, respectively.

Kvasov [5], Subbotin [15] (with different conditions), Subbotin and Chernykh [19] and Shevaldin [10] (in a more general statement) constructed local parabolic splines for functions defined on the axis or on the segment of the axis that preserve linear functions with an arbitrary distinct setting of nodes with good approximation property and their own local preservation of the sign, monotonicity and convexity of approximate functions [11]. Subbotin and Shevaldin [16] developed a general scheme of constructing such structures. These splines and their generalizations are widely used in computational mathematics. The works of Shevaldin [8, 9] also deserve a mention.

In another paper, authors [21] have introduced a non-interpolatory complex parabolic spline on the unit circle and have studied the rate of convergence and the error in approximation corresponding to a complex valued functions.

Following the ideas of Subbotin, Kostousov et.al. [4], taking $W^2_\infty = \{f : f' \in AC, \|f''\|_\infty \leq 1\}$, considered the local trigonometric splines, corresponding to the differential operators $\mathcal{L}_2 = \mathcal{D}^2 + \alpha^2 I$ $()\alpha > 0, \mathcal{D} = \partial_x$ is the first derivative $, I$ is the identity operator) for approximating the class of 1-periodic functions $f \in W^2_\infty$ with $\|\mathcal{L}_2(D)\|_\infty \leq 1$.

In this paper, we consider a function f, which is analytic on K and associate with it an operator $L_2D(f)$ defined as

$$L_2D(f) = (z^2D^2 + zD + \beta^2 I)f$$

with $0 < \beta < \infty$. Then we construct a non interpolatory complex trigonometric spline, say $\mathbb{S}_j(z)$, $j = 1, 2, \dots, n$, on each K_j. Further, taking $W = \{f : \|L_2D(f)\|_\infty \leq 1\}$, we have obtained a quantitative estimate of $S_j(z)$ corresponding to this $f \in W$.

2. Construction of complex trigonometric spline

Let K be a unit circle in the complex plane and $z_1, z_2, \dots, z_n$ be points on K arranged in counterclockwise order, separating K into arcs $\{K_j\}_{j=1}^n$ from z_j to z_{j+1}, where $z_j = \exp\left(\frac{2j\pi i}{n}\right)$. Obviously,

$$z_{j+1} = \exp\left(\frac{2\pi i}{n}\right) z_j.$$

We are interested to construct a non-interpolatory trigonometric spline $\mathbb{S}_\Delta(z)$ on K for the subdivision $\Delta = \{K_j; j = 1, 2, \ldots, n\}$. Here, z_j will be called **main nodes** and $z_{j+\frac{1}{2}} (j = 1(1)n)$ will be called **additional nodes**.
Let $f : \mathbb{C} \to \mathbb{C}$ and $y_j = f(z_j)$.Write $h = \frac{2\pi}{n}$. Associate operators Ω and Λ, on the space of sequences $\{y_j\}$ defined as

$$\Omega(y_{j-1}) := \frac{\beta(y_{j+1} - y_{j-1})}{2\ \sin(i\beta h)z_j}$$

and,

$$\Lambda(y_{j-1}) := y_{j+1} - 2y_j \cos(i\beta h) + y_{j-1}.$$

Here, $\Omega(y_{j-1})$ approximately gives derivative of f at $z = z_j$ because

$$\begin{aligned} f'(z_j) &\approx \frac{f(z_{j+1}) - f(z_{j-1})}{z_{j+1} - z_{j-1}} \\ &= \frac{y_{j+1} - y_{j-1}}{(e^{ih})z_j - (e^{-ih})z_j} \\ &= \frac{y_{j+1} - y_{j-1}}{(2i\,\mathrm{sinh})z_j}. \end{aligned}$$

When h is very small then $\sin h \approx h$

$$\therefore f'(z_j) \approx \frac{y_{j+1} - y_{j-1}}{(2ih)z_j}.$$

Further,

$$\begin{aligned} \Omega(y_{j-1}) &= \beta\,\frac{(y_{j+1} - y_{j-1})}{2\sin(i\beta h)z_j} \\ &= \left(\frac{i\beta h}{\sin(i\beta h)}\right)\left(\frac{y_{j+1} - y_{j-1}}{2ihz_j}\right) \\ &\approx \frac{y_{j+1} - y_{j-1}}{2ihz_j}, \end{aligned}$$

as h is very small.
Note that when h is small then $\cos(i\beta h) \approx 1$

$$\begin{aligned} \therefore \Lambda(y_{j-1}) &\approx y_{j+1} - 2y_j + y_{j-1} \\ &= (y_{j+1} - y_j) - (y_j - y_{j-1}), \end{aligned}$$

which is the second forward difference at y_{j-1}. For $z \in K_j$, the spline $\mathbb{S}_j(z)$ is represented in the form

$$\mathbb{S}_j(z) = C_0^{(j)} + C_1^{(j)} \cos\left(\beta \log \frac{z}{z_j}\right) + C_2^{(j)} \sin\left(\beta \log \frac{z}{z_j}\right) + C_3^{(j)} \phi\left(\frac{z}{z_{j+1/2}}\right)_+ \qquad (2.1)$$

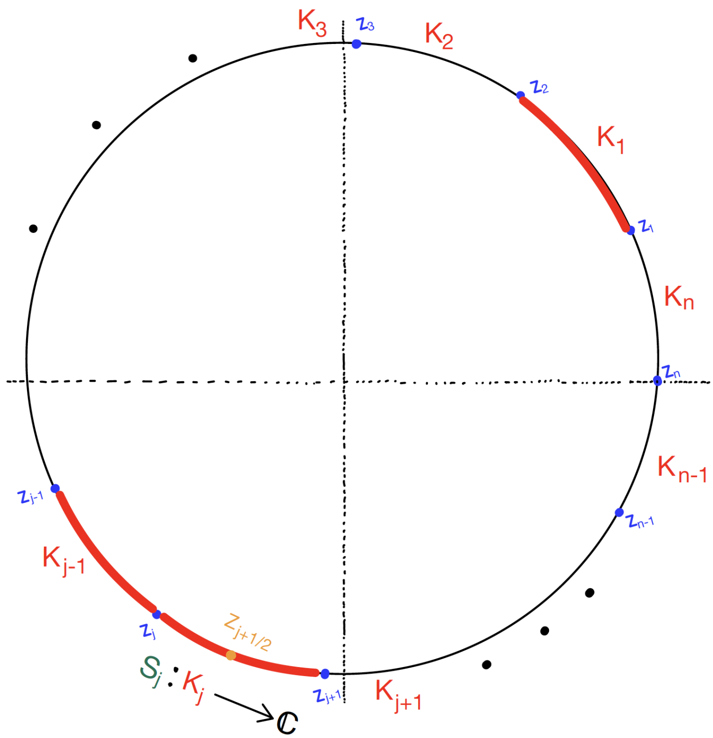

FIGURE 1: Uniform distribution of nodes on unit circle

where, ϕ is a solution of the differential equation $z^2D'' + zD' + \beta^2 D = 1$.
Also,

$$\phi\left(\frac{z}{z_{j+1/2}}\right)_+ = \begin{cases} 0 & , \ \arg z \leq \arg z_{j+\frac{1}{2}} \\ \phi\left(\frac{z}{z_{j+1/2}}\right) & , \ \arg z > \arg z_{j+\frac{1}{2}} \end{cases} \tag{2.2}$$

and, the complex constants $C_0^{(j)}, C_1^{(j)}, C_2^{(j)}, C_3^{(j)}$ can be determined by assuming that the spline $\mathbb{S}_\Delta(z)$ is continuously differentiable and satisfies the differential equations:
For $\arg z \leq \arg z_{j+\frac{1}{2}}$

$$z^2\mathbb{S}_j''(z) + z\mathbb{S}_j'(z) + \beta^2\mathbb{S}_j(z) = \frac{\beta^2\Lambda(y_{j-1})}{2\sin(i\beta h)\sin(i\beta h/2)}, \tag{2.3a}$$

and, for $\arg z > \arg z_{j+\frac{1}{2}}$

$$z^2\mathbb{S}_j''(z) + z\mathbb{S}_j'(z) + \beta^2\mathbb{S}_j(z) = \frac{\beta^2\Lambda(y_j)}{2\sin(i\beta h)\sin(i\beta h/2)}. \tag{2.3b}$$

Take

$$\mathbb{S}_j(z_j) = y_j + b\Lambda(y_{j-1}), \quad \mathbb{S}_j(z_{j+1}) = y_{j+1} + b\Lambda(y_j),$$

for continuity of $\mathbb{S}_\Delta(z)$ on K,
where

$$b = \frac{1}{8\cos^2(i\beta h/4)\cos(i\beta h/2)}.$$

Using above in (2.1) we get,

$$\begin{aligned} &S_j(z_j) = y_j + b\Lambda(y_{j-1}) \\ &\therefore C_0^j + C_1^j \cos 0 + C_2^j \sin 0 + C_3^j \times 0 = y_j + b\Lambda(y_{j-1}) \\ &\Longrightarrow \qquad C_0^j + C_1^j = y_j + b\Lambda(y_{j-1}). \end{aligned} \tag{2.4}$$

Now, take $\mathbb{S}'_j(z_j) = \Omega(y_{j-1})$ and $\mathbb{S}'_j(z_{j+1}) = \Omega(y_j)$ (for differentiability of $\mathbb{S}_\Delta(z)$).
If we take $\phi(z) = \frac{1}{\beta^2}[1-\cos(\beta \log z)]$, which is one of the solution of $z^2D'' + zD' + \beta^2 I = 1$, then the spline (2.1) must be C^1 on the arc K_j and hence,

$$\lim_{z\to z_j} \mathbb{S}'_{j-1}(z) = \lim_{z\to z_j} \mathbb{S}'_j(z) \quad \forall j = 1, 2, \ldots, n.$$

Now, using $\mathbb{S}'_j(z_j) = \Omega(y_{j-1})$ in (2.1), we obtain

$$\begin{aligned} &-\frac{C_1^{(j)}\beta}{z}\sin\left(\beta\log\frac{z}{z_j}\right) + \frac{C_2^{(j)}\beta}{z}\cos\left(\beta\log\frac{z}{z_j}\right) + \frac{C_3^{(j)}}{z_{j+1/2}}\phi'\left(\frac{z}{z_{j+1/2}}\right)_+\Bigg|_{z=z_j} = \Omega(y_{j-1}) \\ &\Rightarrow \qquad -\frac{C_1^{(j)}\beta}{z_j}\sin 0 + \frac{C_2^{(j)}\beta}{z_j}\cos 0 + 0 = \Omega(y_{j-1}) \\ &\Rightarrow \qquad C_2^{(j)} = \frac{y_{j+1} - y_{j-1}}{2\sin(i\beta h)}. \end{aligned} \tag{2.5}$$

On differentiating (2.1) w.r.t z, we get

$$\mathbb{S}'_j(z) = -\frac{C_1^{(j)}\beta}{z}\sin\left(\beta\log\frac{z}{z_j}\right) + \frac{C_2^{(j)}\beta}{z}\cos\left(\beta\log\frac{z}{z_j}\right) + \frac{C_3^{(j)}}{z_{j+1/2}}\phi'\left(\frac{z}{z_{j+1/2}}\right)_+. \tag{2.6}$$

Again differentiating with respect to z, we get

$$\begin{aligned} \mathbb{S}''_j(z) = &\frac{C_1^{(j)}\beta}{z^2}\sin\left(\beta\log\frac{z}{z_j}\right) - \frac{C_1^{(j)}\beta^2}{z^2}\cos\left(\beta\log\frac{z}{z_j}\right) - \frac{C_2^{(j)}\beta}{z^2}\cos\left(\beta\log\frac{z}{z_j}\right) \\ &- \frac{C_2^{(j)}\beta^2}{z^2}\sin\left(\beta\log\frac{z}{z_j}\right) + \frac{C_3^{(j)}}{z_{j+1/2}}\phi''\left(\frac{z}{z^2_{j+1/2}}\right)_+ \end{aligned} \tag{2.7}$$

Using (2.3a) at $z = z_j$ we get

$$(0 - C_1^{(j)}\beta^2 - C_2^{(j)}\beta - 0 - 0) + (-0 + C_2^{(j)}\beta) + (\beta^2 C_0^{(j)} + \beta^2 C_1^{(j)}) = \frac{\beta^2\Lambda(y_{j-1})}{2\sin(i\beta h)\sin(i\beta h/2)}$$

$$\Rightarrow \qquad C_0^{(j)} = \frac{\Lambda(y_{j-1})}{2\sin(i\beta h)\sin(i\beta h/2)}. \tag{2.8}$$

Using (2.4), we have

$$\begin{aligned}
C_1^{(j)} &= y_j + b\Lambda(y_{j-1}) - C_0^{(j)} \\
&= y_j + \frac{\Lambda(y_{j-1})}{8\cos^2(i\beta h/4)\cos(i\beta h/2)} - \frac{\Lambda(y_{j-1})}{2\sin(i\beta h)\sin(i\beta h/2)} \\
&= y_j - \Lambda(y_{j-1})\left[\frac{1}{2\sin(i\beta h)\sin(i\beta h/2)} - \frac{1}{8\cos^2(i\beta h/4)\cos(i\beta h/2)} \times \frac{\sin^2(i\beta h/4)}{\sin^2(i\beta h/4)}\right] \\
&= y_j - \Lambda(y_{j-1})\left[\frac{1}{4\sin^2(i\beta h/2)\cos(i\beta h/2)} - \frac{\sin^2(i\beta h/4)}{2\sin^2(i\beta h/2)\cos(i\beta h/2)}\right] \\
&= y_j - \frac{\Lambda(y_{j-1})}{4\sin^2(i\beta h/2)}\left[\frac{1-2\sin^2(i\beta h/4)}{\cos(i\beta h/2)}\right]
\end{aligned}$$

Thus,

$$C_1^{(j)} = y_j - \frac{\Lambda(y_{j-1})}{4\sin^2(i\beta h/2)} \tag{2.9}$$

Using (2.6) and (2.7) at $z = z_{j+1}$, we get

$$z_{j+1}\mathbb{S}_j'(z_{j+1}) = -C_1^{(j)}\beta\sin(i\beta h) + C_2^{(j)}\beta\cos(i\beta h) + C_3^{(j)}e^{ih/2}\phi'(e^{ih/2}),$$

and,

$$\begin{aligned}
z_{j+1}^2\mathbb{S}_j''(z_{j+1}) = {} & C_1^{(j)}\beta\sin(i\beta h) - C_1^{(j)}\beta^2\cos(i\beta h) - C_2^{(j)}\beta\cos(i\beta h) - C_2^{(j)}\beta^2\sin(i\beta h) \\
& + C_3^{(j)}(e^{ih/2})^2\phi''(e^{ih/2}).
\end{aligned}$$

Also, we have,

$$\beta^2\mathbb{S}_j(z_{j+1}) = C_0^{(j)}\beta^2 + C_1^{(j)}\beta^2\cos(i\beta h) + C_2^{(j)}\beta^2\sin(i\beta h) + C_3^{(j)}\beta^2\phi(e^{ih/2}).$$

Using all above in (2.3b), we get,

$$\begin{aligned}
& C_0^{(j)}\beta^2 + C_3^{(j)}\left[(e^{ih/2})^2\phi''(e^{ih/2}) + e^{ih/2}\phi'(e^{ih/2}) + \beta^2\phi(e^{ih/2})\right] = \frac{\beta^2\Lambda(y_j)}{2\sin(i\beta h)\sin(i\beta h/2)} \\
& \Rightarrow C_0^{(j)}\beta^2 + C_3^{(j)} \times 1 = \frac{\beta^2\Lambda(y_j)}{2\sin(i\beta h)\sin(i\beta h/2)} \\
& \Rightarrow C_3^{(j)} = \frac{\beta^2\Lambda(y_j)}{2\sin(i\beta h)\sin(i\beta h/2)} - \frac{\beta^2\Lambda(y_{(j-1)}}{2\sin(i\beta h)\sin(i\beta h/2)} \\
& \Rightarrow C_3^{(j)} = \frac{\beta^2[\Lambda(y_j - \Lambda(y_{j-1})]}{2\sin(i\beta h)\sin(i\beta h/2)}.
\end{aligned} \tag{2.10}$$

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3. Approximation by complex trigonometric splines

Let K_1 be the arc joining the points z_1 and z_2. Moreover we can assume that z lies in the arc joining $z_1 = e^{ih}$ and $z_{3/2} = e^{3ih/2}$, where $\arg(z) < \arg(z_{3/2})$. Otherwise, we can make a change in variable $z = z_2 v$.
For z, lying in the arc joining z_1 and $z_{3/2}$, the spline has the form

$$\begin{aligned}\mathbb{S}_1(z) &= \frac{\Lambda(y_0)}{2\sin(i\beta h)\sin(i\beta h/2)} + \left[y_1 - \frac{\Lambda(y_0)}{4\sin^2(i\beta h/2)}\right]\cos\left(\beta\log\left(\frac{z}{z_1}\right)\right) \\ &+ \frac{(y_2 - y_0)}{2\sin(i\beta h)}\sin\left(\beta\log\left(\frac{z}{z_1}\right)\right) \\ &= \frac{1}{2\sin(i\beta h)}\left[\frac{y_2 - 2y_1\cos(i\beta h) + y_0}{\sin(i\beta h/2)} + \frac{1}{\sin(i\beta h)}\cos\left(\beta\log\left(\frac{z}{z_1}\right)\right)\times\right. \\ &\left.(2y_1 - y_2 - y_0)(1 + \cos(i\beta h)) + (y_2 - y_0)\sin\left(\beta\log\left(\frac{z}{z_1}\right)\right)\right]. \quad (3.1)\end{aligned}$$

We are interested in getting the integral representation of the spline $\mathbb{S}_1(z)$ for any function f which is analytic in $\mathbb{C}$.
Consider the linear differential equation

$$(z^2D^2 + zD + \beta^2)f(z) = zu(log(z)). \quad (3.2)$$

The general solution of (3.2) has the form

$$f(z) = c_1\cos\left(\beta\ \log\left(\frac{z}{z_1}\right)\right)+c_2\sin\left(\beta\ \log\left(\frac{z}{z_1}\right)\right)+\frac{1}{\beta}\int_{z_1}^{z} u(\log(t))\sin\left(\beta\ \log\left(\frac{z}{t}\right)\right)dt, \quad (3.3)$$

where c_1 and c_2 are arbitrary constants. Thus

$$y_0 = f(z_0) = c_1\cos(i\beta h) - c_2\sin(\beta ih) + \frac{1}{\beta}\int_{z_1}^{z_0} u(\log t)\sin\left(\beta\ \log\left(\frac{z_0}{t}\right)\right)dt, \quad (3.4)$$

$$y_1 = f(z_1) = c_1, \quad (3.5)$$

and,

$$y_2 = f(z_2) = c_1\cos(\beta ih) + c_2\sin(\beta ih) + \frac{1}{\beta}\int_{z_1}^{z_2} u(\log t)\sin\left(\beta\ \log\left(\frac{z_2}{t}\right)\right)dt. \quad (3.6)$$

Let us denote

$$I_0 = \int_{z_0}^{z_1} u(\log t)\sin\left(\beta\ \log\left(\frac{z_0}{t}\right)\right)dt,$$
$$I_2 = \int_{z_1}^{z_2} u(\log t)\sin\left(\beta\ \log\left(\frac{z_2}{t}\right)\right)dt,$$

then, to obtain the integral representation of the spline, we write (3.1) as

$$\mathbb{S}_1(z) = \frac{1}{2\sin(i\beta h)}\left[A_1 + \frac{1}{\sin(i\beta h)}\cos\left(\beta\log\left(\frac{z}{z_1}\right)\right)A_2 + A_3\right] \tag{3.7}$$

where,

$$A_1 = \frac{y_2 - 2y_1\cos(i\beta h) + y_0}{\sin(i\beta h/2)} = \frac{I_2 - I_0}{\beta\sin(i\beta h/2)} \tag{3.8}$$

$$\begin{aligned} A_2 &= (y_2 - y_0)\cos(i\beta h) - (y_3 - y_1) + (y_1 - y_2)(1 + 2\cos(i\beta h)) + y_3 - y_0 \\ &= \frac{1}{\beta}\left[(1 + \cos(i\beta h))(I_0 - I_2) + 2c_1\beta\sin^2(i\beta h)\right] \end{aligned} \tag{3.9}$$

$$A_3 = (y_2 - y_0)\sin\left(\beta\log\left(\frac{z}{z_1}\right)\right) = \frac{1}{\beta}\sin\left(\beta\log\left(\frac{z}{z_1}\right)\right)\left[2c_2\beta\sin(i\beta h) + I_2 + I_0\right]. \tag{3.10}$$

Therefore, due to (3.8), (3.10) and (3.9), after simple calculations, (3.7) reduces to

$$\begin{aligned} \mathbb{S}_1(z) = &\frac{1}{2\beta\sin(i\beta h)}\left[I_0\left\{\frac{-1}{\sin(i\beta h/2)} + \frac{(1+\cos(i\beta h))}{\sin(i\beta h)}\cos\left(\beta\log\left(\frac{z}{z_1}\right)\right) + \sin\left(\beta\log\left(\frac{z}{z_1}\right)\right)\right\}\right. \\ &\left. + I_2\left\{\frac{1}{\sin(i\beta h/2)} - \frac{(1+\cos(i\beta h))}{\sin(i\beta h)}\cos\left(\beta\log\left(\frac{z}{z_1}\right)\right) + \sin\left(\beta\log\left(\frac{z}{z_1}\right)\right)\right\}\right] \\ &+ c_1\cos\left(\beta\log\left(\frac{z}{z_1}\right)\right) + c_2\sin\left(\beta\log\left(\frac{z}{z_1}\right)\right) \\ = &\frac{1}{2\beta\sin(i\beta h)\sin(i\beta h/2)}\left[I_0\left\{-1 + cos\left(\frac{i\beta h}{2} - \beta\log\left(\frac{z}{z_1}\right)\right)\right\}\right. \\ &\left. + I_2\left\{1 - \cos\left(\frac{i\beta h}{2} + \beta\log\left(\frac{z}{z_1}\right)\right)\right\}\right] + c_1\cos\left(\beta\log\left(\frac{z}{z_1}\right)\right) + c_2\sin\left(\beta\log\left(\frac{z}{z_1}\right)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} | f(z) - \mathbb{S}_1(z) |= &\left|\frac{1}{\beta}\int_{z_1}^{z} u(\log t)\sin\left(\beta\log\left(\frac{z}{t}\right)\right)dt\right. \\ &- \frac{-1}{2\beta\sin(i\beta h)\sin(i\beta h/2)}\left[\left\{1 - \cos\left(\beta\log\left(\frac{z}{z_{3/2}}\right)\right)\right\}\int_{z_0}^{z_1} u(\log t)\sin\left(\beta\log\left(\frac{z_0}{t}\right)\right)dt\right. \\ &\left.\left. - \left\{1 - \cos\left(\beta\log\left(\frac{z}{z_{1/2}}\right)\right)\right\}\int_{z_1}^{z_2} u(\log t)\sin\left(\beta\log\left(\frac{z_2}{t}\right)\right)dt\right]\right|. \end{aligned}$$

On taking $z = e^{i\theta}$, $t = e^{i\phi}$ and breaking the second integral, we get

$$|f(z) - \mathbb{S}_1(z)| = \left|\frac{1}{\beta}\int_{h}^{\theta} ie^{i\phi}u(i\phi)\sin(i\beta(\theta - \phi))d\phi\right.$$

$$
\begin{aligned}
&- \frac{-1}{2\beta\sin(i\beta h) sin(i\beta h/2)}\Bigg[\Bigg\{1-\cos\Big(i\beta\Big(\theta-\frac{3h}{2}\Big)\Big)\Bigg\}\int_0^h ie^{i\phi}u(i\phi)\sin(-i\beta\,\phi)d\phi \\
&- \Bigg\{1-\cos\Big(i\beta\Big(\theta-\frac{h}{2}\Big)\Big)\Bigg\}\int_h^\theta ie^{i\phi}u(i\phi)\sin(i\beta\,(2h-\phi))d\phi \\
&- \Bigg\{1-\cos\Big(i\beta\Big(\theta-\frac{h}{2}\Big)\Big)\Bigg\}\int_\theta^{2h} ie^{i\phi}u(i\phi)\sin(i\beta\,(2h-\phi))d\phi\Bigg]\Bigg| \equiv |I_{11}-I_{12}-I_{13}| \\
&\leq |I_{11}|+|I_{12}|+|I_{13}|, \qquad (3.11)
\end{aligned}
$$

where,

$$I_{11} = \int_0^h L_1(\theta,\phi)u(i\phi)d\phi, \qquad (3.12)$$

$$I_{12} = \int_\theta^{2h} L_2(\theta,\phi)u(i\phi)d\phi, \qquad (3.13)$$

$$I_{13} = \int_h^\theta L_3(\theta,\phi)u(i\phi)d\phi, \qquad (3.14)$$

with

$$L_1(\theta,\phi) = -\frac{ie^{i\phi}\sin(i\beta\phi)}{2\beta\sin(i\beta h)\sin(i\beta h/2)}\Bigg[1-\cos\Big(i\beta\Big(\theta-\frac{3h}{2}\Big)\Big)\Bigg], \qquad (3.15)$$

$$L_2(\theta,\phi) = \frac{ie^{i\phi}\sin(i\beta(2h-\phi))}{2\beta\sin(i\beta h)\sin(i\beta h/2)}\Bigg[1-\cos\Big(i\beta\Big(\theta-\frac{h}{2}\Big)\Big)\Bigg], \qquad (3.16)$$

and,

$$L_3(\theta,\phi) = \frac{ie^{i\phi}}{\beta}\Bigg[\sin(i\beta(\theta-\phi)) - \frac{\sin(i\beta(2h-\phi))}{2\sin(i\beta h)\sin(i\beta h/2)}\Bigg[1-\cos\Big(i\beta\Big(\theta-\frac{h}{2}\Big)\Big)\Bigg]\Bigg]. \qquad (3.17)$$

Let us calculate the absolute value of I_{11}, I_{12} and I_{13}.

$$
\begin{aligned}
|\,I_{11}\,| &\leq \frac{|\,1-\cos(i\beta(\theta-3h/2))\,|}{2\,|\,\beta\sin(i\beta h)\sin(i\beta h/2)\,|}\int_0^h |\,e^{i\phi}u(i\phi)\sin(i\beta\phi)\,|\,d\phi \\
&\leq \frac{|\,1-\cos(i\beta(\theta-3h/2))\,|}{2\,|\,\beta\sin(i\beta h)\sin(i\beta h/2)\,|}\,|\,u(ih)\sin(i\beta h)\,|\int_0^h d\phi.
\end{aligned}
$$

If $h\to 0$, then

$$|\,I_{11}\,| \leq \frac{1+|\cos(i\beta\theta)\,|}{\beta^2}\,|\,u(0)\,|, \qquad (3.18)$$

for $h\to 0$,

$$|\,I_{12}\,| \leq \frac{4+4\,|\cos(i\beta\theta)\,|}{\beta^2}\,|\,u(0)\,|\,. \qquad (3.19)$$

Lastly,

$$|\,I_{13}\,| = \Bigg|\int_h^\theta \frac{ie^{i\phi}}{\beta}\Bigg[\sin(i\beta(\theta-\phi)) - \frac{\sin(i\beta(2h-\phi))\,(1-\cos(i\beta(\theta-h/2)))}{2\sin(i\beta h)\sin(i\beta h/2)}\Bigg]u(i\phi)d\phi\Bigg|$$

$$\leq \left|\frac{1}{\beta}\left[\sin(i\beta(\theta-h)) - \frac{(1-\cos(i\beta(\theta-h/2)))}{2\sin(i\beta h/2)}\right]u(ih)(\theta-h)\right|$$
$$= \left|\frac{u(ih)(\theta-h)}{2\beta\sin(i\beta h/2)}\left[\cos(i\beta(\theta-3h/2))-1\right]\right|.$$

If $h \to 0$, $\theta/h \to 0$, hence

$$|I_{13}| \leq \frac{1+|\cos(i\beta\theta)|}{\beta^2}|u(0)|.$$

Thus, we have proved that

THEOREM 3.1. *Let f be analytic on K. Let $L_2D(f)$ be an operator associated with the function f, defined as $L_2D(f) = (z^2D^2 + zD + \beta^2 I)f$ with $0 < \beta < \infty$. Let $W = \{f : \|L_2D(f)\|_\infty \leq 1\}$, and let $\mathbb{S}_j(z)$ be the complex non interpolatory trigonometric spline on K_j, then*

$$\sup_{f\in W}|f(z) - \mathbb{S}_j(z)| \leq \frac{6+6\mid\cos(i\beta\theta)\mid}{\beta^2}.$$

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