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APPROXIMATION OF A FUNCTION $f \in W(L_p, \xi(t))$ CLASS BY $(C,2)[F,d_n]$ MEANS OF ITS FOURIER SERIES

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Abstract

We obtained on approximation of a continuous function belonging to $W\left(L_p, \xi(t)\right)$ Class by $(C,1)\,[F,d_n]$ means of its Fourier series [18]. In this paper we using the product summability method of the $(C,2)$ and $[F,d_n]$ associate with infinite series and we obtain new results on approximation of a function $f \in W(L_p, \xi(t))$ Class by $(C,2)[F,d_n]$ means of its Fourier series.

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1. Introduction

The summability methods $[F,d_n]$ was introduced by Jakimovsky [4] and we studied on approximation of f belong to many classes also $W(L_P,(\xi(t))$. by Cesăro mean, Nörlund mean has been discussed by investigator like respectively Alexits [1], Chandra [2], Khan [6], Qureshi [13], Shrivastava, verma and yadav [19] etc. We also studied about product summability on approximation has been obtained by several researchers like Lal and Kushwaha [8], Nigam [12], Kushwaha [7] etc. Further Rathore and Shrivastava [14] determined on approximation of a function belonging to $W\,(L_r, \xi(t))$ by $(C,2)(E,q)$ product summability. Recently has been established by Rathore, Shrivastava and Mishra ([15], [16], [17]). Further generalizing the result of Rathore and Shrivastava [18] obtained on approximation of continuous function $f \in W\left(L_p, \xi(t)\right)$ by $(C,1)\,[F,d_n]$ mean of Fourier series. We extend the result on approximation of function $f \in W\left(L_p, \xi(t)\right)$ by $(C,2)\,[F,d_n]$ mean has been proved.

2. Definition and Notation

Let f be periodic and $L-$integrable function on $[-\pi,\pi]$. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \operatorname{sinn} x) \tag{2.1}$$

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with n^{th} partial sum $S_n(f; x)$.

Let $d_1, d_2 - - - d_n$, be a fixed sequence and x be a real number. The element P_{nk} of $[F, d_n]$ matrix are defined by the relations

$$\prod_{j=1}^{n} \frac{x + d_j}{1 + d_j} = \sum_{k=0}^{\infty} P_{nk} x^k \tag{2.2}$$

and

$$P_{00} = 1. \tag{2.3}$$

Let

$$\sigma_x(f; x) = \sum_{k=0}^{\infty} P_{nk} S_k(f; x). \tag{2.4}$$

Denote the $[F, d_n]$ mean of $f \in L[-\pi, \pi]$ at x, where $S_k(f; x)$ is the k^{th} partial sum of (2.1). The $[F, d_n]$ method were introduced by Jakimovsky [4] as generalization of both the Euler E_r method and Stirling- Karamata-Lototsky method. When $d_n = \frac{(n-1)}{c}, n = 1, 2, 3, ..., c$, a positive integer the $[F, d_n]$ matrix reduces to the matrix corresponding to the Stirling-Karamata-Lototsky method defined by Karamata [5]. The Euler $E_r(0 < r < 1)$ are obtained with $d_n = \frac{(1-r)}{r}, n = 1, 2, 3, ...$ Lorch and Newman [9] studied the Lebesgue constant for this method. Several fundamental properties of $[F, d_n]$ matrix have been discussed in Meir and Miracle [10, 11]

$$S_n = 2 \sum_{k=1}^{n} \frac{d_k}{\left(1 + d_k^2\right)} \tag{2.5}$$

and

$$U_n = 1 + 2 \sum_{k=1}^{n} \frac{1}{(1 + d_k)}. \tag{2.6}$$

The $[F, d_n]$ matrix is regular by Jakimovsky[4] if $U_n \to \infty$ as $n \to \infty$ we shall consider only regular matrices and indeed assume that for large n then d_n is bounded away from zero.

The $\frac{d_n}{(1+d_n)^2}$ is also bounded and $S_n \to \infty$ as $n \to \infty$.

$n + 1 = [U_n]$ be the integral part of U_n.

A function $f \in W\left(L_p, \xi(t)\right)$ class, if

$$(\int_0^{2\pi} (\left|[f(x + t) - f(x)] \sin^\beta x\right|^p dx))^{1/p} = O(\xi(t)), (\beta \geq 0). \tag{2.7}$$

denotes $\xi(t)$ is a increasing function and $p \geq 1$,

If the $(C, 2)$ transform of S_n is defined as (see Hardy [3])

$$t_n^{(C,2)}(f : x) = \frac{2 \sum_{k=0}^{n} (n - k + 1)}{(n + 1)(n + 2)} S_k \to S, n \to \infty. \tag{2.8}$$

Let $\left\{t_n^{(C,2)}\right\}$ denote the sequence of $(C,2)$ transform of $\{S_n\}$.

$$t_n^{(C,2)[F,d_n]}(f:x) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)} t_k^{[F,d_n]} \to S, n \to \infty \tag{2.9}$$

where $t_n^{(C,2)[F,d_n]}$ denote the sequence of $(C,2)\,[F,d_n]$ transform of S_n, and $\sum_{k=0}^{\infty} u_k$ is summable to S by $(C,2)\,[F,d_n]$ product transform.

We have

$$\mathrm{Lip}\,\alpha \subseteq \mathrm{Lip}(\alpha,p) \subseteq \mathrm{Lip}(\xi(t),p) \subseteq W\left(L_p,\xi(t)\right), \text{ for } 0<\alpha\leq 1, p \geq 1 \tag{2.10}$$

we define

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}, p \geq 1. \tag{2.11}$$

The degree of approximation $E_n(f)$ be given by

$$E_n(f) = \min \|T_n - f\|_p\,, \tag{2.12}$$

where $T_n(x)$ is a trigonometric polynomial of degree n by (Zygmund[21]).

Now using

$$\phi(t) = f(x+t) + f(x-t) - 2f(x) \tag{2.13}$$

3. Main theorem

We prove the following theorem

Theorem: If a function $f : R \to R$ is integrable and periodic function on $[0,2\pi]$ belongs to $W\left(L_p,\xi(t)\right)$ class associate with the approximation of f by $(C,2)\,[F,d_n]$ summability means of its (2.1) satisfies.

$$\left\|t_n^{(C,2)[F,d_n]} - f(x)\right\|_p = O\left[(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right] \tag{3.1}$$

provided $\xi(t)$ satisfies the following condition :

$$\text{decreasing sequence be } \left\{\frac{\xi(t)}{t}\right\} \tag{3.2}$$

$$\left[\int_0^{\frac{\pi}{n+1}} \left\{\frac{t|\phi(t)|\sin^\beta t}{\xi(t)}\right\}^p dt\right]^{1/p} = O\left(\frac{1}{n+1}\right) \tag{3.3}$$

$$\left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{t^{-\delta}|\phi(t)|\sin^\beta t}{\xi(t)}\right\}^p dt\right]^{1/p} = \left\{(n+1)^\delta\right\}. \tag{3.4}$$

The condition $q(1-\delta)-1>0$, when $\frac{1}{p}+\frac{1}{q}=1$, where δ is an arbitrary number then uniformly in x the conditions (3.3) and (3.4) hold and $(C,2)\,[F,d_n]$ is denoted by $t_n^{(C,2)[F,d_n]}$ means of (2.1).

4. Lemma

We will use following lemmas:

Lemma 1. For our theorem required lemmas:

$$\prod_{k=1}^{n} \frac{\exp(it) + d_k}{1 + d_k} = \exp\left\{(U_n - 1)\, it/2 - S_n t^2/4\right\} + O\left(S_n t^3\right). \qquad (4.1)$$

This is due to Lorch and Newman [9]

Lemma 2.

$$K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \frac{\sin\left(\frac{U_n t}{2}\right)\exp\left(\frac{-S_n t^2}{4}\right)}{\sin t/2}$$

then

$$|K_n(t)| = O(n+1), \text{ for } 0 \le t \le \frac{\pi}{(n+1)}$$

Proof. Apply $\sin nt \le n \sin t$ for $0 \le t \le \frac{\pi}{n+1}$

Then

$$\begin{aligned} K_n(t) &= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \exp\left(\frac{-S_n t^2}{4}\right) \frac{U_n \sin t/2}{\text{sint}\,/2} \\ &= O\left(U_n\right) \frac{1}{(n+2)\pi} - \sum_{k=0}^{n} k \\ &= \frac{2}{(n+2)\pi} - \frac{n(n+1)}{2} \\ &= O(n+1) \end{aligned}$$

Lemma 3.

$$K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \frac{\sin\left(\frac{U_n t}{2}\right)\exp\left(\frac{-S_n t^2}{4}\right)}{\sin t/2}$$

then

$$|K_n(t)| = O\left(\frac{1}{t}\right), \text{ for } \frac{\pi}{n+1} \le t \le \pi$$

Proof- Using $\sin \frac{t}{2} \ge \left(\frac{t}{\pi}\right)$ and $\left|\sin \frac{U_n t}{2}\right| \le 1$ for $\frac{\pi}{n+1} \le t \le \pi$

$$\begin{aligned} K_n(t) &= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \exp\left(\frac{-S_n t^2}{4}\right) \frac{1}{t/\pi} \\ &= \frac{2}{(n+2)t} - \sum_{k=0}^{n} k \\ &= \frac{2}{(n+2)t} - \frac{n(n+1)}{2t} \end{aligned}$$

$$= O\left(\frac{1}{t}\right) \tag{4.3}$$

Lemma 4.

$$K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\frac{O\left(S_n t^3\right)}{\sin\frac{t}{2}}$$
$$= O(n+1), \text{ for } 0 \le t \le \frac{\pi}{n+1}$$

Proof- Using $\left|\sin\frac{t}{2}\right| \le \frac{t}{\pi}$ for $0 \le t \le \frac{\pi}{n+1}$

Then

$$K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\frac{O\left(S_n t^3\right)}{t/\pi}$$
$$= \frac{O(S_n)}{(n+1)(n+2)}2\sum_{k=0}^{n}(n-k+1)t^2$$
$$= \left[\frac{2}{(n+2)} - \sum_{k=0}^{n}k\right]t^2$$
$$= \left[\frac{2}{(n+2)} - \frac{n(n+1)}{2}\right]\frac{\pi^2}{(n+1)^2}$$
$$= O(n+1) \tag{4.4}$$

Lemma 5.

$$K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}\frac{O\left(S_n t^3\right)}{\sin\frac{t}{2}}$$
$$= O(n+1) \text{ for } \frac{\pi}{(n+1)} \le t \le \pi$$

Proof - Using $\sin\frac{t}{2} \ge \frac{t}{\pi}$ for $\frac{\pi}{(n+1)} \le t \le \pi$

Then

$$K_n(t) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi}O(S_n)\frac{t^3}{t_{/\pi}}$$
$$= \left[\frac{2}{(n+2)} - \sum_{k=0}^{n}k\right]t^2$$
$$= O(n+1) \tag{4.5}$$

5. Proof of the Main Theorem

Titchmarsh [20] and the k^{th} partial sum $S_k(f;x)$ of (2.1) is

$$S_k(f;x) - f(x) = \frac{1}{\pi}\int_0^{\pi}\frac{1}{\sin t/2}\phi(t)\sin\left(k+\frac{1}{2}\right)tdt \tag{5.1}$$

The $[F, d_n]$ transform $t_n^{[F,d_n]}$ of $S_k(f; x)$ is

$$t_n^{[F,d_n]} - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\phi(t)}{\sin t/2} \sum_{k=0}^{\infty} P_{nk} \sin\left(k + \frac{1}{2}\right) t dt \tag{5.2}$$

The $(C, 1)\,[F, d_n]$ transform of $S_k(f; x)$ by $t_n^{(C,2)[F,d_n]}$ then

$$t_n^{(C,2)[F,d_n]} - f(x) = \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{\sin\frac{t}{2}} \sum_{k=0}^{\infty} P_{nk} \sin(k + \frac{1}{2}) t dt \tag{5.3}$$

$$= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{sin\frac{t}{2}}\, I_m \left\{ \sum_{k=0}^{\infty} P_{nk}\ \exp\left(i(k + \frac{1}{2})\right)\right\} dt$$

$$= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{\sin\frac{t}{2}} I_m \left\{\exp\left(\frac{it}{2}\right) \sum_{k=0}^{\infty} P_{nk} \exp(ikt)\right\} dt$$

$$= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{\sin\frac{t}{2}} I_m \left\{\exp\left(\frac{it}{2}\right) \prod_{k=1}^{n} \frac{\exp(it) + d_k}{1 + d_k}\right\} dt$$

$$|t_n^{(C,2)[F,d_n]} - f(x)| = |\frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{\sin\frac{t}{2}} I_m \left\{\exp\left(\frac{it}{2}\right) \prod_{k=1}^{n} \frac{\exp(it) + d_k}{1 + d_k}\right\} dt| \tag{5.4}$$

$$\leq \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{\sin\frac{t}{2}} I_m \left\{\exp\left(\frac{it}{2}\right) \prod_{k=1}^{n} \frac{\exp(it) + d_k}{1 + d_k}\right\} dt$$

$$= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{\sin\frac{t}{2}}$$

$$\times I_m \left\{\exp\left(\frac{it}{2}\right) \{\exp\left\{\frac{(U_n - 1)\, it}{2} - \frac{S_n t^2}{4}\right\}\} + O\left(S_n t^3\right)\right\} dt$$

$$= \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{\sin\frac{t}{2}} \left\{\sin\left(\frac{U_n t}{2}\right) \exp\left(\frac{-S_n t^2}{4}\right) + O\left(S_n t^3\right)\right\}$$

$$\leq \left| \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{\sin\frac{t}{2}} \sin\left(\frac{U_n t}{2}\right) \exp\left(\frac{-S_n t^2}{4}\right) dt \right|$$

$$+ \left| \frac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \int_0^{\pi} \frac{\phi(x)}{\sin\frac{t}{2}} \left\{O\left(S_n t^3\right)\right\} \right|$$

$$= |I_1| + |I_2| \tag{5.5}$$

Then $|I_1| = \dfrac{2\sum_{k=0}^{n}(n-k+1)}{(n+1)(n+2)\pi} \displaystyle\int_0^{\pi} \frac{|\phi(x)|}{\sin\frac{t}{2}} \sin\left(\frac{U_n t}{2}\right) \exp\left(\frac{-S_n t^2}{4}\right) dt$

Now

$$|I_1| = \int_0^{\pi} |\phi(x)|\,|K_n(x)|\, dt$$

$$= \left[\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi} \cdot\right] |\phi(x)|\,|K_n(x)|\,dt$$
$$= I_{1.1} + I_{1.2} \tag{5.6}$$

Now $I_{1.1} = \displaystyle\int_0^{\frac{\pi}{n+1}} |\phi(x)|\,|K_n(x)|\,dt$ using Lemma 2

Applying Hölder's inequality

$$= \left[\int_0^{\frac{\pi}{n+1}} \left\{\frac{t|\phi(t)|\sin^\beta t}{\xi(t)}\right\}^p dt\right]^{1/p} \left[\int_0^{\frac{\pi}{n+1}} \left\{\frac{K_n(t)\xi(t)}{t\sin^\beta t}\right\}^q dt\right]^{1/q}$$
$$= O\left(\frac{1}{(n+1)}\right) \cdot O(n+1) \left[\int_0^{\frac{\pi}{n+1}} \left\{\frac{\xi(t)}{t^{\beta+1}}\right\}^q dt\right]^{1/q}$$
$$= \xi\left(\frac{1}{(n+1)}\right) \left[\int_0^{\frac{\pi}{n+1}} t^{-(\beta+1)q} dt\right]^{1/q}$$
$$= O(n+1)^{\beta+1/p}\, \xi\left(\frac{1}{(n+1)}\right). \tag{5.7}$$

Then $I_{1.2} = \displaystyle\int_{\frac{\pi}{n+1}}^{\pi} \cdot|\phi(x)|\,|K_n(x)|\,dt$ using Lemma 3

Applying Hölder's inequality

$$= \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{t^{-\delta}|\phi(t)|\sin^\beta t}{\xi(t)}\right\}^p dt\right]^{1/p} \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{|K_n(t)|\,\xi(t)}{t^{-\delta}\sin^\beta t}\right\}^q dt\right]^{1/q}$$
$$= O(n+1)^{\delta} \quad \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{-\delta} t\sin^\beta t}\right\}^q dt\right]^{1/q}$$
$$= O(n+1)^{\delta} \quad \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{\beta+1-\delta}}\right\}^q dt\right]^{1/q}$$
$$= O(n+1)^{\delta} \quad \left[\int_{1/\pi}^{n+1/\pi} \left\{\frac{\xi(1/y)}{y^{-\beta-1+\delta}}\right\}^q \frac{dy}{y^2}\right]^{1/q} \qquad \because t = \frac{1}{y}$$
$$= O(n+1)^{\beta+1/p}\, \xi\left(\frac{1}{(n+1)}\right) \tag{5.8}$$

Now

$$|I_2| = \left[\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}\right] |\phi(x)|\,|K_n(x)|\,dt \text{ using Lemma 4}$$
$$= I_{2.1} + I_{2.2} \tag{5.9}$$

Similarly $\quad I_{2.1} = O(n+1)^{\beta+1/p} \quad \xi\left(\frac{1}{(n+1)}\right)$

Now $\quad I_{2.2} = \int_{\frac{\pi}{n+1}}^{\pi} \cdot|\phi(x)|\,|K_n(x)|\,dt$ using Lemma 5

Applying Hölder's inequality

$$= \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{t^{-\delta}|\phi(t)|\sin^{\beta} t}{\xi(t)}\right\}^{p} dt\right]^{1/p} \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{K_n(t)\xi(t)}{t^{-\delta}\sin^{\beta} t}\right\}^{q} dt\right]^{1/q}$$

$$= O(n+1)^{\delta} \cdot O(n+1) \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{-\delta}\sin^{\beta} t}\right\}^{q} dt\right]^{1/q}$$

$$= O(n+1)^{\delta+1} \xi\left(\frac{1}{(n+1)}\right)\left[t^{(\delta-\beta)+1/q}\right]_{\pi/n+1}^{\pi}$$

$$= O(n+1)^{\beta+1/p} \xi\left(\frac{1}{(n+1)}\right). \tag{5.11}$$

Thus combining (5.5) to (5.11) we set

$$\left|t_n^{(C,2)[F,d_n]} - f(x)\right| = O\left\{(n+1)^{\beta+1/p} \xi\left(\frac{1}{(n+1)}\right)\right\}$$

$$\left\|t_n^{(C,2)[F,d_n]} - f(x)\right\|_p = \left\{\int_0^{2\pi} O\left\{(n+1)^{\beta+1/p} \xi\left(\frac{1}{(n+1)}\right)\right\}^{p} dx\right\}^{1/p}.$$

Hence proof of the main theorem is completed.

Application: From our theorem can be derived following corollaries

Corollary 1. $f \in \mathrm{Lip}(\alpha, p), 0 < \alpha \leq 1$, if $\beta = 0, \xi(t) = t^{\alpha}$,

$$\left\|t_n^{(C,2)[F,d_n]} - f(x)\right\|_p = O\left\{\frac{1}{(n+1)^{\alpha-1/p}}\right\}.$$

Corollary 2. If $p \to \infty$ and condition from corollaries 1 .

$$\left\|t_n^{(C,2)[F,d_n]} - f(x)\right\|_{\infty} = O\left\{\frac{1}{(n+1)^{\alpha}}\right\}.$$

6. Conclusion

We calculate that under weaker condition weighted class and product summability method $(C, 2)\,[F, d_n]$ then interesting result obtained. If we use $(e, c), F(a, q)$, Euler means etc. in place of $[F, d_n]$ method then our result is more efficiently and generalizing for newer method of summability.

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