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SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH FRACTIONAL INTEGRAL OPERATOR

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Abstract

In this article, two subclasses of analytic bi-univalent functions dealing with fractional integral operators are introduced. Furthermore, we have obtained estimates of the coefficients $|a_2|$ and $|a_3|$ for the functions belonging to these subclasses.

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1. Introduction

The subject of fractional calculus (that is, the calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past over four decades, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of mathematical, physical, engineering and statistical sciences. Various operators of fractional-order derivatives as well as fractional-order integrals do indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. Fractional derivatives, fractional integrals and their properties are the subject of study in the field of fractional calculus. Srivastava and Owa [12] gave definitions for fractional operators (derivative and integral) in the complex Z-plane $\mathbb{C}$ as follows.

Definition 1.1. [12] The fractional derivative of order α is defined, for a function f by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} d\zeta; 0 \leq \alpha < 1,$$

where the function f is analytic in simply-connected region of the complex Z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

DEFINITION 1.2. [12] The fractional integral of order α is defined, for a function f by

$$I_z^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \alpha > 0,$$

where the function f is analytic in simply-connected region of the complex Z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

From defns 1.1 and 1.2, we have

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \mu > -1; 0 \leq \alpha < 1$$

and

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \mu > -1; \alpha > 0.$$

Let $\mathcal{J}$ denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let D denote the subclass of $\mathcal{J}$ which consists of functions of the form (1.1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in U.
We know that every function $f \in D$ has an inverse f^{-1}, defined by
$f^{-1}(f(z)) = z, (z \in U)$ and $f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f);\ \ r_0(f) \geq \frac{1}{4}\right),$
where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots\ . \tag{1.2}$$

DEFINITION 1.3. [4] A function f in D is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

Let Σ denote the class of bi-univalent functions in U given by (1.1). In [6] Lewin investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$. Subsequently Brannan and Clunie [1] conjectured that $|a_2| \leq \sqrt{2}$. Also Netanyahu [8] showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the Taylor-Maclaurin coefficients $|a_n|$ $(n \geq 3; n \in \mathbb{N})$ is still an open problem. Several authors investigated various subclasses of the class Σ and obtained estimates for their coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses (see [2, 3, 5, 7, 9, 10]).
In [11] Srivastava, Mishra and Gochhayat introduced the following two subclasses of the bi-univalent functions class Σ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

Definition 1.4. [11] A function f given by (1.1) is said to be in the class $H_{\Sigma}^{(\alpha)}$ $(0 < \alpha \leq 1)$, if the following conditions are satisfied:

$$f \in \Sigma, \qquad \left|arg(f'(z))\right| < \frac{\alpha\pi}{2} \qquad (z \in U)$$

$$and \qquad \left|arg(g'(w))\right| < \frac{\alpha\pi}{2} \qquad (w \in U)$$

where the function g is defined by (1.2).

Definition 1.5. [11] A function f given by (1.1) is said to be in the class $H_{\Sigma}(\beta)$ $(0 \leq \beta < 1)$, if the following conditions are satisfied:

$$f \in \Sigma, \qquad \Re\left(f'(z)\right) > \beta \qquad (z \in U)$$

$$and \qquad \Re\left(g'(w)\right) > \beta \qquad (w \in U)$$

where the function g is defined by (1.2).

Lemma 1.6. *[10] Let $h \in P$, the family of all functions h analytic in U for which $\Re\{h(z)\} > 0$ and have the form $h(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$ for $z \in U$. Then $|p_n| \leq 2$ for each $n \in \mathbb{N}$.*

2. Coefficient bounds for the function class $\mathcal{N}_{\Sigma}(I_z^{\lambda}, \alpha)$

Definition 2.1. A function f given by (1.1) is said to be in the class $\mathcal{N}_{\Sigma}(I_z^{\lambda}, \alpha)$ $(0 < \alpha \leq 1)$, if the following conditions are satisfied:

$$f \in \Sigma, \qquad \left|arg\left(\frac{\Gamma(2+\lambda)I_z^{\lambda}f(z)}{z^{\lambda}}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in U) \tag{2.1}$$

and

$$\left|arg\left(\frac{\Gamma(2+\lambda)I_w^{\lambda}g(w)}{w^{\lambda}}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in U) \tag{2.2}$$

where the function g is defined by (1.2).

Theorem 2.2. *Let f be in the function class $\mathcal{N}_{\Sigma}(I_z^{\lambda}, \alpha)$, then*

$$|a_2| \leq 2\alpha(2+\lambda)\sqrt{\frac{3+\lambda}{12\alpha(2+\lambda) - 4(\alpha-1)(3+\lambda)}} \tag{2.3}$$

and

$$|a_3| \leq \frac{\alpha(2+\lambda)(3+\lambda)}{3} + (2+\lambda)^2\alpha^2. \tag{2.4}$$

Proof. We can write the argument in (2.1) and (2.2) as

$$\left(\frac{\Gamma(2+\lambda)I_z^{\lambda}f(z)}{z^{\lambda}}\right) = [p(z)]^{\alpha} \tag{2.5}$$

and

$$\left(\frac{\Gamma(2+\lambda)I_w^\lambda g(w)}{w^\lambda}\right) = [q(w)]^\alpha \tag{2.6}$$

respectively, where $p,\ q \in P$ have the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \tag{2.7}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots\ . \tag{2.8}$$

Equating the coefficients in (2.5) and (2.6), we get

$$\frac{2a_2}{(2+\lambda)} = \alpha p_1, \tag{2.9}$$

$$\frac{6}{(3+\lambda)(2+\lambda)} a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \tag{2.10}$$

$$\frac{-2a_2}{(2+\lambda)} = \alpha q_1, \tag{2.11}$$

$$\frac{6}{(3+\lambda)(2+\lambda)} (2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_1 = -q_1 \tag{2.13}$$

and

$$\frac{8a_2^2}{(2+\lambda)^2} = \alpha^2 (p_1^2 + q_1^2)\,. \tag{2.14}$$

Adding (2.10) and (2.12), and using (2.14), we obtain

$$a_2^2 = \frac{\alpha^2 (3+\lambda)(2+\lambda)^2 (p_2+q_2)}{12\alpha(2+\lambda) - 4(\alpha-1)(3+\lambda)}. \tag{2.15}$$

Using Lemma 1.6, we get

$$|a_2| \le 2\alpha(2+\lambda)\sqrt{\frac{3+\lambda}{12\alpha(2+\lambda) - 4(\alpha-1)(3+\lambda)}}. \tag{2.16}$$

Next in order to find the bound on $|a_3|$, by subtracting (2.12) from (2.10) and then using (2.13), we obtain

$$a_3 - a_2^2 = \frac{\alpha(p_2 - q_2)(2+\lambda)(3+\lambda)}{12}. \tag{2.17}$$

By (2.12), we get

$$a_3 = \frac{\alpha(p_2 - q_2)(2+\lambda)(3+\lambda)}{12} + \frac{\alpha^2 (2+\lambda)^2 (p_1^2 + q_1^2)}{8}. \tag{2.18}$$

Using Lemma 1.6, we get

$$|a_3| \le \frac{\alpha(2+\lambda)(3+\lambda)}{3} + (2+\lambda)^2\alpha^2. \tag{2.19}$$

This completes the proof of theorem. □

Putting λ =0 in Theorem 2.2 we have

COROLLARY 2.3. *Let f be in the function class $\mathcal{N}_\Sigma(I_z^0, \alpha)$, then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}}$$

and

$$|a_3| \le 2\alpha(2\alpha+1)\,.$$

3. Coefficient bounds for the function class $\mathcal{N}_\Sigma(I_z^\lambda, \beta)$

DEFINITION 3.1. A function f given by (1.1) is said to be in the class $\mathcal{N}_\Sigma(I_z^\lambda, \beta)$ $(0 \le \beta < 1)$, if the following conditions are satisfied:

$$f \in \Sigma, \qquad \Re\left(\frac{\Gamma(2+\lambda)I_z^\lambda f(z)}{z^\lambda}\right) > \beta \qquad (z \in U) \tag{3.1}$$

and

$$\Re\left(\frac{\Gamma(2+\lambda)I_w^\lambda g(w)}{w^\lambda}\right) > \beta \qquad (w \in U) \tag{3.2}$$

where the function g is defined by (1.2).

THEOREM 3.2. *Let f be in the function class $\mathcal{N}_\Sigma(I_z^\lambda, \beta)$, then*

$$|a_2| \le \sqrt{\frac{(1-\beta)(2+\lambda)(3+\lambda)}{3}} \tag{3.3}$$

and

$$|a_3| \le \frac{(2+\lambda)(1-\beta)}{3}\left((3+\lambda) + 3(2+\lambda)(1-\beta)\right). \tag{3.4}$$

PROOF. First of all the argument inequalities in (3.1) and (3.2) can be written in their equivalent forms as,

$$\left(\frac{\Gamma(2+\lambda)I_z^\lambda f(z)}{z^\lambda}\right) = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\left(\frac{\Gamma(2+\lambda)D_w^\lambda g(w)}{w^\lambda}\right) = \beta + (1-\beta)q(z) \tag{3.6}$$

respectively, where $p(z), q(z)$ are given by (2.7) and (2.8). Now equating the coefficients in (3.5) and (3.6), we get

$$\frac{2a_2}{(2+\lambda)} = (1-\beta)p_1, \tag{3.7}$$

$$\frac{6a_3}{(3+\lambda)(2+\lambda)} = (1-\beta)p_2, \tag{3.8}$$

$$\frac{-2a_2}{(2+\lambda)} = (1-\beta)q_1 \tag{3.9}$$

and

$$\frac{6(2a_2^2-a_3)}{(3+\lambda)(2+\lambda)} = (1-\beta)q_2. \tag{3.10}$$

Using (3.7) and (3.9), we obtain

$$p_1 = -q_1 \tag{3.11}$$

and

$$(1-\beta)^2(p_1^2+q_1^2) = \frac{8a_2^2}{(2+\lambda)^2}. \tag{3.12}$$

By (3.8) and (3.10), we get

$$\frac{12a_2^2}{(3+\lambda)(2+\lambda)} = (1-\beta)(p_2+q_2) \tag{3.13}$$

or equivalently,

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)(3+\lambda)(2+\lambda)}{12}. \tag{3.14}$$

Applying Lemma 1.6, we get

$$|a_2| \le \sqrt{\frac{(1-\beta)(2+\lambda)(3+\lambda)}{3}} \tag{3.15}$$

which is the bound on $|a_2|$ as given in (3.3). Next in order to find the bound on $|a_3|$, subtracting (3.10) from (3.8), we get

$$\frac{12}{(3+\lambda)(2+\lambda)}(a_3-a_2^2) = (1-\beta)(p_2-q_2) \tag{3.16}$$

or equivalently,

$$a_3 = a_2^2 + \frac{(3+\lambda)(2+\lambda)}{12}(1-\beta)(p_2-q_2). \tag{3.17}$$

Using (3.12), we obtain

$$a_3 = \frac{(1-\beta)^2(2+\lambda)^2(p_1^2+q_1^2)}{8} + \frac{(3+\lambda)(2+\lambda)}{12}(1-\beta)(p_2-q_2). \tag{3.18}$$

Using Lemma 1.6, we get

$$|a_3| \leq \frac{(2+\lambda)(1-\beta)}{3}\left[(3+\lambda)+3(2+\lambda)(1-\beta)\right]. \tag{3.19}$$

This completes the proof of theorem. □

Putting λ =0 in Theorem 3.2 we have

COROLLARY 3.3. *Let f be in the function class $\mathcal{N}_\Sigma(I_z^0, \beta)$, then*

$$|a_2| \leq \sqrt{2(1-\beta)}$$

and

$$|a_3| \leq 2(1-\beta)(3-2\beta).$$

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