

# CHARACTERIZATION OF THE LP-SASAKIAN MANIFOLDS ADMITTING A NEW TYPE OF SEMI-SYMMETRIC NON-METRIC CONNECTION

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## Abstract

The goal of the present paper is to observe the properties of LP-Sasakian manifolds prepared with new semi-symmetric non-metric connection. We make a plan to study some curvature's properties of LP-Sasakian manifolds admitting new semi-symmetric non-metric connection. Semi-symmetric and locally  $\phi$ -symmetric LP-Sasakian manifolds admitting new semi-symmetric non-metric connection studies as well.

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## 1. Introduction

The microscopic outlook of connections in Riemannian geometry began to a few existing with Christoffel. Later Levi-Civita discovered that a connection also endorsed for a concept of parallel shipping. Levi-Civita focussed on connection as a sort of differential operator. In 20<sup>th</sup> century, Cartan founded connection as a certain form of differential shape. In 1950, an algebraic framework for a connection as a differential operator has been given by Koszul. The principal invariants of an affine connection are its torsion and curvature [9, 11]. A torsion tensor of  $\nabla$  is defined as

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad (1.1)$$

$\forall X, Y \in \chi(M)$ , where  $\chi(M)$  is a set of all smooth vector fields on  $M$ . A connection  $\nabla$  is called torsion free if  $T(X, Y)$  vanishes, and if it is not vanish then it is known as non-symmetric. The concept of semi-symmetric connection on a differentiable manifold was proposed by Friedmann and Schouten. A linear connection  $\tilde{\nabla}$  on  $M$  is called semi-symmetric if

$$\tilde{T}(X, Y) = \eta(Y)X - \eta(X)Y, \quad (1.2)$$

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holds  $\forall X, Y$  on  $M$ , where  $\eta$  is a 1-form associated with the vector field  $\xi$  and satisfies

$$\eta(X) = g(X, \xi), \quad (1.3)$$

$\forall X, Y \in \chi(M)$ . Further, the idea of metric connection along torsion on a Riemannian manifold has been proposed by Hayden [11]. A connection  $\nabla$  on a manifold  $M$  is said to be a metric connection if Riemannian metric  $g$  in  $M$  holds  $\nabla g = 0$ , and if  $\nabla g \neq 0$ , then it is known as non-metric. Yano [30] introduced regular examen semi-symmetric connection in a Riemannian manifold. In 1992, the concept of semi-symmetric non-metric connection has been proposed by Agashe and Chafle [1]. In [10], quarter-symmetric connection in a differentiable manifold with affine connection has been defined and examined by Golab. After that many geometers as Rastogi [25], Mishra and Pandey [16], Yano and Imai [31], Pradeep Kumar, Venkatesha and Bagewadi [19] & many others examined the different characteristics of quarter-symmetric metric connection. In [29], M. M. Tripathi and N. Nakkar studied the semi-symmetric non-metric connection in a Kenmotsu manifold. Inline with this, another semi-symmetric non-metric connection was proposed by S. K. Chaubey et al. [5]. Afterward the several connections were examined by some another authors [6, 17, 18]. The existence of a new connection has been proved by M. M. Tripathi [28] and showed that in particular cases. Motivated by the Sasakian structures, the concept of Lorentzian para-Sasakian structures (briefly, LP-Sasakian structures) has been proposed by Matsumoto [14], in 1989. Mihai et al. [13] offered the identical concept and found many fruitful outcomes. Afterward, the characteristics of LP-Sasakian manifolds were examined by many geometres and acquired multiple geometrical and physical outcomes. We refer [15, 21–23, 32] and the references there in. However semi-symmetric LP-Sasakian manifolds conceding to semi-symmetric non-metric connection have now not been studied to date. In particularly, we have taken trace  $\phi = 0$ , throughout this paper.

Motivated from the above definition we are going to outline LP-Sasakian manifolds conceding to new semi-symmetric non-metric connection. After introduction, we consolidates basic formulas and results of LP-Sasakian manifolds in section 2. The necessary consequences of new semi-symmetric non-metric connection are given in the section 3. We study basic properties of Riemannian curvature tensor admitting new semi-symmetric non-metric connection in section 4. Further, we discuss semi-symmetric LP-Sasakian manifolds admitting new semi-symmetric non-metric connection in section 5. We also study locally  $\phi$ -symmetric LP-Sasakian manifold admitting new semi-symmetric non-metric connection in section 6. In the last section 7, we have given an example satisfying the equations (2.5), (2.7), (2.8), (2.9), (2.10), (2.11), (3.6), (3.7), (4.1), (4.2), (4.5), (4.6), (4.7), (4.8) and (4.9).

## 2. Preliminaries

Let  $M$  be a  $(2n + 1)$ -dimensional differentiable manifold of differentiability class  $C^{r+1}$ . If  $M$  admits a  $(1, 1)$ -type vector valued linear function  $\phi$ , a 1-form  $\eta$ , and the associated vector field  $\xi$ , which satisfies

$$\phi^2 X = X + \eta(X) \xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad (2.1)$$

and  $\text{rank}\phi = n - 1$  then  $M$  is known as Lorentzian almost paracontact manifold [14] and the structure  $(\phi, \xi, \eta, g)$  is admitted as a Lorentzian almost paracontact structure on  $M$ . If the Lorentzian metric  $g$  of  $M$  holds

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.2)$$

$$g(\phi X, Y) = g(X, \phi Y), \quad g(X, \xi) = \eta(X), \quad (2.3)$$

$\forall X, Y \in \chi(M)$ , where  $\chi(M)$  is a set of all smooth vector fields on  $M$ . Then  $(M, g)$  is admitted as a Lorentzian almost paracontact metric manifold.

$$(\nabla_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (2.4)$$

$\forall X, Y \in \chi(M)$ , then it reduces to a LP-Sasakian manifolds (shortly, LP-Sasakian manifold) [14]. Where  $\nabla$  is a Levi-Civita connection of Riemannian metric  $g$  on  $M^{2n+1}$ . From equations (2.1), (2.2), (2.3) and (2.4), we have

$$\nabla_X \xi = \phi X, \quad (2.5)$$

$$(\nabla_X \eta)Y = g(\phi X, Y). \quad (2.6)$$

Also, the following relations hold in a LP-Sasakian manifolds:

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (2.7)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (2.8)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.9)$$

$$R(\xi, X)\xi = \eta(X)\xi + X, \quad (2.10)$$

$$S(X, \xi) = 2n\eta(X), \quad (2.11)$$

$$Q\xi = 2n\xi, \quad (2.12)$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \quad (2.13)$$

$$S(X, Y) = g(QX, Y). \quad (2.14)$$

### 3. New semi-symmetric non-metric connection

Let us define, a linear connection  $\widetilde{\nabla}$  [4] as

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}[\eta(Y)X - \eta(X)Y]. \quad (3.1)$$

satisfying

$$\widetilde{T}(X, Y) = \eta(Y)X - \eta(X)Y, \quad (3.2)$$

and

$$(\widetilde{\nabla}_X g)(Y, Z) = \frac{1}{2}[2\eta(X)g(Y, Z) - \eta(Y)g(X, Z) - \eta(Z)g(X, Y)]. \quad (3.3)$$

for arbitrary vector fields  $X$ ,  $Y$  and  $Z$  is said to be new semi-symmetric non-metric connection. Also, we have

$$(\widetilde{\nabla}_X \phi)(Y) = \frac{1}{2}[2(\nabla_X \phi)(Y) - \eta(Y)\phi(X)], \quad (3.4)$$

$$(\widetilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y), \quad (3.5)$$

$$(\widetilde{\nabla}_X g)(\phi Y, Z) = \frac{1}{2}[2\eta(X)g(\phi Y, Z) - \eta(Z)g(X, \phi Y)]. \quad (3.6)$$

On replacing  $Y$  by  $\xi$  in the equation (3.1), we have

$$\widetilde{\nabla}_X \xi = \nabla_X \xi - \frac{1}{2}\phi^2(X). \quad (3.7)$$

On replacing  $X$  by  $\xi$  in the equation (3.3), we have

$$(\widetilde{\nabla}_\xi g)(Y, Z) = -g(\phi Y, \phi Z). \quad (3.8)$$

Hence we have the following proposition:

**THEOREM 3.1.** *Co-variant differentiation of Riemannian metric  $g$  with respect to contra-variant vector field  $\xi$  is given by the equation (3.8) in a contact metric manifold admitting new semi-symmetric non-metric connection  $\widetilde{\nabla}$ .*

The curvature tensor  $\widetilde{R}$  of  $\widetilde{\nabla}$  defined as follows

$$\widetilde{R}(X, Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X, Y]}Z, \quad (3.9)$$

where  $X, Y, Z \in \chi(M)$ .

Using equation (3.1) in (3.9), we have

$$\begin{aligned} \widetilde{R}(X, Y)Z &= R(X, Y)Z + \frac{1}{2}[(\nabla_X \eta)(Z)Y - (\nabla_X \eta)(Y)Z \\ &\quad - (\nabla_Y \eta)(Z)X + (\nabla_Y \eta)(X)Z] \\ &\quad + \frac{1}{4}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \quad (3.10)$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z \quad (3.11)$$

is the Riemannian curvature tensor [3] of Levi-Civita connection.

**PROPOSITION 3.2.** *The relation between Riemannian curvature tensor admitting new semi-symmetric non-metric connection  $\widetilde{\nabla}$  and with respect to Levi-Civita connection  $\nabla$  is given by the equation (3.10).*

#### 4. Curvature tensor of the LP-Sasakian manifold admitting new semi-symmetric non-metric connection

Now using equation (2.6) in equation (3.10), we have

$$\begin{aligned}\widetilde{R}(X, Y)Z &= R(X, Y)Z + \frac{1}{2}[g(X, \phi Z)Y - g(Y, \phi Z)X] \\ &\quad + \frac{1}{4}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].\end{aligned}\quad (4.1)$$

Contracting equation (4.1) with respect to  $X$ , we have

$$\widetilde{S}(Y, Z) = S(Y, Z) - ng(Y, \phi Z) + \frac{n}{2}\eta(Y)\eta(Z). \quad (4.2)$$

Using equation (2.14) in equation (4.2), we have

$$\widetilde{Q}(Y) = Q(Y) - n\phi Y + \frac{n}{2}\eta(Y)\xi. \quad (4.3)$$

Again contracting equation (4.2), we have

$$\widetilde{r} = r - \frac{n}{2}. \quad (4.4)$$

Where  $\widetilde{S}(Y, Z)$ ;  $S(Y, Z)$ ,  $\widetilde{Q}$ ;  $Q$  and  $\widetilde{r}$ ;  $r$  are the Ricci tensors, Ricci operators and scalar curvatures of new semi-symmetric non-metric connection  $\widetilde{\nabla}$  and Levi-Civita connection  $\nabla$ .

On replacing  $X$  by  $\xi$  in the equation (4.1) and using equations (2.1), (2.3), we have

$$\widetilde{R}(\xi, Y)Z = R(\xi, Y)Z - \frac{1}{2}g(Y, \phi Z)\xi + \frac{1}{4}\eta(Y)\eta(Z)\xi + \frac{1}{4}\eta(Z)Y. \quad (4.5)$$

In view of equations (2.8) and (4.5), we have

$$\widetilde{R}(\xi, Y)Z = \frac{1}{2}[2g(Y, Z)\xi - g(\phi Y, Z)\xi - \frac{3}{2}\eta(Z)Y + \frac{1}{2}\eta(Y)\eta(Z)\xi]. \quad (4.6)$$

Again on replacing  $Z$  by  $\xi$  in equation (4.1) and using equations (2.1), (2.7), (3.2), we have

$$\widetilde{R}(X, Y)\xi = \frac{3}{4}[\eta(Y)X - \eta(X)Y] = \frac{3}{4}R(X, Y)\xi = \frac{3}{4}\widetilde{T}(X, Y) \neq 0. \quad (4.7)$$

Thus we have the following theorem:

**THEOREM 4.1.** *Every  $(2n + 1)$ -dimensional LP-Sasakian manifold equipped with  $\widetilde{\nabla}$  is regular with respect to  $\widetilde{\nabla}$ .*

Now operating  $\eta$  on both side of equation (4.1) and using equation (2.1), we have

$$\begin{aligned}\eta(\widetilde{R}(X, Y)Z) &= \frac{1}{2}[2g(Y, Z)\eta(X) - 2g(X, Z)\eta(Y) \\ &\quad + g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)].\end{aligned}\quad (4.8)$$

On contracting equation (4.7) with respect to  $X$ , we have

$$\widetilde{S}(Y, \xi) = \frac{3}{2}m\eta(Y). \quad (4.9)$$

In view of equations (4.2), (4.3) and (4.4), we have the following lemma:

LEMMA 4.2. *In a LP-Sasakian manifold Ricci tensor, Ricci operator and scalar curvature are given by the equations (4.2), (4.3) and (4.4), respectively for new semi-symmetric non-metric connection  $\widetilde{\nabla}$ .*

THEOREM 4.3. *If Riemannian curvature tensor of the connection  $\widetilde{\nabla}$  in a LP-Sasakian manifold admitting new semi-symmetric non-metric connection  $\widetilde{\nabla}$  vanishes, then the Ricci tensor is of the form (4.12).*

On taking  $\widetilde{R}(X, Y)Z = 0$  in the equation (4.1), we have

$$R(X, Y)Z = -\frac{1}{2}[g(X, \phi Z)Y - g(Y, \phi Z)X] - \frac{1}{4}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \quad (4.10)$$

Thus

$$\begin{aligned} 'R(X, Y, Z, W) &= -\frac{1}{2}g(X, \phi Z)g(Y, W) + \frac{1}{2}g(Y, \phi Z)g(X, W) \\ &\quad -\frac{1}{4}\eta(Y)\eta(Z)g(X, W) + \frac{1}{4}g(Y, W)\eta(X)\eta(Z). \end{aligned} \quad (4.11)$$

Contracting equation (4.11) with respect to vector field  $X$ , we have

$$S(Y, Z) = \frac{n}{2}[2g(Y, \phi Z) - \eta(Y)\eta(Z)], \quad (4.12)$$

$$Q(Y) = n\phi Y - \frac{n}{2}\eta(Y)\xi. \quad (4.13)$$

Again contracting equation (4.12), we have

$$r = -\frac{n}{2}. \quad (4.14)$$

## 5. Semi-symmetric LP-Sasakian manifolds

A  $(2n + 1)$ -dimensional paracontact metric manifold  $M$  with new semi-symmetric non-metric connection is called semi-symmetric [17] if

$$(\widetilde{R}(X, Y) \cdot \widetilde{R})(Z, U)V = 0.$$

i.e.

$$\begin{aligned} &\widetilde{R}(X, Y)\widetilde{R}(Z, U)V - \widetilde{R}(\widetilde{R}(X, Y)Z, U)V \\ &- \widetilde{R}(Z, \widetilde{R}(X, Y)U)V - \widetilde{R}(Z, U)\widetilde{R}(X, Y)V = 0. \end{aligned} \quad (5.1)$$

On replacing  $X$  by  $\xi$ , we have

$$\begin{aligned} &\widetilde{R}(\xi, Y)\widetilde{R}(Z, U)V - \widetilde{R}(\widetilde{R}(\xi, Y)Z, U)V \\ &- \widetilde{R}(Z, \widetilde{R}(\xi, Y)U)V - \widetilde{R}(Z, U)\widetilde{R}(\xi, Y)V = 0. \end{aligned} \quad (5.2)$$

In view of equation (4.6), we obtain

$$\begin{aligned}
 & g(Y, \widetilde{R}(Z, U)V)\xi - \frac{1}{2}g(Y, \phi(\widetilde{R}(Z, U)V))\xi - \frac{3}{4}\eta(\widetilde{R}(Z, U)V)Y \\
 & + \frac{1}{4}\eta(Y)\eta(\widetilde{R}(Z, U)V)\xi - g(Y, Z)\widetilde{R}(\xi, U)V + \frac{1}{2}g(Y, \phi Z)\widetilde{R}(\xi, U)V \\
 & + \frac{3}{4}\eta(Z)\widetilde{R}(Y, U)V - \frac{1}{4}\eta(Y)\eta(Z)\widetilde{R}(\xi, U)V - g(Y, U)\widetilde{R}(Z, \xi)V \\
 & + \frac{1}{2}g(Y, \phi U)\widetilde{R}(Z, \xi)V + \frac{3}{4}\eta(U)\widetilde{R}(Z, Y)V - \frac{1}{4}\eta(Y)\eta(U)\widetilde{R}(Z, \xi)V \\
 & - g(Y, V)\widetilde{R}(Z, U)\xi + \frac{1}{2}g(Y, \phi V)\widetilde{R}(Z, U)\xi + \frac{3}{4}\eta(V)\widetilde{R}(Z, U)Y \\
 & - \frac{1}{4}\eta(Y)\eta(V)\widetilde{R}(Z, U)\xi = 0. \quad (5.3)
 \end{aligned}$$

Now taking  $\eta$  on both side of equation (5.3), we have

$$\begin{aligned}
 & 2'\widetilde{R}(Z, U, V, Y) - \widetilde{R}(Z, U, V, \phi Y) = \\
 & -\frac{3}{2}\eta(\widetilde{R}(Z, U)V)\eta(Y) - \frac{1}{2}\eta(Y)\eta(\widetilde{R}(Z, U)V) - 2g(Y, Z)\eta(\widetilde{R}(\xi, U)V) \\
 & + g(\phi Y, Z)\eta(\widetilde{R}(\xi, U)V) + \frac{3}{2}\eta(Z)\eta(\widetilde{R}(Y, U)V) - \frac{1}{2}\eta(Y)\eta(Z)\eta(\widetilde{R}(\xi, U)V) \\
 & - 2g(Y, U)\eta(\widetilde{R}(Z, \xi)V) + g(\phi Y, U)\eta(\widetilde{R}(Z, \xi)V) + \frac{3}{2}\eta(U)\eta(\widetilde{R}(Z, Y)V) \\
 & - \frac{1}{2}\eta(Y)\eta(U)\eta(\widetilde{R}(Z, \xi)V) - 2g(Y, V)\eta(\widetilde{R}(Z, U)\xi) + g(\phi Y, V)\eta(\widetilde{R}(Z, U)\xi) \\
 & + \frac{3}{2}\eta(V)\eta(\widetilde{R}(Z, U)Y) - \frac{1}{2}\eta(Y)\eta(V)\eta(\widetilde{R}(Z, U)\xi). \quad (5.4)
 \end{aligned}$$

Using equations (2.1), (2.3), (4.6), (4.7) and (4.8) in the above equation (5.4), we have

$$\begin{aligned}
 & 2'\widetilde{R}(Z, U, V, Y) - \widetilde{R}(Z, U, V, \phi Y) = \\
 & 2g(Y, Z)g(U, V) - g(Y, Z)g(\phi U, V) + g(Y, Z)\eta(U)\eta(V) \\
 & - g(\phi Y, Z)g(U, V) + \frac{1}{2}g(\phi Y, Z)g(\phi U, V) - \frac{1}{4}g(\phi Y, Z)\eta(U)\eta(V) \\
 & - 2g(Y, U)g(Z, V) + g(Y, U)g(\phi Z, V) - \frac{1}{2}\eta(Z)\eta(V)g(Y, U) \\
 & + g(\phi Y, U)g(Z, V) - \frac{1}{2}g(\phi Y, U)g(\phi Z, V) + \frac{1}{4}\eta(Z)\eta(V)g(\phi Y, U) \\
 & - \frac{3}{2}g(Y, V)\eta(U)Z + \frac{3}{2}g(Y, V)\eta(Z)U + \frac{3}{4}g(\phi Y, V)\eta(U)Z \\
 & - \frac{3}{4}g(\phi Y, V)\eta(Z) - \frac{3}{8}\eta(U)\eta(V)\eta(Y)Z + \frac{3}{8}\eta(V)\eta(Y)\eta(Z)U, \quad (5.5)
 \end{aligned}$$

$$\begin{aligned}
& 2\widetilde{R}(Z, U) V - \widetilde{R}(Z, U) \phi V = \\
& 2g(U, V) Z - g(\phi U, V) Z + \eta(U) \eta(V) Z \\
& -g(U, V) \phi Z + \frac{1}{2} g(\phi U, V) \phi Z - \frac{1}{4} \eta(U) \eta(V) \phi Z \\
& -2g(Z, V) U + g(\phi Z, V) U - \frac{1}{2} \eta(Z) \eta(V) U \\
& +g(Z, V) \phi U - \frac{1}{2} g(\phi Z, V) \phi U + \frac{1}{4} \eta(Z) \eta(V) \phi U \\
& -\frac{3}{2} \eta(U) Z.V + \frac{3}{2} \eta(Z) U.V + \frac{3}{4} \eta(U) Z.\phi V \\
& -\frac{3}{4} \eta(Z) U.\phi V - \frac{3}{8} \eta(U) \eta(V) \eta(Y) Z + \frac{3}{8} \eta(V) \eta(Y) \eta(Z) U. \quad (5.6)
\end{aligned}$$

Contracting above equation with respect to  $Z$ , we have

$$\begin{aligned}
2\widetilde{S}(U, V) &= \frac{1}{2} \widetilde{S}(U, \phi V) + \frac{n}{4} [8g(U, V) - 4g(\phi U, V) \\
&+ 4\eta(U) \eta(V) - 6\eta(U) V + 3\eta(U) \phi V] \\
&- \frac{1}{8} [2g(U, V) - 4g(\phi U, V) + 3\eta(U) \eta(V) \eta(Y)]. \quad (5.7)
\end{aligned}$$

**THEOREM 5.1.** *In a semi-symmetric LP-Sasakian manifolds admitting new semi-symmetric non-metric connection  $\widetilde{\nabla}$ , the Ricci tensor is given by the equation (5.7).*

## 6. Locally $\phi$ -symmetric LP-Sasakian manifolds

**DEFINITION 6.1.** A LP-Sasakian manifolds  $M^{2n+1}$  admitting new semi-symmetric non-metric connection is called locally  $\phi$ -symmetric [2] if

$$\phi^2 \left( (\widetilde{\nabla}_W \widetilde{R})(X, Y) Z \right) = 0$$

$\forall X, Y, Z, W$  are orthogonal to  $\xi$ .

Taking covariant differentiation of Riemannian curvature tensor with respect to  $W$ , we have

$$\begin{aligned}
(\widetilde{\nabla}_W R)(X, Y) Z &= \widetilde{\nabla}_W R(X, Y) Z - R(\widetilde{\nabla}_W X, Y) Z \\
&- R(X, \widetilde{\nabla}_W Y) Z - R(X, Y) (\widetilde{\nabla}_W Z). \quad (6.1)
\end{aligned}$$

Now using equations (2.7) and (3.1) in equation (6.1), we have

$$\begin{aligned}
(\widetilde{\nabla}_W R)(X, Y) Z &= (\nabla_W R)(X, Y) Z + \frac{1}{2} [2\eta(W) R(X, Y) Z \\
&- \eta(X) R(W, Y) Z - \eta(Y) R(X, W) Z \\
&- \eta(Z) R(X, Y) W + g(Y, Z) \eta(X) W \\
&- g(X, Z) \eta(Y) W]. \quad (6.2)
\end{aligned}$$



Applying covariant differentiation on (4.1) with respect to  $W$ , we have

$$\begin{aligned}
 (\widetilde{\nabla}_W \widetilde{R})(X, Y)Z &= (\widetilde{\nabla}_W R)(X, Y)Z + \frac{1}{2}[(\widetilde{\nabla}_W g)(X, \phi Z)Y \\
 &\quad + g(X, (\widetilde{\nabla}_W \phi)Z)Y - (\widetilde{\nabla}_W g)(Y, \phi Z)X \\
 &\quad - g(Y, (\widetilde{\nabla}_W \phi)Z)X] + \frac{1}{4}[(\widetilde{\nabla}_W \eta)(Y)\eta(Z)X \\
 &\quad + (\widetilde{\nabla}_W \eta)(Z)\eta(Y)X - (\widetilde{\nabla}_W \eta)(X)\eta(Z)Y \\
 &\quad - (\widetilde{\nabla}_W \eta)(Z)\eta(X)Y].
 \end{aligned} \tag{6.3}$$

Using equations (2.4), (2.6), (3.3), (3.4), (3.5) and (6.2), we have

$$\begin{aligned}
 (\widetilde{\nabla}_W \widetilde{R})(X, Y)Z &= (\nabla_W R)(X, Y)Z + \frac{1}{2}[2\eta(W)R(X, Y)Z \\
 &\quad - \eta(X)R(W, Y)Z - \eta(Y)R(X, W)Z \\
 &\quad - \eta(Z)R(X, Y)W + g(Y, Z)\eta(X)W \\
 &\quad - g(X, Z)\eta(Y)W + \eta(W)g(X, \phi Z)Y \\
 &\quad - \eta(X)g(W, \phi Z)Y - \eta(W)g(Y, \phi Z)X \\
 &\quad + \eta(Y)g(W, \phi Z)X + g(W, Z)\eta(X)Y \\
 &\quad + \eta(Z)g(X, W)Y + 2\eta(W)\eta(Z)\eta(X)Y \\
 &\quad - \eta(Z)g(X, \phi W)Y - g(W, Z)\eta(Y)X \\
 &\quad - \eta(Z)g(Y, W)X - 2\eta(W)\eta(Z)\eta(Y)X \\
 &\quad + \eta(Z)g(Y, \phi W)X].
 \end{aligned} \tag{6.4}$$

Now applying  $\phi^2$  on both side of equation (6.4) and using equations (2.1), (2.2), we

have

$$\begin{aligned}
\phi^2 \left( (\widetilde{\nabla}_W \widetilde{R}) (X, Y) Z \right) = & \phi^2 ((\nabla_W R) (X, Y) Z) + \frac{1}{2} [2\eta(W) R(X, Y) Z \\
& + 2\eta(W) \eta(R(X, Y) Z) \xi - \eta(X) R(W, Y) Z \\
& - \eta(X) \eta(R(W, Y) Z) \xi - \eta(Y) R(X, W) Z \\
& - \eta(Y) \eta(R(X, W) Z) \xi - \eta(Z) R(X, Y) W \\
& - \eta(Z) \eta(R(X, Y) W) \xi + \eta(X) g(Y, Z) W \\
& + \eta(X) \eta(W) g(Y, Z) \xi - \eta(Y) g(X, Z) W \\
& - \eta(Y) \eta(W) g(X, Z) \xi + \eta(W) g(X, \phi Z) Y \\
& + \eta(Y) \eta(W) g(X, \phi Z) \xi - \eta(X) g(W, \phi Z) Y \\
& - \eta(W) g(Y, \phi Z) X - \eta(X) \eta(W) g(Y, \phi Z) \xi \\
& + \eta(Y) g(W, \phi Z) X + \eta(X) g(W, Z) Y \\
& + \eta(Z) g(X, W) Y + \eta(Z) \eta(Y) g(X, W) \xi \\
& + 2\eta(X) \eta(Z) \eta(W) Y - \eta(Z) g(X, \phi W) Y \\
& - \eta(Y) \eta(Z) g(X, \phi W) \xi - \eta(Y) g(W, Z) X \\
& - \eta(Z) g(Y, W) X - \eta(X) \eta(Z) g(Y, W) \xi \\
& - 2\eta(Y) \eta(Z) \eta(W) X + \eta(Z) g(Y, \phi W) X \\
& + \eta(X) \eta(Z) g(Y, \phi W) \xi].
\end{aligned} \tag{6.5}$$

Taking  $X, Y, Z$  and  $W$  orthogonal to  $\xi$ , we have

$$\phi^2 \left( (\widetilde{\nabla}_W \widetilde{R}) (X, Y) Z \right) = \phi^2 ((\nabla_W R) (X, Y) Z). \tag{6.6}$$

**THEOREM 6.2.** *Necessary and sufficient condition for a LP-Sasakian manifolds to be locally  $\phi$ -symmetric equipped with new semi-symmetric non-metric connection is that manifold is also locally  $\phi$ -symmetric equipped with the Levi-Civita connection.*

## 7. Example

Suppose a 3-dimensional manifold  $M^3 = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$  with the standard coordinate system  $(x, y, z)$  of  $\mathbb{R}^3$ . Let  $E_1 = e^z \frac{\partial}{\partial x}$ ,  $E_2 = e^z \frac{\partial}{\partial y}$ ,  $E_3 = \frac{\partial}{\partial z} = \xi$  are linearly independent vector fields at every point of  $M^3$  and form a basis of tangent space at every point.

Let us consider  $g$  be a Riemannian metric of  $M^3$  defined by

$$\begin{aligned}
g(E_1, E_2) &= g(E_2, E_3) = g(E_1, E_3) = 0, \\
g(E_1, E_1) &= g(E_2, E_2) = 1 \neq g(E_3, E_3) = -1,
\end{aligned} \tag{7.1}$$

where

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and  $\phi$  is  $(1, 1)$ -tensor field defined by

$$(\phi E_1) = -E_1, \quad (\phi E_2) = -E_2, \quad (\phi E_3) = 0. \quad (7.2)$$

Using linearity of  $\phi$  and  $g$ , we have

$$\begin{aligned} \eta(E_3) &= \eta(\xi) = -1, \quad \phi^2(X) = X + \eta(X)E_3, \\ g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y). \end{aligned} \quad (7.3)$$

$\forall X, Y \in \chi(M)$ , where  $\chi(M)$  is a set of all smooth vector fields on  $M$ . Here  $\eta$  is a 1-form on  $M^3$  given by  $\eta(X) = g(X, \xi) = g(X, E_3)$ ,  $\forall X \in \chi(M)$ . Hence for  $\xi = E_3$ , the quartet  $(\phi, \xi, \eta, g)$  defines a Lorentzian almost paracontact metric quartet on  $M^3$ . The Lie bracket for the example can be estimated by using the definition  $[X, Y]f = X(Yf) - Y(Xf)$ . All feasible Lie brackets for the example are as follows:

$$\begin{aligned} [E_1, E_1] &= 0, \quad [E_1, E_2] = 0, \quad [E_1, E_3] = -E_1, \\ [E_2, E_1] &= 0, \quad [E_2, E_2] = 0, \quad [E_2, E_3] = -E_2, \\ [E_3, E_1] &= E_1, \quad [E_3, E_2] = E_2, \quad [E_3, E_3] = 0. \end{aligned} \quad (7.4)$$

Let us consider  $\nabla$ , a Levi-Civita connection with Riemannian metric  $g$ . Using the Koszul's formula for the Riemannian metric  $g$ , we have

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y), \end{aligned} \quad (7.5)$$

By virtue of equation (7.5), we have

$$\begin{aligned} \nabla_{E_1} E_1 &= -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = -E_1 \\ \nabla_{E_2} E_1 &= 0, \quad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_2} E_3 = -E_2 \\ \nabla_{E_3} E_1 &= 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0. \end{aligned} \quad (7.6)$$

Now for  $X = X^1 E_1 + X^2 E_2 + X^3 E_3$  and  $\xi = E_3$ , we have

$$\begin{aligned} \nabla_X \xi - \frac{1}{2} \phi^2(X) &= \nabla_{X^1 E_1 + X^2 E_2 + X^3 E_3} E_3 - \frac{1}{2} \phi^2(X) \\ &= X^1 \nabla_{E_1} E_3 + X^2 \nabla_{E_2} E_3 + X^3 \nabla_{E_3} E_3 - \frac{1}{2} (X^1 E_1 + X^2 E_2) \\ &= -\frac{3}{2} X^1 E_1 - \frac{3}{2} X^2 E_2, \end{aligned} \quad (7.7)$$

$$\begin{aligned} \phi X &= \phi(X^1 E_1 + X^2 E_2 + X^3 E_3) \\ &= X^1 \phi(E_1) + X^2 \phi(E_2) + X^3 \phi(E_3) \\ &= -X^1 E_1 - X^2 E_2, \end{aligned} \quad (7.8)$$

where  $X^1, X^2, X^3$  are scalars. In perspective of equations (7.7) and (7.8) we can say that the structure  $(\phi, \xi, \eta, g)$  is a Lorentzian almost paracontact structure on  $M^3$ . Therefore

$M^3(\phi, \xi, \eta, g)$  is a LP-Sasakian manifold. In instance of equations (2.1), (2.3), (3.1) and (7.6), we have

$$\begin{aligned}\widetilde{\nabla}_{E_1}E_1 &= -E_3, & \widetilde{\nabla}_{E_1}E_2 &= 0, & \widetilde{\nabla}_{E_1}E_3 &= -\frac{3}{2}E_1 \\ \widetilde{\nabla}_{E_2}E_1 &= 0, & \widetilde{\nabla}_{E_2}E_2 &= -E_3, & \widetilde{\nabla}_{E_2}E_3 &= -\frac{3}{2}E_2 \\ \widetilde{\nabla}_{E_3}E_1 &= \frac{E_1}{2}, & \widetilde{\nabla}_{E_3}E_2 &= \frac{1}{2}E_2, & \widetilde{\nabla}_{E_3}E_3 &= 0.\end{aligned}\tag{7.9}$$

In the equations (3.2) and (3.3), we have

$$\begin{aligned}\widetilde{T}(E_1, E_3) &= \eta(E_3)E_1 - \eta(E_1)E_3 \\ &= g(E_3, E_3)E_1 - g(E_1, E_3)E_3 \\ &= -E_1 \\ &\neq 0,\end{aligned}$$

and

$$\begin{aligned}(\widetilde{\nabla}_{E_1}g)(E_1, E_3) &= \frac{1}{2}\{2\eta(E_1)g(E_1, E_3) - \eta(E_1)g(E_1, E_3) - \eta(E_3)g(E_1, E_1)\} \\ &= \frac{1}{2} \neq 0.\end{aligned}$$

Hence the connection defined in (3.1) is a new semi-symmetric non-metric connection. Again for  $X = X^1E_1 + X^2E_2 + X^3E_3$  and  $\xi = E_3$ , we have

$$\begin{aligned}\widetilde{\nabla}_X\xi &= \widetilde{\nabla}_{X^1E_1+X^2E_2+X^3E_3}E_3 \\ &= X^1\widetilde{\nabla}_{E_1}E_3 + X^2\widetilde{\nabla}_{E_2}E_3 + X^3\widetilde{\nabla}_{E_3}E_3 \\ &= -\frac{3}{2}(X^1E_1 + X^2E_2),\end{aligned}\tag{7.10}$$

In perspective of equations (7.7) and (7.10), we can say that the example validate the equation (3.7).

The Riemannian curvature tensor  $R(E_i, E_j)E_k; i, j, k = 1, 2, 3$  of the connection  $\nabla$  can be estimated by using equations (3.11), (7.4) and (7.6), we have

$$\begin{aligned}R(E_1, E_2)E_1 &= -E_2, & R(E_1, E_2)E_2 &= E_1, & R(E_1, E_2)E_3 &= 0, \\ R(E_1, E_3)E_1 &= -E_3, & R(E_1, E_3)E_2 &= 0, & R(E_1, E_3)E_3 &= -E_1, \\ R(E_2, E_3)E_1 &= 0, & R(E_2, E_3)E_2 &= -E_3, & R(E_2, E_3)E_3 &= -E_2.\end{aligned}\tag{7.11}$$

Along with  $R(E_i, E_i)E_i = 0; \forall i = 1, 2, 3$ . From direct calculations, it can be easily verify the equations (2.5), (2.7), (2.8), (2.9) and (2.10).

Analogously, we can estimate the Riemannian curvature tensor  $\widetilde{R}(E_i, E_j)E_k; i, j, k =$

1, 2, 3 of the connection  $\tilde{\nabla}$  by using equations (3.9), (7.4) and (7.9), we have

$$\begin{aligned}\tilde{R}(E_1, E_2)E_1 &= -\frac{3}{2}E_2, & \tilde{R}(E_1, E_2)E_2 &= \frac{3}{2}E_1, & \tilde{R}(E_1, E_2)E_3 &= 0, \\ \tilde{R}(E_1, E_3)E_1 &= -\frac{3}{2}E_3, & \tilde{R}(E_1, E_3)E_2 &= 0, & \tilde{R}(E_1, E_3)E_3 &= -\frac{3}{4}E_1, \\ \tilde{R}(E_2, E_3)E_1 &= 0, & \tilde{R}(E_2, E_3)E_2 &= -\frac{3}{2}E_3, & \tilde{R}(E_2, E_3)E_3 &= -\frac{3}{4}E_2.\end{aligned}\quad (7.12)$$

Along with  $\tilde{R}(E_i, E_i)E_i = 0; \forall i = 1, 2, 3$ . In conclusion of equations (7.11) and (7.12), we can facile validate the equations (4.1), (4.5), (4.6), (4.7), and (4.8).

The Ricci tensors  $S(E_j, E_k); j, k = 1, 2, 3$  of the connection  $\nabla$  in the LP-Sasakian manifold can be estimated by using equation (7.11) as under:

$$S(E_j, E_k) = \sum_{i=1}^3 g(R(E_i, E_j)E_k, E_i).$$

It follows that

$$S(E_1, E_1) = 2, \quad S(E_2, E_2) = 2, \quad S(E_3, E_3) = -2. \quad (7.13)$$

Along with  $S(E_j, E_k) = 0; \forall j, k (j \neq k) = 1, 2, 3$ .

The Ricci tensors  $\tilde{S}(E_j, E_k); j, k = 1, 2, 3$  of the connection  $\tilde{\nabla}$  in the LP-Sasakian manifold can also be estimated by using equation (7.12) as under:

$$\tilde{S}(E_j, E_k) = \sum_{i=1}^3 g(\tilde{R}(E_i, E_j)E_k, E_i).$$

It follows that

$$\tilde{S}(E_1, E_1) = 3, \quad \tilde{S}(E_2, E_2) = 3, \quad \tilde{S}(E_3, E_3) = -\frac{3}{2}. \quad (7.14)$$

Along with  $\tilde{S}(E_j, E_k) = 0; \forall j, k (j \neq k) = 1, 2, 3$ .

In perspective of equations (7.13) and (7.14), we can say that the example validate the equations (4.2) and (4.9).

Hence, we can say that given example is perfect match of our investigations.

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