

RADICAL TRANSVERSAL SCREEN SEMI-SLANT LIGHTLIKE SUBMERSIONS

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Abstract

We introduce radical transversal screen semi-slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold. We give examples and obtain a characterization theorem for such submersions. Integrability conditions of distributions involved in the definition of these submersions have been studied. Further, we investigate the geometry of foliations which are arisen from above distributions and find necessary and sufficient conditions for these foliations to be totally geodesic.

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1. Introduction

A Riemannian submersion between Riemannian manifolds M and B is defined as the mapping $f : M \rightarrow B$ such that f has maximum rank and the derivative map f_* preserves the length of horizontal vectors. In [1], Sahin defined screen lightlike submersions from lightlike manifolds onto semi-Riemannian manifolds. Later on, Sahin and Gündüzalp [2] defined lightlike submersions from semi-Riemannian manifolds onto lightlike manifolds.

On the other hand, lightlike submanifolds of semi-Riemannian manifolds were introduced by Duggal and Bejancu [3]. Sahin [4, 5] introduced the notion of a slant and screen-slant lightlike submanifold of an indefinite Hermitian manifold [6]. Following this, Shukla and Yadav defined screen semi-slant lightlike submanifolds of an indefinite Kaehler manifold in [7]. The concept of radical transversal, transversal, semi-transversal lightlike submanifolds is dealt in [8]. From [9], we conclude that, contrary to the Riemannian slant submersions [10], slant lightlike submersions do not include complex and screen real subcases. To fill this gap, Shukla and Omar introduce the notion of screen slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold [11], which includes complex (invariant) and screen real (anti-invariant) lightlike submersions. In this article, we introduce the notion of

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radical transversal screen semi-slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold.

The paper is arranged as follows. In section(2), we give basic formulas and definitions related to this paper. In section (3), we introduce radical transversal screen semi-slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold with a non-trivial example, give a characterization theorem for a lightlike submersion to be radical transversal screen semi-slant lightlike submersion and obtain a necessary and sufficient condition for integrability of distributions Δ , D_1 , and D_2 involved in the definition. Section (4) is devoted to the study of foliations determined by distributions involved in the definition of the above submersions from an indefinite Kaehler manifold onto a lightlike manifold.

2. Preliminaries

Suppose (M, J) is a $2m$ -dimensional almost complex manifold with an almost complex structure J and semi-Riemannian metric g of index r , where $0 < r \leq 2m$. In this case, M is said to be an indefinite almost Hermitian manifold, if

$$g(JX, JY) = g(X, Y), \quad \forall X, Y \in \Gamma(TM). \quad (2.1)$$

Further, if J is a complex structure on M , then (J, g) and (M, J, g) are said to be an indefinite Hermitian structure and indefinite Hermitian manifold respectively. Now, we assume that (M, J, g) is an indefinite almost Hermitian manifold and ∇ be the Levi-Civita connection on M . Then M is called an indefinite Kaehler manifold if

$$(\nabla_X J)Y = 0, \quad \forall X, Y \in \Gamma(TM). \quad (2.2)$$

Let (M, g) be a real m -dimensional smooth manifold, then the radical or null space $Rad T_p M$ of $T_p M$ is defined as $Rad T_p M = \{\xi \in T_p M : g(\xi, X) = 0, \forall X \in T_p M\}$. If the mapping $Rad TM : p \in M \rightarrow Rad T_p M$ defines a smooth distribution of rank $r > 0$ on M such that $0 < r \leq m$, then we call $Rad TM$, a radical (or null) distribution of M and the manifold M , an r -lightlike manifold.

Let $f : (M, g) \rightarrow (B, g')$ be a smooth submersion from a semi-Riemannian manifold M onto an r -lightlike manifold B . Then, $Ker f_{*p} = \{X \in T_p M : f_{*p}X = 0\}$. It follows that $(Ker f_{*p})^\perp = \{Y \in T_p M : g(Y, X) = 0, \forall X \in Ker f_{*p}\}$. As $T_p M$ is a semi-Riemannian vector space $(Ker f_{*p})^\perp$ may not be a complementary space to $Ker f_{*p}$. Assume that $Ker f_{*p} \cap (Ker f_{*p})^\perp = \Delta_p \neq \{0\}$. In this case $\Delta : p \rightarrow \Delta_p$ is said to be a radical distribution on M at $p \in M$. As Δ is a lightlike distribution, we have $Ker f_* = \Delta \perp S(Ker f_*)$. Similarly $(Ker f_*)^\perp = \Delta \perp S(Ker f_*)^\perp$. Here $S(Ker f_*)^\perp$ is the complementary distribution to Δ in $(Ker f_*)^\perp$. Assume that $dim(\Delta) = r > 0$. Since $\Delta \subset (S(Ker f_*)^\perp)^\perp$ and $(S(Ker f_*)^\perp)^\perp$ is non-degenerate, then there exists null vectors N_1, N_2, \dots, N_r , such that $g(N_i, N_j) = 0$, $g(\xi_i, N_j) = \delta_{ij}$. Here $\{N_i\}$ are smooth null vector fields of $S(Ker f_*)^\perp$ and $\{\xi_i\}$ is the lightlike basis of Δ . The distribution generated by vector fields N_1, N_2, \dots, N_r is denoted by $ltr(ker f_*)$. Then we

have $tr(ker f_*) = ltr(ker f_*) \perp S(ker f_*)^\perp$. Here $ltr(Ker f_*)$ and $Ker f_*$ are not orthogonal to each other. Moreover, we have the following decomposition

$$TM = (\Delta \oplus ltr(Ker f_*)) \perp S(Ker f_*) \perp S(Ker f_*)^\perp. \quad (2.3)$$

A Riemannian submersion $f : (M, g) \rightarrow (B, g')$ is called

- (a) r -lightlike submersion if $\dim \Delta = \dim\{(Ker f_*) \cap (Ker f_*)^\perp\} = r$,
 $0 < r < \min\{\dim(ker f_*), \dim(ker f_*)^\perp\}$,
- (b) co-isotropic submersion if $\dim \Delta = \dim(Ker f_*)^\perp < \dim(Ker f_*)$,
- (c) isotropic submersion if $\dim \Delta = \dim(Ker f_*) < \dim(Ker f_*)^\perp$,
- (d) totally lightlike submersion if $\dim \Delta = \dim(Ker f_*)^\perp = \dim(Ker f_*)$.

The geometry of lightlike submersions is characterized by O' Neill's tensors T and A given by

$$T_X Y = h\nabla_{vX} vY + v\nabla_{vX} hY, \quad (2.4)$$

$$A_X Y = v\nabla_{hX} hY + h\nabla_{hX} vY. \quad (2.5)$$

Tensors T and A are vertical and horizontal tensors, respectively. Also, T has symmetric property for vertical vector fields, that is,

$$T_X Y = T_Y X, \quad \forall X, Y \in \Gamma(Ker f_*). \quad (2.6)$$

Let f be a lightlike submersion from a real $(m + n)$ -dimensional semi-Riemannian manifold (M, g) onto a lightlike manifold (B, g') , where $m, n > 1$. Next, we assume that $Ker f_*$ is an m -dimensional lightlike distribution on M and $tr(Ker f_*)$ is the complementary distribution of $Ker f_*$ in M with respect to the pair $\{S(Ker f_*), S(Ker f_*)^\perp\}$. Let us denote by ∇ the Levi-Civita connection on M and by \hat{g} the induced metric on $Ker f_*$ of g . Then from (2.4), we have

$$\nabla_U V = \hat{\nabla}_U V + T_U V, \quad (2.7)$$

$$\nabla_U X = T_U X + \nabla_U^\perp X, \quad (2.8)$$

$\forall U, V \in \Gamma(Ker f_*), X \in \Gamma(Ker f_*)^\perp$, where $\hat{\nabla}_U V = v\nabla_U V$ and $\nabla_U^\perp X = h\nabla_U X$. Here $\{\hat{\nabla}_U V, T_U X\}$ and $\{T_U V, \nabla_U^\perp X\}$ belong to $\Gamma(Ker f_*)$ and $\Gamma(tr(Ker f_*))$, respectively. Let $S(Ker f_*)^\perp \neq \{0\}$. Denote by L and S the projection of $tr(Ker f_*)$ on $ltr(Ker f_*)$ and $S(Ker f_*)^\perp$, respectively. Then, for any $U, V \in \Gamma(Ker f_*), N \in \Gamma(ltr(Ker f_*))$ and $W \in \Gamma(S(Ker f_*)^\perp)$, we obtain

$$\nabla_U V = \hat{\nabla}_U V + T_U^l V + T_U^s V, \quad (2.9)$$

$$\nabla_U N = T_U N + \nabla_U^{\perp l} N + D^{\perp s}(U, N), \quad (2.10)$$

$$\nabla_U W = T_U W + D^{\perp l}(U, W) + \nabla_U^{\perp s} W. \quad (2.11)$$

Using (2.9-2.11) and that ∇ is a metric connection, we obtain

$$g(T_U^s V, W) + g(V, D^{\perp l}(U, W)) = -\hat{g}(T_U W, V), \quad (2.12)$$

$$g(D^{\perp s}(U, N), W) = -g(N, T_U W) \quad (2.13)$$

If f is either r -lightlike or co-isotropic submersion and $\phi : Ker f_* \rightarrow S(Ker f_*)$, then we write

$$\hat{\nabla}_U \phi V = \hat{\nabla}_U^* \phi V + T_U^* \phi V, \quad (2.14)$$

$$\hat{\nabla}_U \xi = T_U^* \xi + \nabla_U^{*\perp} \xi, \quad (2.15)$$

$\forall U, V \in \Gamma(Ker f_*)$, $\xi \in \Gamma\Delta$, where $\{\hat{\nabla}_U^* \phi V, T_U^* \xi\}$ and $\{T_U^* \phi V, \nabla_U^{*\perp} \xi\}$ belong to $\Gamma(S(Ker f_*))$ and $\Gamma\Delta$ respectively.

3. Radical Transversal Screen Semi-slant Lightlike Submersions

In this section, we introduce the notion of radical transversal screen semi-slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold, giving a non trivial example and obtain a characterization theorem. We recall the following lemma for later use.

LEMMA 3.1. [11] *Let $f : (M, g) \rightarrow (B, g')$ be a $2r$ -lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold (B, g') . Assume that $Ker f_*$ is a lightlike distribution of M . Then the screen distribution $S(Ker f_*)$ is Riemannian.*

DEFINITION 3.2. Let $f : (M, g, J) \rightarrow (B, g')$ be a lightlike submersion from a real $2m$ -dimensional indefinite Kaehler manifold M onto a lightlike manifold B . We say that f is a radical transversal screen semi-slant lightlike submersion if

- (i) $J\Delta = ltr(ker f_*)$,
- (ii) there exist non-null orthogonal distribution D_1 and D_2 such that

$$S(Ker f_*) = D_1 \oplus D_2,$$

- (iii) the distribution D_1 is invariant with respect to J , that $JD_1 = D_1$,
- (iv) the distribution D_2 is slant with angle $\theta (\neq 0)$, that is, for every point $p \in M$ and each non-zero vector $U \in (D_2)_p$, the angle θ between JU and $(D_2)_p$ is a non-zero constant, which is independent of the choice of p and U .

Here θ is called the slant angle of the distribution D_2 . A radical transversal screen semi-slant lightlike submersion is called proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \frac{\pi}{2}$.

From the definition, following decomposition is clear

$$Ker f_* = \Delta \oplus D_1 \oplus D_2. \quad (3.1)$$

Now, for any $U \in \Gamma(Ker f_*)$, we assume

$$JU = PU + FU, \quad (3.2)$$

where $PU \in \Gamma(Ker f_*)$ and $U \in \Gamma(tr(Ker f_*))$. Also, for any $U \in \Gamma(Ker f_*)$, we assume

$$U = P_1 U + P_2 U + P_3 U, \quad (3.3)$$

where P_1, P_2 and P_3 are projections on Δ, D_1 and D_2 respectively. Applying J on (3.2), we have

$$JU = JP_1 U + JP_2 U + JP_3 U = JP_1 U + JP_2 U + fP_3 U + FP_3 U, \quad (3.4)$$

where fP_3 (resp. FP_3X) denotes the tangential (resp. transversal) component of JP_3X . Thus, we have $JP_1U \in \Gamma(\text{ltr}(Kerf_*))$, $JP_2U \in \Gamma(D_1)$, $fP_3U \in \Gamma(D_2)$ and $FP_3U \in \Gamma(S(Kerf_*)^\perp)$. In the same way, for any $W \in \text{tr}(Kerf_*)$, we write

$$W = Q_1W + Q_2W, \quad (3.5)$$

where Q_1W and Q_2W denote the projections of $\text{tr}(Kerf_*)$ on $\text{ltr}(Kerf_*)$ and $S(Kerf_*)^\perp$ respectively. Equation (3.5) gives

$$JW = JQ_1W + BQ_2W + CQ_2W, \quad (3.6)$$

where $JQ_1W \in \Gamma(\Delta)$, $BQ_2W \in \Gamma(D_2)$ and $CQ_2W \in \Gamma(S(Kerf_*)^\perp)$. In view of (2.1), (3.3)-(3.6), comparing the components of Δ , D_1 , D_2 , $\text{ltr}(Kerf_*)$ and $S(Kerf_*)^\perp$, for any $U, V \in \Gamma(Kerf_*)$, we get the following equations

$$P_1(T_UJP_1V) + P_1(\nabla_UJP_2V) + P_1(\nabla_UfP_3V) + P_1(T_UFP_3V) = J(T_U^lV), \quad (3.7)$$

$$P_2(T_UJP_1V) + P_2(\nabla_UJP_2V) + P_2(\nabla_UfP_3V) + P_2(T_UFP_3V) = JP_2(\nabla_UV), \quad (3.8)$$

$$P_3(T_UJP_1V) + P_3(\nabla_UJP_2V) + P_3(\nabla_UfP_3V) + P_3(T_UFP_3V) = B(T_U^sV) + fP_3(\nabla_UV), \quad (3.9)$$

$$\nabla_U^{\perp l}(JP_1V) + T_U^l(JP_2V) + T_U^l(fP_3V) + D^{\perp l}(U, FP_3V) = JP_1(\nabla_UV), \quad (3.10)$$

$$D^{\perp s}(U, JP_1V) + T_U^s(JP_2V) + T_U^s(fP_3V) + \nabla_U^{\perp s}(FP_3V) = FP_3(\nabla_UV). \quad (3.11)$$

Denote by $\mathbb{R}_{r,q,p}^n$ the space \mathbb{R}^n equipped with the semi-Riemannian metric g defined by $g(e_i, e_j)_{r,q,p} = (G_{r,q,p})_{ij}$, $i \in \{1, \dots, n\}$. Here e_i is the standard basis of \mathbb{R}^n and $G_{r,q,p}$ is the diagonal matrix determined by g , that is, $G_{ij} = \text{diagonal}(\underbrace{0, \dots, 0}_{r\text{-times}}, \underbrace{-1, \dots, -1}_{q\text{-times}}, \underbrace{1, \dots, 1}_{p\text{-times}})$.

EXAMPLE 3.3. Let $\mathbb{R}_{0,2,10}^{12}$ and $\mathbb{R}_{2,0,4}^6$ endowed with the semi-Riemannian metric

$$g = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2 \\ + (dx_7)^2 + (dx_8)^2 + (dx_9)^2 + (dx_{10})^2 + (dx_{11})^2 + (dx_{12})^2$$

and degenerate metric $g' = (dy_3)^2 + (dy_4)^2 + (dy_5)^2 + (dy_6)^2$, where x_1, \dots, x_{12} and y_1, \dots, y_6 are the canonical coordinates on \mathbb{R}^{12} and \mathbb{R}^6 , respectively. Define the map $f : (\mathbb{R}^{12}, g) \rightarrow (\mathbb{R}^6, g')$ as

$$(x_1, \dots, x_{12}) \mapsto \left(\frac{x_1 + x_7}{\sqrt{2}}, \frac{x_2 - x_8}{\sqrt{2}}, \frac{x_3 + x_6}{\sqrt{2}}, x_5, x_{11}, x_{12} \right).$$

Then

$$Kerf_* = \text{Span} \left\{ \xi_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_7} \right), \xi_2 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_8} \right), U_1 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_6}, \right. \\ \left. U_2 = \frac{\partial}{\partial x_4}, U_3 = \frac{\partial}{\partial x_9}, U_4 = \frac{\partial}{\partial x_{10}} \right\}$$

and

$$(Ker f_*)^\perp = Span\{\xi_1, \xi_2, V_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_6}\right), V_2 = \frac{\partial}{\partial x_5}, V_3 = \frac{\partial}{\partial x_{11}}, V_4 = \frac{\partial}{\partial x_{12}}\}.$$

Therefore f is a 2-lightlike submersion with $\Delta = Ker f_* \cap (Ker f_*)^\perp = Span\{\xi_1, \xi_2\}$. Moreover, we also have

$$ltr(Ker f_*) = Span\{N_1 = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_7}\right), N_2 = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_8}\right)\}.$$

By easy computation, we can see that $J\xi_1 = N_2$ and $J\xi_2 = N_1$, therefore $J\Delta = ltr(Ker f_*)$. Further, we have $JU_3 = U_4$, which implies $D_1 = Span\{U_3, U_4\}$ is invariant under J . Finally we observe $D_2 = Span\{U_1, U_2\}$ is a slant distribution with slant angle $\theta = \frac{\pi}{4}$. Hence f is a radical transversal screen semi-slant lightlike submersion.

Now, we give a characterization theorem for radical transversal screen semi-slant lightlike submersions:

THEOREM 3.4. Let $f : (M, g, J) \rightarrow (B, g')$ be a 2r-lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B and $Ker f_*$ be a lightlike distribution of M . Then f is a radical transversal screen semi-slant lightlike submersion if and only if

- (i) $J(ltr(Ker f_*)) = \Delta$,
- (ii) distribution D_1 is invariant with respect to J , i.e. $JD_1 = D_1$,
- (iii) there exists a constant $\lambda \in [0, 1]$ such that $P^2X = -\lambda X$

Moreover, there also exists a constant $\mu \in (0, 1]$ such that $BFX = -\mu X$, for every $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM) = D_1 \oplus D_2$ and $\lambda = \cos^2\theta$, θ is slant angle of D_2 .

PROOF. Let $f : M \rightarrow B$ be a 2r-lightlike submersion and $Ker f_*$ be a lightlike submanifold of M . Then Lemma (3.1) implies that $S(Ker f_*)$ is a Riemannian vector bundle. If f is a radical transversal screen semi-slant lightlike submersion, then ditribution D_1 is invariant with respect to J and $J\Delta = ltr(ker f_*)$. Thus $JX \in ltr(TM)$, for every $X \in \Gamma(\Delta)$, which proves (i) and (ii).

Now for any $X \in \Gamma(D_2)$, we have $|PX| = |JX| \cos \theta$. This imlies that

$$\cos \theta = \frac{|PX|}{|JX|}. \quad (3.12)$$

In view of (3.12), we get $\cos^2 \theta = \frac{|PX|^2}{|JX|^2} = \frac{g(PX, PX)}{g(JX, JX)} = \frac{g(X, P^2X)}{g(X, J^2X)}$, which provides

$$g(X, P^2X) = \cos^2\theta g(X, J^2X). \quad (3.13)$$

Since f is radical transversal screen semi-slant lightlike submersion, $\cos^2\theta = \lambda(\text{constant}) \in (0, 1]$ and therefore from (3.12), we have $g(X, P^2X) = \lambda g(X, J^2X) = g(X, \lambda J^2X)$. It follows that

$$g(X, (P^2 - \lambda J^2)X) = 0.$$

Now, for any $X \in \Gamma(D_2)$, we obtain $J^2X = P^2X + FPX + BFX + CFX$. Taking the tangential components we get $P^2X = -X - BFX \in \Gamma(D_2)$. Hence $(P^2 - \lambda J^2)X \in \Gamma(D_2)$. Now, since the induced metric g on $D_1 \times D_2$ is non-null, we have $(P^2 - \lambda J^2)X = 0$. This implies that

$$P^2X = \lambda J^2X = -\lambda X. \quad (3.14)$$

Next, suppose that $X \in \Gamma(D_2)$, we have

$$JX = PX + FX, \quad (3.15)$$

where PX and FX are tangential and transversal components of JX . Equation (3.15) gives

$$-X = P^2X + BFX. \quad (3.16)$$

From (3.14) and (3.15), we get $BFX = -\mu X$, where $\mu = 1 - \lambda \in (0, 1]$. This proves (iii). Conversely suppose that conditions (i), (ii) and (iii) hold. From (i), we have $JN \in \Delta$, for every $N \in \Gamma(\text{ltr}(\text{Ker } f_*))$. So, $J(JN) \in J(\Delta)$, which gives $-N \in J(\Delta)$, for all $N \in \Gamma(\text{ltr}(\text{Ker } f_*))$. Thus $J\Delta = \text{ltr}(\text{Ker } f_*)$. From (3.16), for any $X \in \Gamma(D_2)$, we get $-X = P^2X - \mu X$, which implies $P^2X = -\lambda X$, where $\lambda = 1 - \mu(\text{constant}) \in [0, 1)$. Now $\cos \theta = \frac{g(JX, PX)}{|JX||PX|} = -\frac{g(X, JPX)}{|JX||PX|} = -\frac{g(X, P^2X)}{|JX||PX|} = -\lambda \frac{g(X, J^2X)}{|JX||PX|} = \lambda \frac{g(JX, JX)}{|JX||PX|}$. From the above equation, we get

$$\cos \theta = \lambda \frac{JX}{PX}. \quad (3.17)$$

Hence (3.12) and (3.17) provide $\cos^2 \theta = \lambda(\text{constant})$. Thus f is a radical transversal screen semi-slant lightlike submersion. \square

COROLLARY 3.5. *Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B with slant angle θ . Then for any $X, Y \in \Gamma(D_2)$, we have*

- (i) $g(PX, PY) = \cos^2 \theta g(X, Y)$,
- (ii) $g(FX, FY) = \sin^2 \theta g(X, Y)$.

PROOF. The proof is following by (2.1) and (3.17). \square

THEOREM 3.6. *Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B . Then Δ is integrable if and only if*

- (i) $P_2(T_U JP_1 V) = P_2(T_V JP_1 U)$ and $P_3(T_U JP_1 V) = P_3(T_V JP_1 U)$,
- (ii) $D^{\perp s}(U, JP_1 V) = D^{\perp s}(V, JP_1 U)$, for all $U, V \in \Gamma(\Delta)$.

PROOF. Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto a lightlike submanifold B . Let $U, V \in \Gamma(\Delta)$. From (3.8), we have $P_2(T_U JP_1 V) = JP_2 \nabla_U V$. It follows that $P_2(T_U JP_1 V) - P_2(T_V JP_1 U) = JP_2[U, V]$. Also, from (3.9), we have $P_3(T_U JP_1 V) - P_3(T_V JP_1 U) = JP_3[U, V]$. In view of (3.11), we have $D^{\perp s}(U, JP_1 V) = FP_3 \nabla_U V + CT_U^s V$, which provides

$D^{\perp s}(U, JP_1V) - D^{\perp s}(V, JP_1V) = FP_3[U, V]$. Now Δ is integrable if and only if $[U, V] \in \Delta$. This concludes the theorem. \square

THEOREM 3.7. *Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B . Then D_1 is integrable if and only if*

- (i) $T_U^l JP_2V = T_V^l JP_2U$ and $T_U^s JP_2V = T_V^s JP_2U$,
- (ii) $P_3(\nabla_U JP_2V) = P_3(\nabla_V JP_2U)$.

PROOF. Let $U, V \in \Gamma(D_1)$. From (3.10), we have $T_U^l JP_2V = JP_1\nabla_U V$. It follows that $T_U^l JP_2V - T_V^l JP_2U = JP_1[U, V]$. Also, from (3.11), we have $T_U^s JP_2V = FP_3\nabla_U V + CT_U^s V$, which gives $T_U^s JP_2V - T_V^s JP_2U = FP_3[U, V]$. In view of (3.9), $P_3\nabla_U JP_2V = fP_3\nabla_U V + B(T_U^s V)$, which provides $P_3\nabla_U JP_2V - P_3\nabla_V JP_2U = fP_3[U, V]$. Using the fact D_1 is integrable if and only if $[U, V] \in D_1$, the proof follows. \square

THEOREM 3.8. *Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B . Then D_2 is integrable if and only if*

- (i) $T_U^l fP_3V + D^{\perp l}(U, FP_3V) = T_V^l fP_3U + D^{\perp l}(V, FP_3U)$,
- (ii) $P_2(T_U fP_3V - T_V fP_3U) = P_2(T_U FP_3V - T_V FP_3U)$.

PROOF. Let f be a radical screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B . Let $U, V \in \Gamma(D_2)$. From (3.10), we have $T_U^l(fP_3V) + D^{\perp l}(U, FP_3V) = JP_1\nabla_U V$, which gives $T_U^l(fP_3V) + D^{\perp l}(U, FP_3V) - T_V^l(fP_3U) - D^{\perp l}(V, FP_3U) = JP_1[U, V]$. Also from (3.8), we get $P_2(\nabla_U fP_3V) + P_2(T_U FP_3V) = JP_2(\nabla_U V)$, which implies $P_2(\nabla_U fP_3V - \nabla_V fP_3U) + P_2(T_U FP_3V - T_V FP_3U) = JP_2[U, V]$. Using the fact D_2 is integrable if and only if $[U, V] \in D_2$, for all $U, V \in D_2$, we obtain the required result. \square

THEOREM 3.9. *Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold onto a lightlike manifold B . Then the induced connection $\hat{\nabla}$ is a metric connection if and only if*

- (i) $fP_3T_UN = -BD^{\perp s}(U, N)$,
- (ii) $JP_2T_UN = 0$, for all $U \in \Gamma(Ker f_*)$ and $N \in \Gamma ltr(Ker f_*)$

PROOF. Let f be a radical screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B . Then the induced connection $\hat{\nabla}$ is a metric connection if and only if Δ is parallel distribution with respect to $\hat{\nabla}$. That is, $\hat{\nabla}$ is metric connection if and only if $\hat{\nabla}_U JN \in \Delta$, for all $U \in \Gamma(Ker f_*)$ and $N \in \Gamma ltr(Ker f_*)$. Let $U \in \Gamma(Ker f_*)$ and $N \in \Gamma ltr(Ker f_*)$. From (2.2), (2.9) and (2.10), we obtain $\hat{\nabla}_U JN + T_U^l JN + T_U^s JN = JT_UN + J\nabla_U^{\perp l} N + JD^{\perp s}(U, N)$. Now, on comparing the tangential components of both sides of above equation, we have $\hat{\nabla}_U JN = JP_2T_UN + fP_3T_UN + J\nabla_U^{\perp l} N + BD^{\perp s}(U, N)$, which completes the proof. \square

4. Foliations determined by distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold onto a lightlike manifold B to be totally geodesic.

THEOREM 4.1. *Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto lightlike manifold B . Then Δ admits a geodesic foliation on $\text{Ker } f_*$ if and only if $g(T_U F P_3 W, JV) = g(\nabla_U J P_2 W + \nabla_U f P_3 W, JV)$, for all $U, V \in \Gamma(\Delta)$ and $W \in \Gamma(S(\text{Ker } f_*))$.*

PROOF. Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold onto lightlike manifold B. To prove Δ admits a totally geodesic foliation, it is sufficient to show that $\hat{\nabla}_U V \in \Gamma(\Delta)$, for all $U, V \in \Gamma(\Delta)$. Since ∇ is a metric connection, using (2.1), (2.2), (2.9) and (3.4), for any $U, V \in \Delta$ and $W \in \Gamma(S(\text{Ker } f_*))$, we obtain $g(\nabla_U V, W) = g(\nabla_U JV, JW) = -g(\nabla_U JW, JV) = -g(\nabla_U J P_2 W + \nabla_U f P_3 W + \nabla_U F P_3 W, JV) = g(P_1 T_U F P_3 W - P_1 \hat{\nabla}_U J P_2 W - P_1 \nabla_U f P_3 W, JV)$, which completes the proof. \square

THEOREM 4.2. *Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B. Then D_1 defines a totally geodesic foliation if and only if*

- (i) $g(T_U F W, JV) = g(\nabla_U f W, JV)$, for every $U, V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$,
- (ii) $T_U^* J N = 0$ on D_1 , for each $N \in \Gamma(\text{ltr}(\text{Ker } f_*))$.

PROOF. Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B. The distribution D_1 defines a totally geodesic foliation if and only if $\nabla_U V \in \Gamma(D_1)$, for all $U, V \in \Gamma(D_1)$. Since ∇ is a metric connection, from (2.1), (2.2) and (2.9), we have $g(\hat{\nabla}_U V, W) = -g(\nabla_U JW, JV) = -g(T_U F W + \hat{\nabla}_U f W, JV)$. Also from (2.1), (2.2), (2.9) and (2.14), for any $U, V \in \Gamma(D_1)$ and $N \in \Gamma(\text{ltr}(\text{Ker } f_*))$, we have $g(\hat{\nabla}_U V, N) = -g(\nabla_U J N, JV) = -g(T_U^* J N, JV)$. Thus we obtain the required result. \square

THEOREM 4.3. *Let f be a radical transversal lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B. Then D_2 defines a totally geodesic foliation on $\text{Ker } f_*$ if and only if*

- (i) $g(fV, \nabla_U JZ) = -g(FV, T_U^s JZ)$,
- (ii) $g(fV, \nabla_U JN) = -g(FV, T_U^s JN)$, for all $U, V \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(\text{ltr}(\text{Ker } f_*))$.

PROOF. Let f be a radical transversal screen semi-slant lightlike submersion from an indefinite Kaehler manifold M onto a lightlike manifold B. The distribution D_2 admits a totally geodesic foliation if and only if $\hat{\nabla}_U V \in \Gamma(D_2)$, for all $U, V \in \Gamma(D_2)$. Since Δ is a metric connection, using (2.1), (2.2) and (2.9), we have $g(\hat{\nabla}_U V, Z) = -g(\nabla_U JZ, JV) = -g(\nabla_U JZ, fV) - g(T_U^s JZ, FV)$ for any $U, V \in \Gamma(D_2)$ and $Z \in$

$\Gamma(\text{ltr}(\text{Ker } f_*))$. Again from (2.1), (2.2) and (2.9), for any $U, V \in \Gamma(D_2)$ and $N \in \Gamma(\text{ltr}(\text{Ker } f_*))$, we get $g(\hat{\nabla}_U V, N) = -g(\nabla_U JN, JV) = -g(\nabla_U JN, fV) - g(T_U^s JN, FV)$. This completes the proof. \square

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