GANITA,Vol.73(2), 2023, 187-193

CHARACTERIZATION OF F-PSEUDOCOMPACTNESS OF A TOPOLOGICAL SPACE X VIA u-TOPOLOGY, m-TOPOLOGY AND r-TOPOLOGY ON $C(X, F)$

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Abstract

Let F be a totally ordered field equipped with its order topology and X, a Hausdorff completely F-regular topological space(CFR space in short) in the sense that, points and closed sets in X could be separated by F-valued continuous functions on X. Suppose $C(X, F)$ is the ring of all F-valued continuous functions on X, $B(X, F) = \{f \in C(X, F) : |f| < \lambda$ for some $\lambda > 0$ in $F\}$ and $C^*(X, F) = \{f \in C(X, F) : \mathrm{cl}_F f(X)$ is compact$\}$. A topological space X is said to F-pseudocompact if $C(X, F) = B(X, F)$. It is shown that a topological space X is F-pseudocompact if and only if $C(X, F)$ with the u-topology is a topological ring/topological vector space if and only if the u-topology and the relative m-topology on $B(X, F)$ coincide. Also it is shown that if X(CFR space) is a F-pseudocompact, almost P-space then u-topology, m-topology and r-topology on $C(X, F)$ coincide.

2010 *Mathematics subject classification:* Primary 54C40; Secondary 46E25.
Keywords and phrases: F-pseudocompact spaces, totally ordered field, completely F-regular spaces, u-topology, m-topology, r-topology.

1. Introduction

Let F be a totally ordered field equipped with its order topology. For any topological space X, the set $C(X, F) = \{f : X \to F \mid f$ is continuous on $X\}$ makes a commutative lattice ordered ring with 1, if the relevant operations are defined pointwise on X. The set $B(X, F) = \{f \in C(X, F)$: there exists $\lambda > 0$ in F with $|f| \leq \lambda$ on $X\}$ and $C^*(X, F) = \{f \in C(X, F) : \mathrm{cl}_F f(X)$ is compact$\}$ turn out to be subrings and sublattices of $C(X, F)$ with the inclusion relation $C^*(X, F) \subseteq B(X, F) \subseteq C(X, F)$. With $F = \mathbb{R}$, $C^*(X, F)$ is the same as $B(X, F)$. However with $F \neq \mathbb{R}$, it may well happen that these two rings are different. This can be illustrated by choosing $X = F = \mathbb{Q}$ and observing that the function $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = \frac{x}{1+|x|}$ where $x \in \mathbb{Q}$, belongs to $B(X, F)$, without belonging to $C^*(X, F)$. Indeed for this function f, $\mathrm{cl}_F f(X)$ is the set $\{x \in \mathbb{Q} : -1 \leq x \leq 1\}$, which is never compact. It is well known that, there is a nice interaction between the topological structure of X and the algebraic ring and order structure of $C(X)$ and $C^*(X)$ both. An excellent account of this interplay can be found in [5]. It is worth mentioning in this context that a good many results related to this interaction are still valid if $C(X)$ (respectively $C^*(X)$) is replaced by

$C(X, F)$ (respectively $B(X, F)$ and $C^*(X, F)$) for any totally ordered field F and this is best realised if one sticks to the completely F-regular spaces. X is called completely F-regular if it is Hausdorff and given a point $x \in X$ and a closed set K in X with $x \notin K$, there is an $f \in B(X, F)$ such that $f(x) = 0$ and $f(K) = 1$. Thus complete F-regularity reduces to Tychonoffness in case $F = \mathbb{R}$. Incidentally if $F \neq \mathbb{R}$ then complete F-regularity of X and zero-dimensionality of X are equivalent conditions. Problems of this kind are already investigated by Acharyya, Chattopadhyay and Ghosh in an earlier paper [2]. A seemingly similar kind of problem, albeit treated differently is also addressed by Bachman, Beckenstein, Narici and Warner in [11]. For brevity completely F-regular Hausdorff spaces will be termed as CFR spaces. The class of all pseudo-compact spaces plays a significant role in many a problem related to rings of continuous functions. A space is called pseudo-compact if $C^*(X) = C(X)$. There are numerous equivalent descriptions of pseudo-compact spaces in the literature. One particularly interesting such description reads as follows : X is pseudo-compact if and only if $C(X)$ with the u-topology (or the topology of uniform convergence) is a topological ring/topological vector space if and only if the u-topology and the m-topology on $C^*(X)$ coincide (See Exercise, Chapter 2, [5]). Now a topological space X is said to be an almost P-space if every non-empty G_δ-set of X has non-empty interior. With this notion another interesting description reads as follows: X (Tychonoff) is pseudo-compact and almost P-space if and only if u-topology, m-topology and r-topology on $C(X)$ coincide (See Theorem 1.5, Theorem 1.8 and Theorem 1.9, [3]). In this article, my principal intention is to show that these results could be deduced as special cases of some analogous facts involving the rings $C(X, F)$ and $B(X, F)$. .

2. Preliminaries

With this incur mind, we call a space X (not necessarily CFR) F-pseudocompact if $C(X, F) = B(X, F)$. It is clear that F-pseudocompact spaces with $F = \mathbb{R}$ are nothing other than pseudocompact spaces. For any $f \in C(X, F)$ and $r \in F^+$, let $u(f, r) = \{g \in C(X, F) : \mid f(x) - g(x) \mid \leq r$ for all $x \in X\}$. Then there is a unique topology on $C(X, F)$ for which for any $f \in C(X, F)$, the family $\{u(f, r) : r \in F^+\}$ forms a base for the neighbourhood system of 'f'. We call this topology as in the classical case for $C(X)$, the u-topology on $C(X, F)$. A typical basic neighbourhood in the u-topology restricted to the subring $B(X, F)$ of $C(X, F)$ will be denoted by $u^*(f, r)$ where $f \in B(X, F)$ and we note that, $u^*(f, r) = u(f, r) \cap B(X, F)$. Also for any $f \in C(X, F)$ and $u \in U^+$ let us set $m(f, u) = \{g \in C(X, F) : \mid f(x) - g(x) \mid \leq u(x)$, for all $x \in X\}$ here U^+ is the set of all positive units in $C(X, F)$. Then there is a unique topology on $C(X, F)$ for which for any $f \in C(X, F)$, the family $\{m(f, u) : u \in U^+\}$ forms a base for the neighbourhood system of 'f'. We call this topology as in the classical situation, the m-topology on $C(X, F)$. For any $f \in C(X, F)$, f is said to be a regular element of $C(X, F)$ if $coz(f)$ (i.e., $X - Z(f)$) is a dense subset of X. Let $R^+ = \{f \in C(X, F) : f(x) > 0$ and f is regular element of $C(X, F)\}$, the set of all positive regular elements of $C(X, F)$. For any $f \in C(X, F)$ and $\lambda \in R^+$, let $r(f, \lambda) = \{g \in C(X, F) : \mid f(x) - g(x) \mid \leq \lambda(x)$, for all

$x \in coz(\lambda)\}$. Then there is a unique topology on $C(X, F)$ for which for any $f \in C(X, F)$, the family $\{r(f, \lambda) : \lambda \in R^+\}$ forms a base for the neighbourhood system of 'f'. We call this topology as in the classical case for $C(X)$, the r-topology on $C(X, F)$. Since $U^+ \subseteq R^+$ we can say that the r-topology is finer than the m-topology on $C(X, F)$.

Let A be a subring of $C(X, F)$ on which a topology has been defined. Then A is called a topological ring if the following two operations are continuous:

1. '$-$' : $A \times A \longrightarrow A$ defined by $-((g, h)) = g - h$
2. '$*$' : $A \times A \longrightarrow A$ defined by $*((g, h)) = g.h$

If A contains the constant functions, then it is a topological vector space if the following two operations are continuous:

1. '$+$' : $A \times A \longrightarrow A$ defined by $+((g, h)) = g + h$.
2. '$\odot$' : $F \times C(X, F) \longrightarrow C(X, F)$ defined by $\odot((\alpha, h)) = \alpha h$.

Lemma 2.1. *Let U^* be the set of all units of $B(X, F)$. Then U^* is an open subset of $B(X, F)$ in the u-topology.*

Proof. Choose $u \in U^*$; then u is bounded away from zero on X which means that there exists $\lambda > 0$ in F such that $| u | \geq \lambda$. i.e, for all $x \in X$, $u(x) \geq \lambda$ or $u(x) \leq -\lambda$. Let us consider the set $E = \{f \in B(X, F) : | f(x) - u(x) | \leq \frac{\lambda}{2}$, for all $x \in X\}$. Then E is a neighbourhood of 'u'. Also it is clear that, each member of E is an unit of $B(X, F)$. Thus 'u' is an interior point of U^*. Hence U^* is open in $B(X, F)$ with respect to u-topology. □

Lemma 2.2. *$B(X, F)$ with u-topology is a topological ring as well as a topological vector space over F.*

Proof. We have to show that, addition and multiplication on $B(X, F)$ are continuous. So let for $f,\ g \in B(X, F)$, $u^*(f + g, r)$ and $u^*(fg, r)$ be arbitrary neighbourhoods of $f + g$ and fg respectively where $r \in F^+$ = the set of all positive elements in F, surely without loss of generality we can choose the same r for both the cases. Then $u^*(f, \frac{r}{2})$ and $u^*(g, \frac{r}{2})$ are neighbourhoods of 'f' and 'g' respectively and $u^*(f, \frac{r}{2}) + u^*(g, \frac{r}{2}) \subseteq u^*(f + g, r)$. Since '$f$', '$g$' $\in B(X, F)$ there exists $n, m \in F^+$ such that $| f(x) | \leq n$ and $| g(x) | \leq m$, for all $x \in X$. It is routine to check that, $u^*(f, \frac{r}{2(\frac{r}{2n}+m)}).u^*(g, \frac{r}{2n}) \subseteq u^*(fg, r)$. In a similar way it can be shown that scalar multiplication is continuous. Hence the result. □

Lemma 2.3. *If X is not F-pseudocompact then the set U of all units of $C(X, F)$ is not an open subset of $C(X, F)$.*

Proof. Since X is not F-pseudocompact we can choose an $f_0 \in C(X, F)$ with $f_0 > 0$ such that $f_0 \notin B(X, F)$. Take $f = \frac{1}{f_0^2+1}$, then $f \in U$. Clearly f doesn't vanish anywhere on X but it takes values arbitrarily near to zero. We claim that, f is not an interior point of U – indeed for any $\lambda > 0$ in F, $u(f, \lambda) \nsubseteq U$. Because corresponding to $\lambda > 0$ in F there exists $a \in X$ such that $0 < f(a) < \lambda$, hence $f - f(a) \in u(f, \lambda)$ but $f - f(a) \notin U$

since $(f - f(a))(a) = 0$ implies $f - f(a)$ is not a unit of $C(X, F)$. Therefore U is not open in $C(X, F)$ with respect to u-topology. □

LEMMA 2.4. *If X is not pseudocompact then $C(X, F)$ with u-topology is neither a topological ring nor a topological vector space over F.*

PROOF. Let X be not F-pseudocompact i.e, $B(X, F) \nsubseteq C(X, F)$ so there exists, $f \in C(X, F) \setminus B(X, F)$ such that $f \geq 1$ on X. We claim that the function

$\psi : C(X, F) \times C(X, F) \longrightarrow C(X, F)$ defined by $\psi((g, h)) = gh$ is not continuous at the point $(\underline{0}, f)$ where '$\underline{0}$' stands for the function identically equal to zero (and this will prove that $C(X, F)$ is not a topological ring). The set $E = \{g \in C(X, F) : | g(x) |\leq 1, \text{for all } x \in X\}$ is surely a neighbourhood of '$\underline{0}$' in $C(X, F)$ and $E \subseteq B(X, F)$. Choose any neighbourhood $u(\underline{0}, \lambda_1)$ of '$\underline{0}$' in $C(X, F)$ and any neighbourhood $u(f, \lambda_2)$ of 'f' in $C(X, F)$. It is enough to check that, $u(\underline{0}, \lambda_1).u(f, \lambda_2) \nsubseteq E$– indeed for the constant function '$\frac{\lambda_1}{2}$', $\frac{\lambda_1}{2} \in u(\underline{0}, \lambda_1)$, $f.\frac{\lambda_1}{2} \notin B(X, F)$ so that $f.\frac{\lambda_1}{2} \notin E$, while $f.\frac{\lambda_1}{2} \in u(\underline{0}, \lambda_1).u(f, \lambda_2)$.

An almost analogous argument can be adopted to show that, $\phi : F \times C(X, F) \longrightarrow C(X, F)$ defined by $\phi((\alpha, f)) = \alpha.f$ is not continuous at the point $(\underline{0}, f)$. Thus $C(X, F)$ is not a topological vector space over F. □

3. Characterization of *F*-pseudocompactness via *u*-topology and *m*-topology

Combining the above four lemmas we can establish the following two theorems:

THEOREM 3.1. *For any topological space X and any ordered field F the following statements are equivalent:*

1. *X is F-pseudocompact.*
2. *The set U of all units of $C(X, F)$ is open in $C(X, F)$ with respect to u-topology.*
3. *$C(X, F)$ with u-topology is a topological ring.*
4. *$C(X, F)$ with u-topology is a topological vector space over F.*

PROOF. Proof follows from Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.4. □

THEOREM 3.2. *For any topological space X and any ordered field F the following statements are equivalent:*

1. *X is F-pseudocompact.*
2. *The u-topology and the relative m-topology on $B(X, F)$ coincide.*

PROOF. It is easy to see that the u-topology on $B(X, F)$ is weaker than the relative m-topology on $B(X, F)$.

(1)⇒(2) : Let X be F-pseudocompact. Then any positive unit u of $C(X, F)$ is a unit of $B(X, F)$ so that it is bounded away from zero meaning that $u(x) \geq \lambda > 0$ for all $x \in X$ for some $\lambda \in F^+$ and hence for any $g \in C(X, F)$, $m(g, \lambda) \subseteq m(g, u)$ and $m(g, \lambda) = u(g, \lambda)$. Hence the relative m-topology on $B(X, F) \subseteq$ the u-topology on $B(X, F)$ and so the relative m-topology on $B(X, F) =$ the u-topology on $B(X, F)$.

(2)⇒(1) : Let us assume that X be not F-pseudocompact. We shall show that, $B(X,F)$ with the relative m-topology is not a topological vector space over F and this proves that, the u-topology on $B(X,F) \subset$ the relative m-topology on $B(X,F)$, because $B(X,F)$ with respect to u-topology is essentially a topological vector space over F. Now the assumption that X is not F-pseudocompact guarantees that there exists $k \in C(X,F)$ such that 'k' is a positive unit of $C(X,F)$ which takes values arbitrarily near to zero. Then for any pair of distinct 'r', 's' in F it will never happen that $\mid \underline{r} - \underline{s} \mid \leq k$ where '$\underline{r}$', '$\underline{s}$' meaning the functions identically equal to r and s respectively. Hence for any $r \in F$, $m(\underline{r}, k) \cap \{\underline{s} : s \in F\} = \{\underline{r}\}$. In other words the set $\{\underline{r} : r \in F\}$ of constant functions on X is a discrete subset of $C(X,F)$ and hence a discrete subset of $B(X,F)$ also. Therefore the scalar multiplication map $\phi : F \times B(X,F) \longrightarrow B(X,F)$ defined by $\phi((\alpha, f)) = \alpha.f$ is not continuous at points like (α, r) where '$\underline{r}$' stands for the constant function identically equal to r. Hence $B(X,F)$ with the relative m-topology is not a topological vector space over F. □

Corollary 3.3 (See Exercise, Chapter 2, [5]). *For any topological space X following statements are equivalent:*

1. *X is pseudocompact.*
2. *The u-topology and the relative m-topology on* $C^*(X)$ *coincide.*

Proof. The proof follows on choosing $F = \mathbb{R}$ in Theorem 3.2. □

4. Characterization of F-pseudocompact and almost P-spaces via u-topology, m-topology and r-topology on C(X,F)

Theorem 4.1. *For any topological space X and any ordered field F the following statements are equivalent:*

1. *X is F-pseudocompact.*
2. *The u-topology and the m-topology on* $C(X,F)$ *coincide.*

Proof. It clearly follows from Theorem 3.2. □

A topological space X is said to be an almost P-space if every non-empty G_δ-set of X has non-empty interior. We recall some equivalent conditions for a space X to be an almost P-space.

Proposition 4.2. *For any completely F-regular space X where F is any ordered field the following statements are equivalent:*

1. *X is an almost P-space.*
2. *Each non-empty zero-set of X has non-empty interior.*
3. *Each zero-set of X is a regular closed subset of X.*

Proof. It follows after closely monitoring the proof of the Proposition 1.1 of [7]. □

In terms of elements of $C(X,F)$, X is an almost P-space if and only if every regular element of $C(X,F)$ is a unit. This leads to the following direction.

THEOREM 4.3. *Let X be a completely F-regular space. The following statements are equivalent.*

1. *The r-topology and m-topology of $C(X, F)$ coincides.*
2. *X is an almost P-space.*
3. $R^+ = U^+$.

PROOF. It is clear that (2) and (3) are equivalent. Since $R^+ = U^+$ the family which forms the base for the neighbourhood system of any $f \in C(X, F)$ for both topologies are equal. So the r-topology and m-topology of $C(X, F)$ coincides. Now we want to prove (1)⇒(2) by method of contradiction. Let us assume that X is not an almost P-space. Then by Proposition 4.2 there exists $f \in C(X, F)$ such that $Z(f)$ is non-empty for which its interior is also non-empty. Now since $Z(f) = Z(|\ f\ |)$ so without loss of generality we assume $f \geq 0$ and hence $f \in R^+$. Consider $r(\underline{0}, f)$ where $\underline{0}$ stands for the identically 0 function. Now if (1) holds then there exists some $g \in U^+$ such that $r(\underline{0}, g) \subseteq r(\underline{0}, f)$. Let $x \in Z(f)$. Then $0 < \frac{g(x)}{2} < f(x) = 0$, a contradiction. Therefore (1)⇒(2) is proved. □

THEOREM 4.4. *Let X be a completely F-regular space. The following statements are equivalent.*

1. *The r-topology and u-topology of $C(X, F)$ coincides.*
2. *X is a F-pseudocompact, almost P-space.*

PROOF. The proof follows from Theorem 4.1 and Theorem 4.3. □

We are now in a position to prove our main result of this section.

THEOREM 4.5. *Let X be a completely F-regular space. The following statements are equivalent.*

1. *The u-topology, r-topology and m-topology of $C(X, F)$ coincides.*
2. *X is a F-pseudocompact, almost P-space.*

PROOF. The proof follows from Theorem 4.3 and Theorem 4.4. □

COROLLARY 4.6 ([3]). *Let X be a Tychonoff space. The following statements are equivalent:*

1. *The u-topology, r-topology and m-topology of $C(X)$ coincides.*
2. *X is a pseudocompact, almost P-space.*

PROOF. It follows from Theorem 4.5 on choosing $F = \mathbb{R}$. □

Acknowledgement: The author expresses his gratitude to Prof. S. K. Acharyya for making some valuable comments towards the improvement of the original version of this paper.

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