RESULTS ON UNIQUENESS OF MEROMORPHIC FUNCTIONS REGARDING THEIR SHIFT AND DIFFERENTIAL POLYNOMIAL

HARINA P. WAGHAMORE [✉] **and MEGHA M. MANAKAME**

Abstract

In this paper, we investigate the uniqueness of meromorphic function with its shift and differential polynomial. The results in this paper generalize and improve the results due to C. Meng and G. Liu[14].

1. Introduction, definitions and main results

Let f be a meromorphic function in the complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the standard notation and fundamental results of Nevanlinna theory of meromorphic functions, such as $m(r, f)$, $N(r, f)$, $T(r, f)$, $\overline{N}(r, f)$ etc (see [8], [25]). The notation $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set E of r of finite logarithmic measure. Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a, f) = \{z : f(z) - a = 0\}$ where a zero point with multiplicity k is counted k times in the set. If these zero points are counted only once, then we denote by $\overline{E}(a, f)$. Let f and g be two non constant meromorphic functions. If $E(a, f) = E(a, g)$, then we say that f and g share the value a CM. If $\overline{E}(a, f) = \overline{E}(a, g)$, then we say that f and g share the value a IM. We denote by $E_{k)}(a, f)$ the set of all $a-$ points of f with multiplicities not exceeding k, where an $a-$point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a, f)$ the set of distinct $a-$ points of f with multiplicities not greater than k [25].

DEFINITION 1.1. [24] Let $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer k, we denote by

(i) $N_{k)}(r, \frac{1}{f-a})$ the counting function of $a-$ points of f with multiplicity $\leq k$.

(ii) $N_{(k}(r, \frac{1}{f-a})$ the counting function of $a-$ points of f with multiplicity $\geq k$. Similarly the reduced counting function $\overline{N}_{k)}(r, \frac{1}{f-a})$ and $\overline{N}_{(k}(r, \frac{1}{f-a})$ are defined.

DEFINITION 1.2. [11] For a complex number $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all $a-$ points of f where an $a-$ point with multiplicity m is counted m times if $m \leq k$

and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$ for a complex number $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g share the value a with weight k.

The definition implies that if f and g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f and g share (a, k) to mean that f and g share the value a with weight k. Clearly if f and g share (a, k) then f and g share (a, p) for all integer p, $0 \leq p \leq k$. Also we note that f and g share a value a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively.

DEFINITION 1.3. [20] Let f and g be two meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f of order p, and a 1-point of g of order q. We denote by $\overline{N}_L(r, \frac{1}{f-1})$ the counting function of those 1-points of f and g such that $q < p$, by $\overline{N}_E^{(2}(r, \frac{1}{f-1})$ the counting function of those 1-points of f and g such that $2 \leq q = p$, by $\overline{N}_E^{1)}(r, \frac{1}{f-1})$ the counting function of those 1-points of f and g such that $p = q = 1$, and by $\overline{N}_{f>2}(r, \frac{1}{g-1})$ the counting function of those 1-points of f and g such that $p > q = 2$, each point in these counting functions is counted only once. In the same way, we can define $\overline{N}_L(r, \frac{1}{g-1})$, $\overline{N}_E^{(2}(r, \frac{1}{g-1})$, $\overline{N}_{g>2}(r, \frac{1}{f-1})$.

DEFINITION 1.4. [13] Let $n_{0j}, n_{1j}.....n_{kj}$ be non-negetive integers. The expression $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}}...(f^{(k)})^{n_{kj}}$ is called differential monomial generated by f of degree $d_{M_j} = d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i + 1)n_{ij}$. The sum $H[f] = \sum_{j=1}^{t} b_j M_j[f]$ is called differential polynomial generated by f of degree

$$\overline{d}(H) = max\{d(M_j) : 1 \leq j \leq t\},$$

and weight

$$\Gamma_H = max\{\Gamma_{M_j} : 1 \leq j \leq t\}.$$

where $T(r, b_j) = S(r, f)$ for $j = 1, 2...t$. The numbers $\underline{d}(H) = min\{d(M_j) : 1 \leq j \leq t\}$ and k (the highest order of the derivative of f in $H[f]$ are called respectively the lower degree and order of $H[f]$.

$H[f]$ is said to be homogeneous if $\overline{d}(H) = \underline{d}(H)$.

$H[f]$ is called a linear differential polynomial generated by f if $\overline{d}(H) = 1$. Otherwise $H[f]$ is called a non-linear differential polynomial.

We denote by $Q = max\{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = max\{n_{1j} + 2n_{2j} + ... + kn_{kj} : 1 \leq j \leq t\}$.

THEOREM 1.5. *[18] If f be a non-constant entire function in the finite complex plane and if f and $f^{'}$ share two distinct values(counting multiplicity), then $f^{'} = f$.*

Thus, L. A. Rubel and C. C. Yang [18], showed that a derivative is worth two values. Since then the study of uniqueness of meromorphic functions sharing values

with derivatives and recently with shifts, difference operator became a subject of much interest ([15], [21], [22]).

In 2018, Qi, Li and Yang[17], considered the value sharing problem related to $f'(z)$ and $f(z + c)$, where c is a complex number. They obtained the following result.

Theorem 1.6. *[17] Let f be a non-constant meromorphic function of finite order and $n \geq 9$ be an integer. If $[f'(z)]^n$ and $f^n(z + c)$ share $a(\neq 0)$ and ∞ CM, then $f'(z) = t f(z + c)$, for a constant t that satisfies $t^n = 1$.*

In 2020, C. Meng and G. Liu[14], extended the above result by considering the kth derivative of f and obtained the following results.

Theorem 1.7. *[14] Let f be a non-constant meromorphic function of finite order and n be a positive integer. If one of the following conditions is satisfied:*

i) $[f^{(k)}(z)]^n$ and $f^n(z + c)$ *share* $(1, 2), (\infty, 0)$ *and* $n \geq 2k + 8$;
ii) $[f^{(k)}(z)]^n$ and $f^n(z + c)$ *share* $(1, 2), (\infty, \infty)$ *and* $n \geq 2k + 7$;
iii) $[f^{(k)}(z)]^n$ and $f^n(z+c)$ *share* $(1, 0), (\infty, 0)$ *and* $n \geq 3k+14$; *then* $f^{(k)}(z) = t f(z+c)$, *for a constant t that satisfies $t^n = 1$.*

Corollary 1.8. *[14] Let f be a non-constant entire function of finite order and $n \geq 5$ be an integer. If $[f^{(k)}(z)]^n$ and $f^n(z + c)$ share $(1, 2)$ then $f^{(k)}(z) = t f(z + c)$, for a constant t that satisfies $t^n = 1$.*

C. Meng and G. Liu [14], further studied the same problem by replacing the shifts $f(z + c)$ by q-difference, i.e., $f(qz)$ and obtained the following results.

Theorem 1.9. *[14] Let f be a non-constant meromorphic function of zero order and n be a positive integer. If one of the following conditions is satisfied:*

i) $[f^{(k)}(z)]^n$ and $f^n(qz)$ *share* $(1, 2), (\infty, 0)$ *and* $n \geq 2k + 8$;
ii) $[f^{(k)}(z)]^n$ and $f^n(qz)$ *share* $(1, 2), (\infty, \infty)$ *and* $n \geq 2k + 7$;
iii) $[f^{(k)}(z)]^n$ and $f^n(qz)$ *share* $(1, 0), (\infty, 0)$ *and* $n \geq 3k + 14$; *then* $f^{(k)}(z) = t f(qz)$, *for a constant t that satisfies $t^n = 1$.*

Corollary 1.10. *[14] Let f be a non-constant entire function of zero order and $n \geq 5$ be an integer. If $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1, 2)$, then $f^{(k)}(z) = t f(qz)$, for a constant t that satisfies $t^n = 1$.*

In this paper, we replace the term $[H(f)]^n$ and $\Delta_c F$ in Theorem C and $[H(f)]^n$ and $\Delta_c F(qz)$ in Theorem D and obtained the following results, where $\Delta_c F(z) = F(z + c) - F(z)$ and $F(z) = f^n(z)$, then $\Delta_c F = f^n(z + c) - f^n(z)$.

Theorem 1.11. *Let f be a non-constant meromorphic function of finite order and n be a positive integer. If one of the following conditions is satisfied:*

i) $[H(f)]^n$ and $\Delta_c F$ *share* $(1, 2), (\infty, 0)$ *and* $n \geq \sigma + \overline{d}(p) + 5$;
ii) $[H(f)]^n$ and $\Delta_c F$ *share* $(1, 2), (\infty, \infty)$ *and* $n \geq \sigma + \overline{d}(p) + 4$;
iii) $[H(f)]^n$ and $\Delta_c F$ *share* $(1, 0), (\infty, 0)$ *and* $n \geq \frac{1}{2}(3\overline{d}(p) + 3\sigma + 20)$. *then* $[H(f)] = t f(z + c)$, *for a constant t that satisfies $t^n = 1/2$.*

THEOREM 1.12. *Let f be a non-constant meromorphic function of zero order and n be a positive integer. If one of the following conditions is satisfied:*

i) $[H(f)]^n$ *and* $\Delta_c F(qz)$ *share* $(1, 2), (\infty, 0)$ *and* $n \geq \sigma + \overline{d}(p) + 5$;

ii) $[H(f)]^n$ *and* $\Delta_c F(qz)$ *share* $(1, 2), (\infty, \infty)$ *and* $n \geq \sigma + \overline{d}(p) + 4$;

iii) $[H(f)]^n$ *and* $\Delta_c F(qz)$ *share* $(1, 0), (\infty, 0)$ *and* $n \geq \frac{1}{2}(3\overline{d}(p) + 3\sigma + 20)$. *where* $\Delta_c F = f^n(qz + c) - f^n(qz)$ *then* $[H(f)] = tf(qz)$, *for a constant t that satisfies* $t^n = 1/2$.

2. Preliminaries

In this section, we present some necessary lemmas.

Denote Ω by the following function.

$$\Omega = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right).$$

LEMMA 2.1. *[2] Let F, G be two non-constant meromorphic functions. If F, G share* $(1, 2)$ *and* (∞, k), *where* $0 \leq k \leq \infty$, *and* $\Omega \not\equiv 0$, *then*

$$T(r, F) \leq N_2\left(r, \frac{1}{F} \right) + N_2\left(r, \frac{1}{G} \right) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G),$$

where $\overline{N}_*(r, \infty; F, G)$ *denotes the reduced counting function of those poles of F whose multiplicities of the corresponding poles of G.*

LEMMA 2.2. *[23] Let f be a non-constant meromorphic function and let* $a_1, a_2, ...a_n$ *be finite complex numbers,* $a_n \neq 0$. *Then*

$$T(r, a_n f^n + ... + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f).$$

LEMMA 2.3. *[6, 19] Let f be a meromorphic function of finite order* $\rho(f)$, *and let c be a nonzero constant. Then*

$$T(r, f(z + c)) = T(r, f(z)) + O(r^{\rho(f)-1+\epsilon}) + O(log r),$$

for any arbitrary $\epsilon > 0$.

We mention that Lemma 2.3 holds also for $c = 0$ *as in the case* $T(r, f(z + c)) = T(r, f(z))$.

LEMMA 2.4. *[12] Let f be a non-constant meromorphic function and p, k be positive integers, then*

$$N_p\left(r, \frac{1}{f^{(k)}} \right) \leq N_{p+k}\left(r, \frac{1}{f} \right) + k\overline{N}(r, f) + S(r, f),$$

where $N_p\left(r, \frac{1}{f^{(k)}} \right)$ *denotes the counting function of the zeros of* $f^{(k)}$ *where a zero of multiplicity m is counted m times if* $m \leq p$ *and p times if* $m > p$ *and* $\overline{N}\left(r, \frac{1}{f^{(k)}} \right) = N_1\left(r, \frac{1}{f^{(k)}} \right)$.

LEMMA 2.5. *[7] Let f be a non-constant meromorphic function of finite order and let $c \in \mathbb{C}$ and $\delta \in (0, 1)$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r,f)}{r^\delta}\right) = S(r,f).$$

LEMMA 2.6. *[26] Suppose that two non-constant meromorphic functions F and G share 1 and ∞ IM. Let Ω be given as above. If $\Omega \not\equiv 0$, then*

$$T(r, F) + T(r, G) \leq 3\overline{N}(r, F) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_E^{1)}\left(r, \frac{1}{F-1}\right)$$

$$+ 2N_E^{(2}\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right)$$

$$+ S(r, F) + S(r, G).$$

LEMMA 2.7. *[27] Let f be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then*
$$T(r, f(qz)) = (1 + o(1))T(r, f(z)) \text{ and}$$
$$N(r, f(qz)) = (1 + o(1))N(r, f(z))$$
on a set of lower logarithmic density 1.

LEMMA 2.8. *[4] Let f be a zero order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*
$$m\left(r, \frac{f(qz)}{f(z)}\right) = S(r, f).$$
on a set of lower logarithmic density 1.

LEMMA 2.9. *[3] Let f be a non-constant meromorphic function and $H[f]$ be a differential polynomial in f. Then*

$$m\left(r, \frac{H[f]}{f^{\bar{d}(p)}}\right) \leq (\bar{d}(p) - \underline{d}(p))m\left(r, \frac{1}{f}\right) + S(r, f),$$

$$N\left(r, \frac{H[f]}{f^{\bar{d}(p)}}\right) \leq (\bar{d}(p) - \underline{d}(p))N\left(r, \frac{1}{f}\right) + \sigma\left(\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right) + S(r, f),$$

$$N(r, H[f]) \leq \bar{d}(p)N(r, f) + \sigma\overline{N}(r, f) + S(r, f),$$

$$T(r, H[f]) \leq \bar{d}(p)T(r, f) + \sigma\overline{N}(r, f) + S(r, f),$$

where $\sigma = max\{n_{1j} + 2n_{2j} + 3n_{3j} + ... + kn_{kj}; 1 \leq j \leq m\}$.

LEMMA 2.10. *Let f be a non-constant meromorphic function and n be a positive integer, if $F = f^n(z+c) - f^n(z)$, where c is a finite complex number and $G = \{H[f]\}^n$, where $H[f]$ is a differential polynomial, then $FG \neq 1$.*

PROOF. On contrary, suppose $FG = 1$, i.e.,

$$(f^n(z+c) - f^n(z))\{H[f]\}^n = 1,$$

from above it is clear that the function f can't have any zero and poles. Therefore $\overline{N}(r, \frac{1}{f}) = S(r, f) = \overline{N}(r, f)$. So by the First Fundamental Theorem and Lemma 2.9 we have,

$$
\begin{aligned}
(2n + n\overline{d}(p))T(r, f) &= T\left(r, \frac{1}{(f^n(z + c) - f^n(z))f^{n\overline{d}(p)}}\right) + S(r, f) \\
&\leq T\left(r, \frac{\{H[f]\}^n}{f^{n\overline{d}(p)}}\right) + S(r, f) \\
&\leq nT\left(r, \frac{H[f]}{f^{\overline{d}(p)}}\right) + S(r, f) \\
&\leq n\left[m\left(r, \frac{H[f]}{f^{\overline{d}(p)}}\right) + N\left(r, \frac{H[f]}{f^{\overline{d}(p)}}\right)\right] + S(r, f) \\
&\leq n\left[((\overline{d}(p) - \underline{d}(p))T(r, f)\right] + n\sigma\left[\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + S(r, f) \\
(2n + n\overline{d}(p))T(r, f) &\leq n\left[((\overline{d}(p) - \underline{d}(p))T(r, f)\right] + S(r, f) \\
(2n + n\underline{d}(p))T(r, f) &\leq S(r, f),
\end{aligned}
$$

which is contradiction. In a similar way we can prove the following Lemma.   □

LEMMA 2.11. *Let f be a non-constant meromorphic function and n be a positive integer, if $F = f^n(qz + c) - f^n(qz)$ where c is a finite complex number and $G = \{H[f]\}^n$, where $H[f]$ is a differential polynomial, then $FG \neq 1$.*

3. Proof of Theorems

Proof of Theorem 1.11.

PROOF.
$$
F = f^n(z + c) - f^n(z), G = \{H[f]\}^n. \tag{3.1}
$$

Case 3.1. Let $\{H[f]\}^n$ and $\Delta_c F$ share $(1, 2), (\infty, 0)$ and $n \geq \sigma + \overline{d}(p) + 5$. Then it follows directly from the assumption of the theorem that F and G share $(1, 2), (\infty, 0)$. Let Ω be defined as above. Suppose that $\Omega \not\equiv 0$. It follows from Lemma 2.1 that

$$
T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G). \tag{3.2}
$$

According to Lemma 2.2 and Lemma 2.3 we have

$$
T(r, F) = nT(r, f(z + c)) + nT(r, f(z)) + S(r, f) = 2nT(r, f) + O(r^{\rho(f) - 1 + \epsilon}) + S(r, f). \tag{3.3}
$$

It is obvious that

$$
\begin{aligned}
N_2\left(r, \frac{1}{F}\right) &= 2\overline{N}\left(r, \frac{1}{f^n(z + c) - f^n(z)}\right) \\
&\leq 4T(r, f) + O(r^{\rho(f) - 1 + \epsilon}) + S(r, f).
\end{aligned} \tag{3.4}
$$

$$\overline{N}(r, F) = \overline{N}(r, f^n(z + c) - f^n(z))$$
$$\leq 2T(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f). \tag{3.5}$$

$$\overline{N}_*(r, \infty; F, G) \leq \overline{N}(r, F) \leq 2T(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f). \tag{3.6}$$

Since $\overline{E}(\infty, f^{(k)}) = \overline{E}(\infty, f)$, we have

$$\overline{N}(r, G) = \overline{N}(r, H[f]) = \overline{N}(r, f). \tag{3.7}$$

From Lemma 2.9 we have ,

$$N_2\left(r, \frac{1}{G}\right) = 2\overline{N}\left(r, \frac{1}{H[f]}\right) \leq 2T(r, H[f]) + S(r, f)$$
$$\leq 2(\overline{d}(p) + \sigma)T(r, f) + S(r, f). \tag{3.8}$$

By combining (3.2) to (3.8), we deduce,

$$(2n - 2\sigma - 2\overline{d}(p) - 9)T(r, f) \leq O(r^{\rho(f)-1+\epsilon}) + S(r, f), \tag{3.9}$$

which contradicts that $n \geq \frac{1}{2}(2\sigma + 2\overline{d}(p) + 10) \geq \sigma + \overline{d}(p) + 5$.
Thus, we have $\Omega \equiv 0$ and hence,

$$\left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) = \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right).$$

By integrating twice, we get

$$\frac{1}{F - 1} = \frac{A}{G - 1} + B. \tag{3.10}$$

where $A \neq 0$ and B are constants, From (3.10) we have,

$$G = \frac{(B - A)F + (A - B - 1)}{BF - (B + 1)} \tag{3.11}$$

Now, we have the following three subcases:
Subcase 3.1.1. Suppose that $B \neq 0, -1$. Then from (3.11), we have,

$$\overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) = \overline{N}(r, G). \tag{3.12}$$

From the Second Fundamental Theorem, Lemma 2.3 and (3.7), we have,

$$2nT(r, f) = T(r, F) + S(r, f)$$
$$\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) + S(r, f) \tag{3.13}$$
$$\leq (4 + \overline{d}(p) + \sigma)T(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f),$$

which contradicts $n \geq \sigma + \overline{d}(p) + 5$.

Subcase 3.1.2 Suppose that $B = -1$. From (3.11) we have

$$G = \frac{(A+1)F - A}{F} \tag{3.14}$$

i) If $A \neq -1$, we obtain from (3.14), we get,

$$\overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{3.15}$$

From the Second Fundamental Theorem, Lemma 2.3 and Lemma 2.9, we get

$$2nT(r, f) = T(r, F) + S(r, f) \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) + S(r, f),$$

$$\leq \overline{N}(r, \Delta_c F) + \overline{N}\left(r, \frac{1}{\Delta_c F}\right) + \overline{N}\left(r, \frac{1}{H[f]}\right) + S(r, f), \tag{3.16}$$

$$2nT(r, f) \leq (4 + \overline{d}(p) + \sigma)T(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f),$$

which contradicts $n \geq \sigma + \overline{d}(p) + 5$.

ii) If $A = -1$ and from (3.14), we get $FG = 1$,

$$[f^n(z+c) - f^n(z)][\{H[f]\}^n] = 1. \tag{3.17}$$

which is a contradiction from Lemma 2.10.

Subcase 3.1.3. Suppose that $B = 0$. From (3.11)

$$G = AF - (A - 1) \tag{3.18}$$

If $A \neq 1$, from (3.18) we obtain

$$\overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) = \overline{N}\left(r, \frac{1}{G}\right) \tag{3.19}$$

Then from the Second Fundamental Theorem, Lemma 2.3 and Lemma 2.9 we have

$$2nT(r, f) = T(r, F) + S(r, f)$$

$$\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) + S(r, f)$$

$$\leq \overline{N}(r, \Delta_c F) + \overline{N}\left(r, \frac{1}{\Delta_c F}\right) + \overline{N}\left(r, \frac{1}{H[f]}\right) + S(r, f), \tag{3.20}$$

$$2nT(r, f) \leq (4 + \overline{d}(p) + \sigma)T(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f),$$

which contradicts $n \geq \sigma + \overline{d}(p) + 5$.

Hence $A = 1$. From (3.18), we have $F = G$, i.e.,

$$[f^n(z+c) - f^n(z)] = [\{H[f]\}^n].$$

Hence $H[f] = tf(z + c)$, for a constant t, such that $t^n = \frac{1}{2}$.

This proves Case 3.1 of Theorem 1.11.

Case 3.2. Suppose $\{H[f]\}^n$ and $\Delta_c F$ share $(1, 2), (\infty, \infty)$ and $n \geq \sigma + \overline{d}(p) + 4$. Then it follows directly from the assumption of the theorem that F and G share $(1, 2)$ and (∞, ∞). Let Ω be defined as above. Suppose that $\Omega \not\equiv 0$. It follows from Lemma 2.1 we have,

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G). \tag{3.21}$$

It is obvious that

$$\overline{N}_*(r, \infty; F, G) = 0 \tag{3.22}$$

By combining (3.21), (3.22), (3.3), (3.4), (3.5), (3.7) and (3.8), we have,

$$(2n - 2\sigma - 2\overline{d}(p) - 7)T(r, f) \leq O(r^{\rho(f)-1+\epsilon}) + S(r, f), \tag{3.23}$$

which contradicts that $n \geq \sigma + \overline{d}(p) + 4$. Therefore $\Omega \equiv 0$. Similar to the proof in Case 3.1, we can get the conclusion of Theorem 1.11.

Case 3.3. Suppose $\{H[f]\}^n$ and $\Delta_c F$ share $(1, 0), (\infty, 0)$ and $n \geq \frac{1}{2}(3\overline{d}(p) + 3\sigma + 20)$. Then it follows directly from the assumption of the theorem that F and G share $(1, 0), (\infty, 0)$. Let Ω be defined as above. Suppose that $\Omega \not\equiv 0$. It follows from Lemma 2.6, we have,

$$\begin{aligned} T(r, F) + T(r, G) \leq &\ 3\overline{N}(r, F) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_E^{1)}\left(r, \frac{1}{F-1}\right) \\ &+ 2N_E^{(2}\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right) \\ &+ S(r, F) + S(r, G). \end{aligned} \tag{3.24}$$

Since,

$$\begin{aligned} N_E^{1)}\left(r, \frac{1}{F-1}\right) + 2N_E^{(2}\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{F-1}\right) + 2N_L\left(r, \frac{1}{G-1}\right) \\ \leq N\left(r, \frac{1}{G-1}\right) \leq T(r, G) + O(1). \end{aligned} \tag{3.25}$$

From (3.24) and (3.25) we get,

$$\begin{aligned} T(r, F) \leq &\ 3\overline{N}(r, F) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 2N_L\left(r, \frac{1}{F-1}\right) \\ &+ N_L\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G). \end{aligned} \tag{3.26}$$

According to Lemma 2.2 and Lemma 2.3,

$$T(r, F) = nT(r, f(z + c)) + nT(r, f(z)) + S(r, f) = 2nT(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f). \tag{3.27}$$

It is obvious that

$$N_2\left(r, \frac{1}{F}\right) = 2\overline{N}\left(r, \frac{1}{f^n(z+c) - f^n(z)}\right) \tag{3.28}$$
$$\leq 4T(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f).$$

$$\overline{N}(r, F) = \overline{N}(r, f^n(z+c) - f^n(z)) \tag{3.29}$$
$$\leq 2T(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f).$$

$$N_L\left(r, \frac{1}{F-1}\right) \leq N\left(r, \frac{F}{F'}\right) \leq N\left(r, \frac{F'}{F}\right) + S(r, f)$$

$$\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f)$$

$$\leq \overline{N}(r, \Delta_c F) + \overline{N}\left(r, \frac{1}{\Delta_c F}\right) + S(r, f) \tag{3.30}$$

$$\leq 4T(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f).$$

$$N_2\left(r, \frac{1}{G}\right) = 2\overline{N}\left(r, \frac{1}{H[f]}\right) \leq 2T(r, H[f]) + S(r, f) \tag{3.31}$$
$$\leq 2(\overline{d}(p) + \sigma)T(r, f) + S(r, f).$$

$$N_L\left(r, \frac{1}{G-1}\right) \leq N\left(r, \frac{G}{G'}\right) \leq N\left(r, \frac{G'}{G}\right) + S(r, f)$$

$$\leq \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) + S(r, f)$$

$$\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{H[f]}\right) + S(r, f) \tag{3.32}$$

$$\leq (\overline{d}(p) + \sigma + 1)T(r, f) + S(r, f).$$

By combining (3.26)-(3.32), we deduce

$$(2n - 3\overline{d}(p) - 3\sigma - 19)T(r.f) \leq O(r^{\rho(f)-1+\epsilon}) + S(r, f), \tag{3.33}$$

which contradicts with $n \geq \frac{1}{2}(3\overline{d}(p) + 3\sigma + 20)$.

Now by following the steps of the proof in Case 3.1, we can get the conclusion of Theorem 1.11. This completes the proof of Theorem 1.11.   □

Proof of Theorem 1.12.

Proof. Let us consider,

$$\Delta_c F = f^n(qz+c) - f^n(qz), G = \{H[f]\}^n. \tag{3.34}$$

then $[H(f)] = tf(qz)$, for a constant t that satisfies $t^n = \frac{1}{2}$.

In a similar manner to the proof of Theorem 1.11, we will get the proof of Theorem 1.12.   □

References

[1] Al-Khaladi and H. H Amer,On meromorphic functions that share one value with their derivative, *Analysis 25*, **(2)** (2005),131-140.

[2] A. Banerjee, Uniqueness of Meromorphic Functions That Share Two Sets, *Southeast Asian Bulletin of Mathematics*, **31(1)**, (2007).

[3] S. S. Bhoosnurmath and A. J. Patil, On the Growth And Value Distribution Of Meromorphic Functions and their Differential Polynomials,*Journal of the Indian Mathematical Society*, **74(3)**, (2007), 167.

[4] D. C. Barnett, R. G. Halburd, W. Morgan and R. J. Korhonen, Nevanlinna theory for the-difference equations, *Proceedings of the Royal Society of Edinburgh Section A Mathematics*, **137(3)**, (2007), 457-474.

[5] Z. Chen, On the difference counterpart of bruck conjecture, *Acta Mathematica Scientia*, **34(3)**, (2014), 653-659.

[6] Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of f(z+ c) and difference equations in the complex plane, *The Ramanujan Journal*, **16(1)**, (2008), 105-129.

[7] R. G. Halburd and Korhonen, Nevanlinna theory for the difference operator, *Annales Academiae Scientiarum Fennicae Mathematica*, **31(2)**, (2006), 463-478.

[8] W. K. Hayman, Meromorphic functions,*Clarendon Press*,Oxford,**(78)**,1964.

[9] J. K. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, and J. Zhang, Value sharing results for shifts of meromorphic functions and sufficient conditions for periodicity, *Journal of Mathematical Analysis and Applications*, **355(1)**, (2009), 352-363.

[10] C. S. Kumar, S. Rajeshwari and T. Bhuvaneshwari, Results on unicity of meromorphic function with its shift and q-difference,*J. Math. Comput. Sci*, **12**, (2021).

[11] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, *Nagoya Mathematical Journal*, **161**, (2001), 193-206.

[12] I. Lahiri and A. Sarkar, Uniqueness of a meromorphic function and its derivative,*J. Inequal. Pure Appl. Math*, **5(1)**, (2004).

[13] I. Lahiri, Value distribution of certain differential polynomials, *International Journal of Mathematics and Mathematical Sciences*, **28(2)**, (2001), 83-91.

[14] C. Meng and G. Liu, Uniqueness of meromorphic functions concerning the shifts and derivatives, *Journal of applied mathematics and informatics*, **37(12)**, (2019), 133-148.

[15] B. N. Naveenkumar and H. P. Waghamore, Some Results on Uniqueness of Meromorphic Functions Regarding Shifts and Derivatives, *International Journal of Difference Equations (IJDE)*, **18(1)**, (2023), 251-265.

[16] L. Ping, Entire functions that share one value with their linear differential polynomials, *Kodai Mathematical Journal*, **22(3)**, (1999), 446-457.

[17] X. Qi, N. Li and L. Yang, Uniqueness of meromorphic functions concerning their differences and solutions of difference Painlevé equations, *Computational Methods and Function Theory*, **18(4)**, (2018), 567-582.

[18] L. A. Rubel and C. C. Yang, Values shared by an entire function and its derivative, *In Complex analysis,*, Springer, Berlin, Heidelberg, (1977), 101-103.

[19] L. Sheng and Z. Gao, Entire functions sharing one or two finite values CM with their shifts or difference operators, *Archiv der Mathematik*, **97(5)**, (2011), 475-483.

[20] A. C. Thamir and X. Yi. Hong, Weighted value sharing and a question of I. Lahiri,Complex Variables theory and Application, *An International Journal*, **49(15)**, (2004), 1063-1078.

[21] H. P. Waghamore and P. N. Raj, Uniqueness results on meromorphic functions concerning their shift and differential polynomial, *Serdica Math. J* , **47**, (2021), 191-212.

[22]   H. P. Waghamore and P. N. Raj, Uniqueness of meromorphic functions with finite logarithmic order regarding their q-shift difference and differential polynomial, *Journal of Fractional Calculus and Applications*, **13(2)**, (2022), 89-99.

[23]   C. C. Yang, On deficiencies of differential polynomials II, *Mathematische Zeitschrift*, **125(2)**, (1972), 107-112.

[24]   C. C. Yang and H. X. Yi, Uniqueness theory of meromorphic functions ser, *Mathematics and its Applications, Dordrecht*,Kluwer Academic Publishers Group, **557**, (2003).

[25]   L. Yang, Value Distribution Theory Springer, (1993).

[26]   J. Zhang, Meromorphic functions sharing a small function with their differential polynomials, *Kyungpook Mathematical Journal*, **50(3)**, (2010), 345-355.

[27]   J. Zhang and R. Korhonen, On the Nevanlinna characteristic of f(qz) and its applications, *Journal of Mathematical Analysis and Applications*, **369(2)**, (2010), 537-544.

Harina P. Waghamore, Department of Mathematics, Jnanabharathi Campus, Bangalore University,Bengaluru 560056,India
e-mail: harinapw@gmail.com

Megha M. Manakame, Department of Mathematics, Jnanabharathi Campus, Bangalore University,Bengaluru 560056,India
e-mail: megha.manakame80@gmail.com