HANKEL DETERMINANTS FOR STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRICAL POINTS RELATED TO NEPHROID AND CARDIOID DOMAINS

SAMBASIVA RAO S. [✉] **and R. BHARAVI SHARMA**

Abstract

The aim of this paper is to estimate an upper bound for third Hankel determinants for certain analytic functions with respect to symmetrical points associated with nephroid and cardioid domains. Some results concerning third Hankel determinant proved in this paper improve the existing bounds available in the literature. Examples to illustrate sharpness of certain results are provided.

1. Introduction

1.1. Preliminary Definitions Let $\mathcal{A}$ be the family of analytic functions f defined on the open unit disk Δ in the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$ and $f'(0) = 1$. The collection of univalent functions $f \in \mathcal{A}$ will be denoted by $\mathcal{S}$. The well known classes of starlike, convex and bounded turning functions are respectively denoted by $\mathcal{S}^*$, C and $\mathcal{R}$, are subclasses of $\mathcal{S}$. Let B_o be the family of analytic functions w in Δ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$. The members of B_o are called Schwarz functions. The functions z, z^2 are well known members of the class B_0. A function $f \in \mathcal{A}$ is said to subordinate to $g \in \mathcal{A}$ if there exists a $w \in B_0$ such that $f(z) = g(w(z))$ for all $z \in \Delta$. In this case, we write $f \prec g$. If g is univalent, then $f \prec g$ if and only if $g(0) = f(0)$ and $f(\Delta) \subset g(\Delta)$. For a function $f \in \mathcal{A}$ with Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ for all } z \in \Delta, \tag{1.1}$$

the qth Hankel determinant of index $n \geq 1$ for $f \in \mathcal{A}$, will be denoted by $H_{q,n}(f)$ (or simply $H_q(n)$) and is defined as

$$
H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix} \tag{1.2}
$$

for $q \geq 2$ with $a_1 = 1$ (see [2], [3]). In particular, $H_2(1) = a_3 - a_2^2$, $H_2(2) = a_2 a_4 - a_3^2$ and

$$
H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \tag{1.3}
$$

are first, second and third Hankel determinants respectively.

1.2. Literature Review and Motivation   The notion of Hankel determinants were initially studied by D.G Cantor to characterize certain meromorphic functions to be of bounded characteristic in Δ (see [4]). These results of Cantor have been extended to meromorphic univalent functions in Δ by Ch. Pommerenke(see [3]). Further, C. Pommerenke studied Hankel determinants for functions in class $\mathcal{S}$ (see [3]) and W.K. Hayman investigated $H_2(2)$ for mean univalent functions (see [2] and [22]). Subsequently, several researchers investigated $H_2(2)$ and $H_3(1)$ for various subclasses of $\mathcal{S}$ (for instance, see [6],[8],[12],[22] and references therein). Recently, Lateef Ahmad Wani and Swaminathan introduced and studied the classes $S_{Ne}^* = \mathcal{S}^*(\varphi_{Ne}) = \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_{Ne}(z)\}$ and $C_{Ne} = C(\varphi_{Ne}) = \{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi_{Ne}(z)\}$, where $\varphi_{Ne}(z) = 1 + z - \frac{z^3}{3}$ is analytic and univalent in Δ such that $\varphi_{Ne}(0) = 1$, $\varphi_{Ne}'(0) = 1$ and φ_{Ne} maps Δ onto the interior of the nephroid domain in the right half plane (see [9]). The function $\varphi_C(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$ for all $z \in \Delta$ is analytic and univalent in Δ such that $\varphi_C(0) = 1$, $\varphi_C'(0) = \frac{4}{3}$ and φ_C maps Δ onto the interior of cardioid domain in the right half plane. The classes $\mathcal{S}^*(\varphi_C)$, $C(\varphi_C)$ associated with φ_C have been introduced and studied by Sharma et al. (see [18]). In [7], Sakaguchi introduced the concept of starlike functions with respect to symmetrical points and shown that such functions f are in $\mathcal{S}$ if and only if $\Re\{\frac{zf'(z)}{f(z)-f(-z)}\} > 0$. In [17], Das and Singh studied the properties of functions convex with respect to symmetric points. R. Bharavi Sharma et al. estimated an upper bound of $|H_3(1)|$ for starlike and convex functions with respect to symmetric points related to shell-shaped region, exponential function and k-Fibonacci sequence (see [5],[13],[16]). The classes $\mathcal{S}_s^*(\varphi)$ and $C_s(\varphi)$ defined as $\mathcal{S}_s^*(\varphi) = \{f \in \mathcal{S} : \frac{2zf'(z)}{f(z)-f(-z)} \prec \varphi(z)\}$ and $C_s(\varphi) = \{f \in \mathcal{S} : \frac{2(zf'(z))'}{f'(z)+f'(-z)} \prec \varphi(z)\}$, where φ is a Ma-Minda type function were introduced and studied by V. Ravichandran (see [21]). Motivated by the above mentioned research work, the objective of our present work is to compute an upper bound to $|H_3(1)|$ for certain analytic functions associated with nephroid and cardioid domains.

A set of useful lemmas are presented in section 2. The bounds of initial coefficients and an upper bound of Hankel determinants for the functions in $S_s^*(\varphi_{Ne})$ and in $S_s^*(\varphi_C)$ are presented in section 3 and section 4 respectively. Section 5 is dedicated to the results concerning improvement of an upper bound of $|H_3(1)|$ for the functions in S_s^* followed by concluding remarks and scope of further research in last section.

2. A Set of Useful Lemmas

The collection of analytic functions p in Δ with $p(0) = 1$ and $\Re\{p(z)\} > 0$ is called the class of functions with positive real part, and it will be denoted by $\mathcal{P}$. The Taylor series expansion of $p \in \mathcal{P}$ is of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \ \text{ for all } z \in \Delta. \tag{2.1}$$

Unless otherwise stated throughout this paper we assume the series expansion of $p \in \mathcal{P}$ is of the form (2.1) and of the function f in the subclasses of $\mathcal{S}$ considered in this paper is of the form (1.1).

LEMMA 2.1. *([14]) Let $p \in \mathcal{P}$. Then $\mid c_n \mid \leq 2$ for any positive integer n. This inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$.*

LEMMA 2.2. *([10]) Let $p \in \mathcal{P}$ and $\mu \in \mathbb{C}$. Then $|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}$. This inequality is sharp for the functions $p(z) = \frac{1+z}{1-z}$ and $p(z) = \frac{1+z^2}{1-z^2}$.*

LEMMA 2.3. *([11]) Let $p \in \mathcal{P}$. Then for any real numbers J, K and L,*
$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|.$

LEMMA 2.4. *([15]) Let $p \in \mathcal{P}$. Then for positive integers n, m,*

$$|\mu c_n c_m - c_{n+m}| \leq \begin{cases} 2 & \text{if } 0 \leq \mu \leq 1 \\ 2|2\mu - 1| & \text{otherwise.} \end{cases}$$

This inequality is sharp.

DEFINITION 2.5. Let $f \in \mathcal{A}$. Then we say that

1. f is in S_s^* if $\Re\{\frac{2zf'(z)}{f(z)-f(-z)}\} > 0$ for all $z \in \Delta$.
2. f is in $S_s^*(\delta)$ if $\Re\{\frac{2zf'(z)}{f(z)-f(-z)}\} > \delta$ for all $z \in \Delta$, where $\delta \geq 0$.
3. f is in $S_s^*(\varphi_{Ne})$ if $\frac{2zf'(z)}{f(z)-f(-z)} \prec \varphi_{Ne}(z) = 1 + z - \frac{z^3}{3}$ for all $z \in \Delta$.
4. f is in $S_s^*(\varphi_C)$ if $\frac{2zf'(z)}{f(z)-f(-z)} \prec \varphi_C(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^3$ for all $z \in \Delta$.

It is clear that S_s^*, $S_s^*(\delta)$, $S_s^*(\varphi_{Ne})$ and $S_s^*(\varphi_C)$ are subclasses of $\mathcal{S}$.

3. Initial coefficient Bounds, Hankel determinants of the functions f in $\mathcal{S}_s(\varphi_{Ne})$

Let $f \in S_s^*(\varphi_{Ne})$. Then there exists $w \in B_0$ such that $\frac{2zf'(z)}{f(z)-f(-z)} = \varphi_{Ne}(w(z))$ for all $z \in \Delta$. If we define $p(z) = \frac{1+w(z)}{1-w(z)}$ for all $z \in \Delta$, then $p(z) \in \mathcal{P}$ such that

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + \Big(\frac{p(z)-1}{p(z)+1}\Big) - \frac{1}{3}\Big(\frac{p(z)-1}{p(z)+1}\Big)^3 \text{ for all } z \in \Delta. \tag{3.1}$$

But,

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + 2a_2z + 2a_3z^2 + (4a_4 - 2a_2a_3)z^3 + (4a_5 - 2a_3^2)z^4 + \ldots \tag{3.2}$$

and

$$1 + \Big(\frac{p(z)-1}{p(z)+1}\Big) - \frac{1}{3}\Big(\frac{p(z)-1}{p(z)+1}\Big)^3 = 1 + \frac{c_1}{2}z + \Big(\frac{c_2}{2} - \frac{c_1^2}{4}\Big)z^2 +$$
$$\Big(\frac{c_1^3}{12} - \frac{c_1c_2}{2} + \frac{c_3}{2}\Big)z^3 + \Big(\frac{c_4}{2} - \frac{c_1c_3}{2} - \frac{c_2^2}{4} + \frac{c_1^2c_2}{4}\Big)z^4 + \ldots \tag{3.3}$$

Thus,

$$a_2 = \frac{c_1}{4}, \tag{3.4}$$

$$a_3 = \frac{1}{4}\Big(c_2 - \frac{c_1^2}{2}\Big), \tag{3.5}$$

$$a_4 = \frac{1}{192}(c_1^3 - 18c_1c_2 + 24c_3), \tag{3.6}$$

$$a_5 = \frac{1}{4}\Big(\frac{1}{2}(c_4 - c_1c_3) + \frac{c_1^4}{32} - \frac{1}{8}c_2(c_2 - c_1^2)\Big). \tag{3.7}$$

THEOREM 3.1. *Let $f \in S_s^*(\varphi_{Ne})$. Then $|a_2| \leq \frac{1}{2}, |a_3| \leq \frac{1}{2}, |a_4| \leq \frac{1}{4}$ and $|a_5| \leq \frac{1}{2}$.*

PROOF. The first three inequalities follow by taking modulus on both sides of Equations (3.4), (3.5) and (3.6) followed by applying Lemmas (2.1), (2.2) and (2.3) respectively, whereas using Lemmas (2.4), (2.1) and (2.2), we get

$$|a_5| = \frac{1}{4}\Big|\frac{1}{2}(c_4 - c_1c_3) + \frac{c_1^4}{32} - \frac{1}{8}c_2(c_2 - c_1^2)\Big|$$
$$\leq \frac{1}{4}\Big(\frac{1}{2}|c_4 - c_1c_3| + \frac{1}{8}|c_2||c_2 - c_1^2| + \frac{|c_1^4|}{32}\Big) \tag{3.8}$$
$$\leq \frac{1}{2}.$$

$\square$

EXAMPLE 3.2. *If we choose $p(z) = \frac{1+z}{1-z}$, $p(z) = \frac{1+z^2}{1-z^2}$ and $p(z) = \frac{1+z^3}{1-z^3}$ in Equation (3.1) respectively, then we get the functions $f_1(z) = z + \frac{z^2}{2} - \frac{z^4}{12} + \ldots, f_2(z) = z + \frac{z^3}{2} + \frac{z^5}{8} + \ldots$ and $f_3(z) = z + \frac{z^4}{4} + \ldots$, which are in $S_s^*(\varphi_{Ne})$ and acts as extremal function for coefficient bounds of $|a_2|, |a_3|$ and $|a_4|$ proved in Theorem (3.1) respectively.*

We now establish results concerning $H_2(1)$ and $H_2(2)$ for $f \in S_s^*(\varphi_{Ne})$.

THEOREM 3.3. *Let* $f \in S_s^*(\varphi_{Ne})$. *Then for any* $\mu \in \mathbb{C}$, $|a_3 - \mu a_2^2| \leq \frac{1}{2} \max\{1, \frac{|\mu|}{2}\}$. *In particular,* $|H_2(1)| = |a_3 - a_2^2| \leq \frac{1}{2}$ *and the sharpness is attained.*

PROOF. The required inequality follows by applying Lemma (2.2) to right hand side of $\left|a_3 - \mu a_2^2\right| = \frac{1}{4}\left|c_2 - \frac{(2+\mu)}{4}c_1^2\right|$ for any $\mu \in \mathbb{C}$ where a_2, a_3 are taken from Equations (3.4) and (3.5). If we choose $\mu = 1$, then $|H_2(1)| \leq \frac{1}{2}$. The function $f_2(2)$ as in Example (3.2) is an extremal function for these inequalities. □

THEOREM 3.4. *Let* $f \in S_s^*(\varphi_{Ne})$. *Then* $|H_2(2)| \leq \frac{3}{8}$.

PROOF. From Equations (3.4), (3.5) and (3.6), we have $\left|H_2(2)\right| \leq \frac{1}{768}\left(\left|c_1\right|\left|11c_1^3 - 30c_1c_2 + 24c_3\right| + 48\left|c_1c_3 - c_2^2\right|\right)$. But, in view of Lemmas (2.1), (2.3) and (2.4), $|c_1| \leq 2$, $|11c_1^3 - 30c_1c_2 + 24c_3| \leq 48$ and $|c_1c_3 - c_2^2| \leq 4$ respectively. Hence, $|H_2(2)| \leq \frac{3}{8}$. □

We now prove the main result of this section.

THEOREM 3.5. *Let* $f \in S_s^*(\varphi_{Ne})$. *Then* $|H_3(1)| \leq \frac{7}{16}$.

PROOF. On substituting the coefficients a_2 to a_5 from equations (3.4) to (3.7) in the definition of $H_3(1)$ followed by grouping the terms, we get

$$
\begin{aligned}
|H_3(1)| &= \left|\frac{1}{36864}\left(1152(c_4 - \frac{3}{4}c_2^2)(c_2 - \frac{3}{4}c_1^2) - 72c_1(c_2 - \frac{3}{4}c_1^2)(c_1^3 - 12c_1c_2 + 16c_3)\right.\right. \\
&\quad \left.\left. - (7c_1^3 - 30c_1c_2 + 24c_3)^2\right)\right| \\
&\leq \frac{1}{36864}\left(1152\left|c_4 - \frac{3}{4}c_2^2\right|\left|c_2 - \frac{3}{4}c_1^2\right| + 72|c_1|\left|c_2 - \frac{3}{4}c_1^2\right|\left|c_1^3 - 12c_1c_2 + 16c_3\right|\right. \\
&\quad \left. + \left|7c_1^3 - 30c_1c_2 + 124c_3\right|^2\right).
\end{aligned}
$$

But, $|c_1| \leq 2$, $|c_4 - \frac{3}{4}c_2^2| \leq 2$, $|c_2 - \frac{3}{4}c_1^2| \leq 2$, $|c_1^3 - 12c_1c_2 + 16c_3| \leq 32$ and $|7c_1^3 - 30c_1c_2 + 24c_3| \leq 48$ in view of Lemmas (2.1), (2.2), (2.3) and (2.4). Hence, $|H_3(1)| \leq \frac{7}{16}$. □

REMARK 3.6. *The function* $f_2(z) = z + \frac{z^3}{2} + \frac{z^5}{8} + \ldots \in S_s^*(\varphi_{Ne})$ *suggests that one can still improve an upper bound (up to sharpness) of* $|H_2(2)|$ *and of* $|H_3(1)|$ *to* $\frac{1}{4}$ *and* $\frac{1}{16}$ *respectively.*

4. Initial coefficient Bounds, Hankel determinants of the functions f in $\mathcal{S}^*_s(\varphi_C)$

Let $f \in S^*_s(\varphi_C)$. Then there exists $w \in B_0$ such that $\frac{2zf'(z)}{f(z)-f(-z)} = \varphi_C(w(z))$ for all $z \in \Delta$. If we take $p(z) = \frac{1+w(z)}{1-w(z)}$ for all $z \in \Delta$, then $p(z) \in \mathcal{P}$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \varphi_C\Big(\frac{p(z) - 1}{p(z) + 1}\Big) \text{ for all } z \in \Delta. \tag{4.1}$$

On substituting the series expansions of $f(z)$ and $p(z)$ in Equation (4.1) and comparing like coefficients yield

$$a_2 = \frac{c_1}{3}, \tag{4.2}$$

$$a_3 = \frac{1}{3}\Big(c_2 - \frac{c_1^2}{4}\Big), \tag{4.3}$$

$$a_4 = \frac{1}{4}\Big(\frac{2}{3}c_3 - \frac{c_1 c_2}{3} + \frac{2}{9}c_1(c_2 - \frac{c_1^2}{4})\Big), \tag{4.4}$$

$$a_5 = \frac{1}{4}\Big(\frac{2}{3}\big(c_4 - \frac{c_1 c_3}{2}\big) - \frac{c_1^2}{9}\big(c_2 - \frac{1}{2}c_1^2\big) + \frac{1}{18}c_2^2\Big). \tag{4.5}$$

THEOREM 4.1. *Let* $f \in S^*_s(\varphi_C)$. *Then* $|a_2| \le \frac{2}{3}, |a_3| \le \frac{2}{3}, |a_4| \le \frac{7}{18}$ *and* $|a_5| \le \frac{4}{9}$.

PROOF. Bounds of $|a_2|$ and $|a_3|$ follow by applying Lemmas (2.1) and (2.2) to the absolute values of a_2 and a_3 as in the Equations (4.2) and (4.3) respectively, whereas using Lemmas (2.3), we get

$$\begin{aligned} |a_4| &= \frac{1}{4}\Big|\frac{2}{3}c_3 - \frac{c_1 c_2}{9} - \frac{c_1^3}{18}\Big| \\ &= \frac{1}{72}\Big|c_1^3 + 2c_1 c_2 - 12c_3\Big| \\ &\le \frac{7}{18}. \end{aligned} \tag{4.6}$$

On similar lines, one can show that $|a_5| \le \frac{4}{9}$. $\qquad\qquad\square$

EXAMPLE 4.2. *If we choose* $p(z) = \frac{1+z}{1-z}$ *and* $p(z) = \frac{1+z^2}{1-z^2}$ *in Equation (4.1), then we get the functions* $f_1(z) = z + \frac{2}{3}z^2 + \frac{1}{3}z^3 + \frac{1}{9}z^4 + \frac{1}{18}z^5 + \ldots$, $f_2(z) = z + \frac{2}{3}z^3 + \frac{2}{9}z^5 + \ldots$, *which are in* $S^*_s(\varphi_C)$ *and acts as an extremal function for coefficient bounds of* $|a_2|, |a_3|$ *proved in Theorem (4.1) respectively.*

THEOREM 4.3. *Let* $f \in S^*_s(\varphi_C)$. *Then for any* $\mu \in \mathbb{C}$, $|a_3 - \mu a_2^2| \le \frac{2}{3}\max\{1, \big|\frac{4\mu-3}{6}\big|\}$. *In particular,* $|H_2(1)| = |a_3 - a_2^2| \le \frac{2}{3}$ *and the sharpness is attained.*

PROOF. From Equations (4.2) and (4.3), we have $\big|a_3 - \mu a_2^2\big| = \frac{1}{3}\big|c_2 - \frac{(3+4\mu)}{12}c_1^2\big|$ for any $\mu \in \mathbb{C}$. Applying Lemma (2.2) to above equation yield the required inequality. If we choose $\mu = 1$, then $|H_2(1)| \le \frac{2}{3}$. The function $f_2(2)$ as in Example (4.2) is an extremal function for these inequalities. $\qquad\square$

Theorem 4.4. *Let* $f \in S_s^*(\varphi_C)$. *Then* $|H_2(2)| \leq \frac{2}{3}$

Proof. From Equations (4.2), (4.3) and (4.4), we have
$|H_2(2)| = \left| \frac{-c_1}{432}(5c_1^3 - 20c_1c_2 + 24c_3) + \frac{1}{9}(c_1c_3 - c_2^2) \right|$. Now, the result follow by applying Triangle inequality and using the fact that $|c_1| \leq 2$, $|5c_1^3 - 20c_1c_2 + 24c_3| \leq 48$ and $|c_1c_3 - c_2^2| \leq 4$. $\qquad\square$

Theorem 4.5. *Let* $f \in S_s^*(\varphi_C)$. *Then* $|H_3(1)| \leq \frac{17}{27}$.

Proof. On substituting the coefficients a_2 to a_5 from Equations (4.2) to (4.5) in the definition of $H_3(1)$ followed by grouping the terms, we get

$$
\begin{aligned}
|H_3(1)| = & \left| \frac{1}{18}(c_4 - \frac{7}{12}c_2^2)(c_2 - \frac{7}{12}c_1^2) - \frac{1}{36}c_1(c_3 - \frac{1}{3}c_1c_2)(c_2 - \frac{7}{12}c_1^2) \right. \\
& + \left. \frac{1}{432}c_1^4(c_2 - \frac{7}{12}c_1^2) - \frac{1}{5184}(c_1^3 - 10c_1c_2 + 12c_3)^2 \right| \\
\leq & \frac{1}{5184}\left(288\left|c_4 - \frac{7}{12}c_2^2\right|\left|c_2 - \frac{7}{12}c_1^2\right| + 144|c_1|\left|c_3 - \frac{1}{3}c_1c_2\right|\left|c_2 - \frac{7}{12}c_1^2\right| \right. \\
& + \left. 12|c_1^4|\left|c_2 - \frac{7}{12}c_1^2\right| + \left|c_1^3 - 10c_1c_2 + 12c_3\right|^2 \right).
\end{aligned}
$$

But, $|c_1| \leq 2$, $|c_4 - \frac{7}{12}c_2^2| \leq 2$, $|c_2 - \frac{7}{12}c_1^2| \leq 2$, $|c_3 - \frac{1}{3}c_1c_2| \leq 2$ and $|c_1^3 - 10c_1c_2 + 12c_3| \leq 24$ in view of Lemmas (2.1), (2.2), (2.3) and (2.4). This completes the proof. $\qquad\square$

Remark 4.6. *The function* $f_2(z) = z + \frac{2}{3}z^3 + \frac{2}{9}z^5 + \ldots \in S_s^*(\varphi_C)$ *suggests that one can still improve an upper bound (up to sharpness) of* $|H_2(2)|$ *and of* $|H_3(1)|$ *to* $\frac{4}{9}$ *and* $\frac{4}{27}$ *respectively.*

5. Improved bounds of $|H_{3,1}(f)|$ for $f \in S_s^*$

Remark 5.1. *The approach of estimating an upper bound of*

$$|H_3(1)| = |a_5a_3 - a_5a_2^2 + 2a_2a_3a_4 - a_3^3 - a_4^2| \tag{5.1}$$

by substituting coefficients a_i *for* $1 \leq i \leq 5$ *in terms of coefficients* c_i *for* $1 \leq i \leq 5$ *of function* $p \in \mathcal{P}$ *followed by grouping the terms, thereby applying preliminary lemmas stated earlier in section 2 provide us better bounds more often rather than finding in a standard way using the inequality* $|H_3(1)| \leq |a_5||H_2(1)| + |a_4||a_4 - a_2a_3| + |a_3||H_2(2)|$.

Theorem 5.2. *Let* $f \in S_s^*(\delta)$. *Then for* $0 \leq \delta < 2$, $a_2 = \frac{(1-\delta)c_1}{2}$, $a_3 = \frac{(1-\delta)c_2}{2-\delta}$, $a_4 = \frac{(1-\delta)(2-\delta)c_3 + (1-\delta)^2 c_1 c_2}{4(2-\delta)}$ *and* $a_5 = \frac{(1-\delta)(2-\delta)c_3 + (1-\delta)^2 c_2^2}{(2-\delta)(4-\delta)}$.

Proof. Refer Theorem 3.1 of ([19]). $\qquad\square$

Theorem 5.3. *Let* $f \in S_s^*(\delta)$. *Then* $|H_3(1)| \leq \frac{(1-\delta)^2(4-\delta)(6+\delta)}{4(4+\delta)(2-\delta)}$ *for* $0 \leq \delta < 2$.

PROOF. It is clear that for $0 \leq \delta < 2$, the inequalities $|c_2 - \frac{(2-\delta)(1-\delta)}{4}c_1^2| \leq 2$, $|c_4 - \frac{2(1-\delta)}{2-\delta}c_2^2| \leq 2$ and $|c_3 - \frac{(1-\delta)}{2(2-\delta)}c_1 c_2| \leq 2$ hold in view of Lemmas (2.2) and (2.4). Hence, on substituting the coefficients a_2 to a_5 in terms of c_i from Theorem (5.2) in Equation (5.1), thereby grouping the terms gives

$$
\begin{aligned}
|H_3(1)| &= \Big| \frac{(1-\delta)^2}{(2-\delta)(4+\delta)} \Big(c_4 - \frac{2(1-\delta)}{2-\delta}c_2^2 \Big)\Big(c_2 - \frac{(2-\delta)(1-\delta)}{4}c_1^2 \Big) \\
&\quad - \frac{(1-\delta)^2}{16}\Big(c_3 - \frac{(1-\delta)}{2(2-\delta)}c_1 c_2 \Big)^2 \Big| \\
&\leq \frac{(1-\delta)^2}{(2-\delta)(4+\delta)}\Big| c_4 - \frac{2(1-\delta)}{2-\delta}c_2^2 \Big\|c_2 - \frac{(2-\delta)(1-\delta)}{4}c_1^2 \Big| \\
&\quad + \frac{(1-\delta)^2}{16}\Big| c_3 - \frac{(1-\delta)}{2(2-\delta)}c_1 c_2 \Big|^2 \leq \frac{(1-\delta)^2(4-\delta)(6+\delta)}{4(4+\delta)(2-\delta)}.
\end{aligned}
$$

$\square$

COROLLARY 5.4. *If $f \in S_s^*$, then $|H_3(1)| \leq \frac{3}{4}$.*

PROOF. Choose $\delta = 0$ in the proof of Theorem (5.3). $\square$

REMARK 5.5. *Upper bound of $|H_3(1)|$ for $f \in S_s^*$ obtained in Corollary (5.4) is better than $|H_3(1)| \leq \frac{5}{2}$ the one proved by Mishra et al.(see Theorem 2.2 of [1]).*

6. Concluding Remarks and Scope of Further Research

In this paper, we estimated an upper bound of Hankel determinants for functions in $\mathcal{S}^*_s(\varphi_{Ne})$ and $\mathcal{S}^*_s(\varphi_C)$. Examples have been provided to illustrate the sharpness of certain results. Further, upper bound of $|H_3(1)|$ for $f(z) \in S_s^*$ obtained in this paper is better than the one existing in the literature. Finally, one can also attempt to find sharp bounds of $|H_2(2)|$ and $|H_3(1)|$ for the functions $f \in S_s^*(\varphi_{Ne})$ and $f \in S_s^*(\varphi_C)$ as pointed out in Remark (3.6) and Remark (4.6).

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SAMBASIVA RAO S., Assistant Professor of Mathematics, Department of Humanities & Sciences, SVS Group of Institutions, Hanumakonda, Telangana, India.
e-mail: ssrao.siginam@gmail.com

R. BHARAVI SHARMA, Department of Mathematics, Kakatiya University, Warangal, Telangana, India.
e-mail: rbsharma005@gmail.com