# A COMMON BEST PROXIMITY POINT THEOREM FOR SPECIAL GENERALIZED PROXIMAL WEAK $\beta$-QUASI CONTRACTIVE MAPPINGS 

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#### Abstract

In this paper, we obtain some common best proximity point results for a new class of non-self mappings $S, T: A \rightarrow B$ called special generalized proximal weak $\beta$-quasi contractive. Our results illustrated by an example. Several consequences are derived.


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## 1. Introduction

In 1922, Polish mathematician [1] proved a very important result regarding a contraction mapping, known as the Banach contraction principle. The main interesting studies deal with the extension of Banach's contraction to non-self-mappings $T: A \rightarrow B$, where $(A, B)$ is a pair of subsets of a metric space $(X, d)$. In fact such mappings do not necessarily have fixed points. The idea is to look for points where $d(x, T x)=d(A, B)$. This points are called best proximity points. In 1969, a best approximation theorem was introduced by Fan [3]. Sadish Bacha [4] proposed necessary and sufficient conditions for the existence of proximal contractions of first and second kind for such points. Several variants of non-self-contractions for the existence of a best proximity point were studied in [5-7].

In this paper, we introduced a new family of non-self-mappings called special generalized proximal weak $\beta$-quasi contractive mappings and we obtain some common best proximity point theorem. As an applications to the self-mapping, the present work generalizes several existing results on fixed point theory as the Banach contraction principle [1] and the generalization of such a principle by Ćirić in [2].

The paper is divided into five sections. Section 2 introduces the notion used herein, presents some definitions, and recalls some useful results. The best proximity point theorem with its proof is stated in Section 3. Finally, several consequences on the existence and uniqueness of best proximity points and fixed point results are given in Section 4.

## 2. Preliminaries and definitions

Let $(A, B)$ be a pair of nonempty subsets of a metric space $(X, d)$. Throughout this work we consider the following notations:

$$
\begin{aligned}
& d(A, B):=\inf \{d(a, b): a \in A, b \in B\} ; \\
& A_{0}:=\{a \in A: \text { there exists } b \in B \text { such that } d(a, b)=d(A, B)\} ; \\
& B_{0}:=\{b \in B: \text { there exists } a \in A \text { such that } d(a, b)=d(A, B)\} .
\end{aligned}
$$

Definition 2.1. [4] Let $T: A \rightarrow B$ be a mapping. An element $x^{*}$ is said to be a best proximity point of $T$ if $d\left(x^{*}, T x^{*}\right)=d(A, B)$.

Definition 2.2. [9] Let $\beta \in(0, \infty)$. A $\beta$-comparison function is a map $\varphi:[0,+\infty) \rightarrow$ $[0,+\infty)$ satisfying the following properties:

1. $\varphi$ is nondecreasing;
2. $\lim _{n \rightarrow \infty} \varphi_{\beta}^{n}(t)=0$ for all $t>0$, where $\varphi_{\beta}^{n}$ denotes the $n^{\text {th }}$ iterate of $\varphi_{\beta}$ and $\varphi_{\beta}(t)=\varphi(\beta t)$;
3. there exists $s \in(0,+\infty)$ such that $\sum_{n=1}^{\infty} \varphi_{\beta}^{n}(s)<\infty$.

The set of all $\beta$-comparison functions $\varphi$ satisfying (1), (2) and (3) will be denotes by $\Phi_{\beta}$.

Remark 2.3. Let $\alpha, \beta \in(0,+\infty)$. If $\alpha<\beta$, then $\Phi_{\beta} \subset \Phi_{\alpha}$.
Lemma 2.4. [9] Let $\beta \in(0,+\infty)$ and $\varphi \in \Phi_{\beta}$. Then

1. $\varphi_{\beta}$ is nondecreasing;
2. $\varphi_{\beta}(t)<t$ for all $t>0$;
3. $\quad \sum_{n=1}^{\infty} \varphi_{\beta}^{n}(t)<\infty$ for all $t>0$.

Lemma 2.5. [13] Let $(X, d)$ be a pair of non-empty closed subsets of a complete metric space $(X, d)$. Suppose that $T: A \rightarrow B$ be a mapping such that $A_{0}$ is empty. Then $T\left(A_{0}\right) \subset B_{0}$.

Lemma 2.6. [13] Let $(A, B)$ be a pair of non-empty closed subsets of a complete metric space $(X, d)$. Suppose that the following conditions are satisfied:

1. $A_{0} \neq \emptyset$,
2. The pair $(A, B)$ has P-property.

Then, the set $B_{0}$ is closed.
Definition 2.7. [10] Let $(A, B)$ be a pair of nonempty subsets of a metric space $(X, d)$ such that $A_{0}$ is nonempty. Then the pair $(A, B)$ is said to have the $P$-property iff $d\left(x_{1}, y_{1}\right)=d\left(x_{2}, y_{2}\right)=d(A, B) \Rightarrow d\left(x_{1}, x_{2}\right)=d\left(y_{1}, y_{2}\right)$, where $x_{1}, x_{2} \in A$ and $y_{1}, y_{2} \in B$.

Defintion 2.8 . We say that $B$ is approximately compact with respect to $A$ iff every sequence $\left\{y_{n}\right\} \subset B$ satisfying $\lim _{n \rightarrow+\infty} d\left(x, y_{n}\right)=d(x, B)$ for some $x \in A$ has a convergent subsequence.

## 3. Main results and theorems

First, we introduce the following concept.
Defintion 3.1. Let $S, T: A \rightarrow B$ be a non-self mappings. A pair $(S, T)$ is said to be a generalized proximal weakly $\beta$-quasi contractive mapping. Let $\beta \in(0, \infty)$, if there exist $\varphi \in \Phi_{\beta}$ and $a, b, c, d^{\prime}, e>0$ such that

$$
\begin{equation*}
d(S x, T y) \leq \varphi\left(M_{S}(x, y)\right), \forall x, y \in A \tag{3.1}
\end{equation*}
$$

where

$$
\begin{aligned}
M_{S}(x, y)= & \max \{\operatorname{ad}(x, y), b(d(x, S x)-\operatorname{dist}(A, B)), \\
& c(d(y, T y)-\operatorname{dist}(A, B)), d^{\prime}(d(y, S x)-\operatorname{dist}(A, B)), \\
& e(d(x, T y)-\operatorname{dist}(A, B))\}
\end{aligned}
$$

Theorem 3.2. Let $(A, B)$ be a pair of non-empty closed subsets of a complete metric space $(X, d)$ such that $A_{0} \neq \emptyset$ and the pair $(A, B)$ has the P-property. Let $S, T: A \rightarrow B$ be two non-self mappings satisfies the following conditions.

1. $T$ or $S$ is continuous.
2. $B$ is approximately compact with respect to $A$.
3. There exists $\beta \geq \max \left\{a, b, c, d^{\prime}, 2 e\right\}$ such that $S, T: A \rightarrow B$ is proximal generalized weakly $\beta$-quasi contractive.
Moreover, assume the following conditions holds;

- $\quad \varphi$ is continuous;
- $\quad \beta>\max \left\{b, d^{\prime}\right\}$.

Then $S$ and $T$ have a unique common best proximity point $x^{*} \in A$ such that $d\left(x^{*}, T x^{*}\right)=d\left(x^{*}, S x^{*}\right)=\operatorname{dist}(A, B)$.

## Proof.

Since $A_{0} \neq \emptyset$ we can choose $x_{0} \in A_{0}$ and fix it. By Lemma $2.5 S x_{0} \in S\left(A_{0}\right) \subset B_{0}$ and then definition of $A_{0}$ we can find $x_{1} \in A_{0}$ such that $d\left(x_{1}, S x_{0}\right)=\operatorname{dist}(A, B)$. Since $T x_{1} \in T A_{0} \subset B_{0}$, we can find $x_{2} \in A_{0}$ such that $d\left(x_{2}, T x_{1}\right)=\operatorname{dist}(A, B)$. Considering that $x_{2} \in A_{0}$ and $S\left(A_{0}\right) \subset B_{0}$ we can find $x_{3} \in A_{0}$ such that $d\left(x_{3}, S x_{2}\right)=\operatorname{dist}(A, B)$.

In this way we can find $x_{4} \in A_{0}$ such that $d\left(x_{4}, T x_{3}\right)=\operatorname{dist}(A, B)$ as $T x_{3} \in T\left(A_{0}\right) \subset$ $B_{0}$. By continuing this process we can get the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ in $A_{0}$ such that for any $n \in \mathbb{N}$

$$
\begin{array}{ll}
d\left(x_{n+1}, T x_{n}\right)=\operatorname{dist}(A, B) & \text { for all } n \in \mathbb{N} \cup\{0\} \\
d\left(x_{n}, S x_{n-1}\right)=\operatorname{dist}(A, B) & \text { for all } n \in \mathbb{N} . \tag{3.3}
\end{array}
$$

Since $(A, B)$ has $P$-property we can write

$$
d\left(x_{n}, x_{n+1}\right)=d\left(S x_{n-1}, T x_{n}\right) .
$$

Since $S, T: A \rightarrow B$ is generalized proximal weak $\beta$-quasi contractive, we obtain

$$
\begin{equation*}
d\left(x_{n}, x_{n+1}\right)=d\left(S x_{n-1}, T x_{n}\right) \leq \varphi\left(M_{S}\left(x_{n-1}, x_{n}\right)\right) \tag{3.4}
\end{equation*}
$$

On the other hand, using (3.2), (3.3), and the triangular inequality, we get

$$
\begin{aligned}
M_{S}\left(x_{n-1}, x_{n}\right)= & \max \left\{\operatorname{ad}\left(x_{n-1}, x_{n}\right), b\left(d\left(x_{n-1}, S x_{n-1}\right)-\operatorname{dist}(A, B)\right)\right. \\
& c\left(d\left(x_{n}, T x_{n}\right)-\operatorname{dist}(A, B)\right), d^{\prime}\left(d\left(x_{n}, S x_{n-1}\right)-\operatorname{dist}(A, B)\right) \\
& \left.e\left(d\left(x_{n-1}, T x_{n}\right)-\operatorname{dist}(A, B)\right)\right\} \\
\leq & \max \left\{\operatorname{ad}\left(x_{n-1}, x_{n}\right), b\left(d\left(x_{n-1}, S x_{n-1}\right)-\operatorname{dist}(A, B)\right),\right. \\
& c\left(d\left(x_{n}, T x_{n}\right)-\operatorname{dist}(A, B)\right), d^{\prime}\left(d\left(x_{n}, x_{n}\right)+d\left(x_{n}, S x_{n-1}\right)-\operatorname{dist}(A, B)\right), \\
& \left.e\left(d\left(x_{n-1}, T x_{n}\right)-\operatorname{dist}(A, B)\right)\right\}, \\
\leq & \max \left\{\operatorname{ad}\left(x_{n-1}, x_{n}\right), b d\left(x_{n-1}, x_{n}\right), c d\left(x_{n}, x_{n+1}\right), 0,\right. \\
& \left.e d\left(x_{n-1}, x_{n}\right)+e d\left(x_{n}, x_{n+1}\right)\right\} \\
\leq & \beta \max \left\{d\left(x_{n-1}, x_{n}\right), d\left(x_{n}, x_{n+1}\right)\right\} .
\end{aligned}
$$

Hence

$$
\begin{equation*}
M_{S}\left(x_{n-1}, x_{n}\right) \leq \beta \max \left\{d\left(x_{n-1}, x_{n}\right), d\left(x_{n}, x_{n+1}\right)\right\}, \tag{3.5}
\end{equation*}
$$

where $\beta \geq \max _{0 \leq k \leq 3}\left\{a, b, c, d^{\prime}, 2 e\right\}$. Using inequalities (3.4) and (3.5) and taking into consideration the fact that $\varphi$ is nondecreasing, we get that

$$
d\left(x_{n+1}, x_{n}\right) \leq \varphi\left(\beta \max \left\{d\left(x_{n-1}, x_{n}\right), d\left(x_{n}, x_{n+1}\right)\right\}\right)=\varphi_{\beta}\left(\max \left\{d\left(x_{n-1}, x_{n}\right), d\left(x_{n}, x_{n+1}\right)\right\}\right) .
$$

Suppose that, for some $n$, we have $d\left(x_{n-1}, x_{n}\right) \leq d\left(x_{n}, x_{n+1}\right)$. It follows that $d\left(x_{n+1}, x_{n}\right) \leq \varphi_{\beta}\left(d\left(x_{n+1}, x_{n}\right)<d\left(x_{n+1}, x_{n}\right)\right.$, which is a contradiction.

Then, for all $n \geq 0$, we necessary have $d\left(x_{n-1}, x_{n}\right)>d\left(x_{n}, x_{n+1}\right)$, and it follows that

$$
\begin{equation*}
d\left(x_{n+1}, x_{n}\right) \leq \varphi_{\beta}\left(d\left(x_{n-1}, x_{n}\right)\right), \forall n \in \mathbb{N} . \tag{3.6}
\end{equation*}
$$

Then, by induction, we obtain that

$$
\begin{equation*}
d\left(x_{n+1}, x_{n}\right) \leq \varphi_{\beta}^{n}\left(d\left(x_{1}, x_{0}\right)\right), \forall n \in \mathbb{N} \cup\{0\} . \tag{3.7}
\end{equation*}
$$

Let $\epsilon>0$ be fixed. Since the numerical series $\sum_{n=1}^{+\infty} \varphi_{\beta}^{n}\left(d\left(x_{1}, x_{0}\right)\right)$ converges, there exists a positive integer $N$ such that $\left.\sum_{n \geq N}^{\infty} \varphi_{\beta}^{n} d\left(x_{1}, x_{0}\right)\right)<\epsilon$. For $m>n>N$, using the triangular inequality, the convergence of the series, and (3.7), we obtain

$$
\begin{align*}
d\left(x_{n}, x_{m}\right) & \leq \sum_{k=n}^{m-1} d\left(x_{k}, x_{k+1}\right)  \tag{3.8}\\
& \leq \sum_{k=n}^{m-1} \varphi_{\beta}^{k}\left(d\left(x_{1}, x_{0}\right)\right) \tag{3.9}
\end{align*}
$$

since the series $\sum_{n=1}^{+\infty} \varphi_{\beta}^{n}(t)$ converges for all $t \geq 0$, as a result

$$
\sum_{k=n}^{m-1} \varphi_{\beta}^{k}\left(d\left(x_{1}, x_{0}\right)\right) \rightarrow 0 \text { as } n, m \rightarrow+\infty
$$

Hence $\left\{x_{n}\right\}$ is a Cauchy sequence in $A$. As $A$ is a closed subset of a complete metric space so $A$ is complete. Therefore there exists $x \in A$ such that $x_{n} \rightarrow x$ as $n \rightarrow \infty$.

Now we prove that $x^{*}$ is a common best proximity point of $T$ and $S$.
Without loss of generality, assume that the mapping $T$ is continuous. Since $x_{n} \rightarrow x^{*}$, we obtain that $T x_{n} \rightarrow T x^{*}$. On the other hand, $x_{n+1} \rightarrow x^{*}$.

Hence the continuity of the metric function $d$ implies that $d\left(x_{n+1}, T x_{n}\right)=$ $d\left(x^{*}, T x^{*}\right)$. But (3.2) shows that the sequence $d\left(x_{n+1}, T x_{n}\right)$ is a constant sequence with the value $\operatorname{dist}(A, B)$. Therefore, $d\left(x^{*}, T x^{*}\right)=\operatorname{dist}(A, B)$.

On the other hand, we have

$$
\begin{aligned}
d\left(x^{*}, B\right) & \leq d\left(x^{*}, S x_{n}\right) \\
& \leq d\left(x^{*}, x_{n-1}\right)+d\left(x_{n-1}, S x_{n}\right) \\
& =d\left(x^{*}, x_{n-1}\right)+\operatorname{dist}(A, B) \\
& \leq d\left(x^{*}, x_{n-1}\right)+d\left(x^{*}, B\right)
\end{aligned}
$$

As $n \rightarrow \infty$, we get that the sequence $d\left(x^{*}, S x_{n}\right)$ converges to $d\left(x^{*}, B\right)$. Since $B$ is approximately compact with respect to $A$, there exists a subsequence $\left\{x_{n_{k}}\right\}$ of $\left\{x_{n}\right\}$ such that $S\left(x_{n_{k}}\right)$ converges to some $\eta \in B$. Hence

$$
\begin{aligned}
\operatorname{dist}(A, B) & \leq d\left(x^{*}, \eta\right) \\
& \leq d\left(x^{*}, x_{n_{k}-1}\right)+d\left(x_{n_{k}-1}, S x_{n_{k}}\right)+d\left(S x_{n_{k}}, \eta\right) \\
& =d\left(x^{*}, x_{n_{k}-1}\right)+\operatorname{dist}(A, B)+d\left(S x_{n_{k}}, \eta\right)
\end{aligned}
$$

As $n \rightarrow \infty$, we get that $d\left(x^{*}, \eta\right)=\operatorname{dist}(A, B)$ and therefore $x^{*} \in A_{0}$. By Lemma 2.5 , there exists $u \in A$ such that $d\left(u, T x^{*}\right)=\operatorname{dist}(A, B)$. Consequently, we obtain $d\left(u, T x^{*}\right)=d\left(x_{n-1}, S x_{n}\right)=\operatorname{dist}(A, B)$. Using the $P$-property, we deduce that $d\left(u, x_{n-1}\right)=d\left(T x^{*}, S x_{n}\right)$.

Since $T$ is generalized proximal weakly $\beta$ quasi-contractive, we obtain

$$
\begin{align*}
d\left(u, x_{n-1}\right) & =d\left(T x^{*}, S x_{n}\right) \\
& \leq \varphi\left(M_{S}\left(x^{*}, x_{n}\right)\right), \quad \forall n \in \mathbb{N}, \tag{3.10}
\end{align*}
$$

where

$$
\begin{align*}
M_{S}\left(x^{*}, x_{n}\right)= & \max \left\{\operatorname{ad}\left(x^{*}, x_{n}\right), b\left(d\left(x^{*}, S x^{*}\right)-\operatorname{dist}(A, B)\right),\right. \\
& c\left(d\left(x_{n}, T x_{n}\right)-\operatorname{dist}(A, B)\right), d^{\prime}\left(d\left(x_{n}, S x^{*}\right)-\operatorname{dist}(A, B)\right), \\
& \left.e\left(d\left(x^{*}, T x_{n}\right)-\operatorname{dist}(A, B)\right)\right\} \tag{3.11}
\end{align*}
$$

On the other hand, using the triangular inequality and (3.1), we have

$$
\begin{align*}
M_{S}\left(x^{*}, x_{n}\right)= & \max \left\{\operatorname{ad}\left(x^{*}, x_{n}\right), b\left(d\left(x^{*}, S x^{*}\right)-\operatorname{dist}(A, B)\right),\right. \\
& \operatorname{cd}\left(x_{n}, x_{n+1}\right), d^{\prime}\left(d\left(x_{n}, x^{*}\right)+d\left(x^{*}, S x^{*}\right)-\operatorname{dist}(A, B)\right), \\
& \left.e d\left(x^{*}, x_{n+1}\right)\right\} \tag{3.12}
\end{align*}
$$

Moreover, using the triangular inequality, we get

$$
\begin{align*}
d\left(x^{*}, S x^{*}\right) & \leq d\left(x^{*}, x_{n+1}\right)+d\left(x_{n+1}, T x_{n}\right)+d\left(T x_{n}, S x^{*}\right) \\
& =d\left(x^{*}, x_{n+1}\right)+\operatorname{dist}(A, B)+d\left(T x_{n}, S x^{*}\right), \forall n \in \mathbb{N} . \tag{3.13}
\end{align*}
$$

Using inequality (3.13) and (3.1), we obtain that

$$
\begin{equation*}
d\left(x^{*}, S x^{*}\right)-d\left(x^{*}, x_{n+1}\right)-\operatorname{dist}(A, B) \leq \varphi\left(M_{S}\left(x^{*}, x_{n}\right)\right), \forall n \in \mathbb{N} . \tag{3.14}
\end{equation*}
$$

Let $s=d\left(x^{*}, S x^{*}\right)=\operatorname{dist}(A, B)$, letting $n \rightarrow+\infty$ in inequality (3.12), we get

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} M_{S}\left(x^{*}, x_{n}\right) \leq \max \left\{b, d^{\prime}\right\} s \tag{3.15}
\end{equation*}
$$

Suppose that $s>0$. If $\varphi$ is continuous, letting $n \rightarrow+\infty$ in inequality (3.14) and using the fact that $\varphi$ is nondecreasing, we get

$$
s \leq \varphi\left(\max \left\{b, d^{\prime}\right\} s\right) \leq \varphi(\beta s)=\varphi_{\beta}(s)<s,
$$

which is a contradiction. If $\beta>\max \left\{b, d^{\prime}\right\}$. We claim that also $s=0$. Suppose that $s>0$. Using inequality (3.15) and the definition of the limit, there exists $\epsilon>0$ and $N>0$ such that, for all $n>N$, we have $M_{S}\left(x_{n}, x^{*}\right)<\left(\max \left\{b, d^{\prime}\right\}+\epsilon\right) s$. Since $\varphi$ is nondecreasing, from (3.15), we get

$$
\begin{align*}
d\left(x^{*}, S x^{*}\right)-d\left(x^{*}, x_{n+1}\right)-\operatorname{dist}(A, B) & \leq \varphi\left(M_{S}\left(x^{*}, x_{n}\right)\right) \\
& \leq \varphi\left(\left(\max \left\{b, d^{\prime}\right\}+\epsilon\right) s\right) \\
& =\varphi_{\beta}\left(\frac{\max \left\{b, d^{\prime}\right\}+\epsilon}{\beta} s\right) \\
& <\frac{\max \left\{b, d^{\prime}\right\}+\epsilon}{\beta} s<s . \tag{3.16}
\end{align*}
$$

By letting $n \rightarrow+\infty$ in (3.16), we get

$$
s<\frac{\max \left\{b, d^{\prime}\right\}+\epsilon}{\beta} s<s,
$$

which is a contraction. Therefore $s=0$ and so $d\left(x^{*}, S x^{*}\right)-\operatorname{dist}(A, B)=0$, which implies that $d\left(x^{*}, S x^{*}\right)=\operatorname{dist}(A, B)$. So we get that $d\left(x^{*}, T x^{*}\right)=d\left(x^{*}, S x^{*}\right)=$ $\operatorname{dist}(A, B)$, that is $x^{*}$ is a common best proximity point for the pair of mappings $(S, T)$ from the pair of $(A, B)$.

Finally we prove that $x^{*}$ is a unique common best proximity point of $T$ and $S$.
Let $x_{1}^{*}$ be another element in A such that

$$
d\left(x_{1}^{*}, S x_{1}^{*}\right)=d\left(x_{1}^{*}, T x_{1}^{*}\right)=\operatorname{dist}(A, B) .
$$

Since

$$
\begin{aligned}
d\left(x^{*}, T x^{*}\right) & =d\left(x^{*}, S x^{*}\right)=\operatorname{dist}(A, B), \\
d\left(x_{1}^{*}, S x_{1}^{*}\right) & =d\left(x_{1}^{*}, T x_{1}^{*}\right)=\operatorname{dist}(A, B) .
\end{aligned}
$$

Using $P$-property, we get

$$
d\left(x^{*}, x_{1}^{*}\right)=d\left(T x^{*}, S x_{1}^{*}\right),
$$

since $T$ is generalized proximal weakly $\beta$-quasi contractive, we deduce that

$$
d\left(x^{*}, x_{1}^{*}\right)=d\left(T x^{*}, S x_{1}^{*}\right) \leq \varphi\left(M_{S}\left(x_{1}^{*}, x^{*}\right)\right),
$$

where

$$
\begin{aligned}
M_{S}\left(x_{1}^{*}, x^{*}\right)= & \max \left\{\operatorname{ad}\left(x_{1}^{*}, x^{*}\right), b\left(d\left(x_{1}^{*}, S x_{1}^{*}\right)-\operatorname{dist}(A, B)\right)\right. \\
& c\left(d\left(x^{*}, T x^{*}\right)-\operatorname{dist}(A, B)\right), d^{\prime}\left(d\left(x^{*}, S x_{1}^{*}\right)-\operatorname{dist}(A, B)\right), \\
& \left.e\left(d\left(x_{1}^{*}, T x^{*}\right)-\operatorname{dist}(A, B)\right)\right\} \\
= & \max \left\{a, d^{\prime}, e\right\} d\left(x^{*}, x_{1}^{*}\right) .
\end{aligned}
$$

Let $r=d\left(x^{*}, x_{1}^{*}\right)$. Since $\varphi$ is nondecreasing, we obtain

$$
\left.r \leq \varphi\left(\max \left\{a, d^{\prime}, e\right\}\right) r\right) \leq \varphi(\beta r)=\varphi_{\beta}(r)<r,
$$

which is a contraction. Therefore $x^{*}$ is a unique common best proximity point of the pair mappings ( $S, T$ ) of the pair $(A, B)$. Hence proved.

Example 3.1. Consider the complete metric space $X=\mathbb{R}$ with metric $d(x, y)=|x-y|$. Let $A=[0,2]$ and $B=[4,6]$. These sets are closed on the metric space $(X, d)$. Also, since $B$ is compact, then $B=[4,6]$ is approximately compact with respect to $A=[0,2]$. Also, let $T: A \rightarrow B$ and $S: A \rightarrow B$ be defined by $T x=2+x$ and $S x=6-x$. Then it is easy to see that $\operatorname{dist}(A, B)=2$ and $A_{0}=\{2\}, B_{0}=\{4\}$. Thus, $T\left(A_{0}\right)=S\left(A_{0}\right)=T(\{2\})=S(\{2\})=\{4\}=B_{0}$ Now we shall to show that $T$ is generalized proximal weakly $\beta$-quasi contractive.

Case 1: Take $x=y=0$ with $\varphi(t)=\frac{1}{4} t, \beta=4, b=d^{\prime}=4$ and $a=c=e=0$

$$
d(S x, T y)=|4-(x+y)| \leq \frac{1}{4} \max \left\{0 a, 4 b, 0 c, 4 d^{\prime}, 0 e\right\}
$$

The function $\varphi(t)$ is continuous mappings as well as $\beta=4>\max \{a, c, e\}=0$.
Case 2: Take $x=y=2$ with $\varphi(t)=\frac{1}{4} t, \beta=4, b=d^{\prime}=4$ and $a=c=e=0$

$$
0=d(S x, T y)=|4-(x+y)| \leq \frac{1}{4} \max \{0 a, 0 b, 0 c, 0 d, 0 e\}=0
$$

The function $\varphi(t)$ is continuous mappings as well as $\beta=4>\max \{a, c, e\}=0$.
Case 3: Take $x=0, y=2$ with $\varphi(t)=\frac{1}{4} t, \beta=2, b=c=2, a=d^{\prime}=e=4$

$$
d(S x, T y)=|4-(x+y)| \leq \frac{1}{4} \max \left\{2 a, 4 b, 0 c, 2 d^{\prime}, 2 e\right\}
$$

The function $\varphi(t)$ is continuous mappings as well as $\beta=4>\max \{b, c\}=2$. We deduce using our Theorem 3.2, that $S$ and $T$ have a unique best proximity point which is $x^{*}=2$ in this example.

$$
\begin{aligned}
& d\left(x^{*}, T x^{*}\right)=d(2, T(2))=d(2,4)=\operatorname{dist}(A, B)=2 \\
& d\left(x^{*}, S x^{*}\right)=d(2, S(2))=d(2,4)=\operatorname{dist}(A, B)=2 .
\end{aligned}
$$

## 4. Consequences

Several consequence of the main results of Section 3 are established next.
First, as an application to best proximity points, we propose the following results, which are an immediate consequence of our main Theorem 3.2.

Corollary 4.1. Let $(A, B)$ be a pair of non-empty closed subsets of a complete metric space $(X, d)$ such that $A_{0} \neq \emptyset$ and the pair $(A, B)$ has the P-property. Let $S, T: A \rightarrow B$ be two non-self mappings satisfies the following conditions.

1. $T$ or $S$ is continuous.
2. $B$ is approximately compact with respect to $A$.
3. there exists $\phi \in \Phi_{2}$ such that

$$
\begin{equation*}
d(S x, T y) \leq \phi(M(x, y)), \forall x, y \in A, \tag{4.1}
\end{equation*}
$$

where

$$
\begin{aligned}
& M(x, y)=\max \{d(x, y), d(x, S x)-\operatorname{dist}(A, B), d(y, T y)-\operatorname{dist}(A, B), \\
& \left.\quad \frac{d(y, S x)+d(x, T y)}{2}-\operatorname{dist}(A, B)\right\}
\end{aligned}
$$

Then $S$ and $T$ have a unique common best proximity point $x^{*} \in A$ such that $d\left(x^{*}, T x^{*}\right)=d\left(x^{*}, S x^{*}\right)=\operatorname{dist}(A, B)$.

Proof. The main idea is that

$$
M(x, y) \leq M_{S}(x, y),
$$

where

$$
\begin{aligned}
M_{S}(x, y)= & \max \{d(x, y), d(x, S x)-\operatorname{dist}(A, B), \\
& d(y, T y)-\operatorname{dist}(A, B), d(y, S x)-\operatorname{dist}(A, B), \\
& d(x, T y)-\operatorname{dist}(A, B)\} .
\end{aligned}
$$

Here $a=b=c=d^{\prime}=e=1$. So for $\beta \geq 2>\max \left\{b, d^{\prime}\right\}=1$. According to our Theorem 3.2, if the comparison function $\phi \in \Phi_{2}$, then $S$ and $T$ have a unique common best proximity point in $A$.

Corollary 4.2. Let $(A, B)$ be a pair of non-empty closed subsets of a complete metric space $(X, d)$ such that $A_{0} \neq \emptyset$ and the pair $(A, B)$ has the P-property. Let $S, T: A \rightarrow B$ be two non-self mappings satisfies the following conditions.

1. $T$ or $S$ is continuous.
2. $B$ is approximately compact with respect to $A$.
3. there exists $q \in[0,1)$ such that

$$
\begin{equation*}
d(S x, T y) \leq q M(x, y), \forall x, y \in A \tag{4.2}
\end{equation*}
$$

where

$$
\begin{aligned}
& M(x, y)=\max \{d(x, y), d(x, S x)-\operatorname{dist}(A, B), d(y, T y)-\operatorname{dist}(A, B), \\
& \left.\quad \frac{d(y, S x)+d(x, T y)}{2}-\operatorname{dist}(A, B)\right\}
\end{aligned}
$$

Then $S$ and $T$ have a unique common best proximity point $x^{*} \in A$ such that $d\left(x^{*}, T x^{*}\right)=d\left(x^{*}, S x^{*}\right)=\operatorname{dist}(A, B)$.

Proof. Let $\phi=q t$, which belongs to $\Phi_{1}$ and is continuous. According our Theorem 3.2, $S$ and $T$ have a unique common best proximity point in $A$. Before proposing consequences of our result to the existence and uniqueness of fixed points for selfmappings, we introduce the following definition.

Defintion 4.3. Let $A$ be a nonempty set of a metric space ( $X, d$ ). A self-mappings $S, T: A \rightarrow A$ is called generalized proximal weakly $\beta$-quasi contractive if there exists a function $\phi \in \Phi_{\beta}$, where $\beta>0$ such that, for all $x, y \in A$, we have

$$
d(S x, T y) \leq \phi\left(M_{S}(x, y)\right),
$$

where

$$
M_{S}(x, y)=\max \left\{\operatorname{ad}(x, y), b d(x, S x), c d(y, T y), d^{\prime} d(y, S x), e d(x, T y)\right\}
$$

with $a, b, c, d^{\prime} \geq 0$.
Several papers dealt with fixed point theory in the context of the generalizing of Banach's principle as in [14-16]. By our special generalized $\beta$-quasi contractive mapping, we can propose some theorems on the existence and uniqueness of fixed points in complete spaces in a simple way.

Corollary 4.4. Let $(X, d)$ be a nonempty complete metric space. Consider a selfmapping $T: X \rightarrow X$. Suppose that there exists $\beta \geq \max \left\{a, b, c, d^{\prime}, 2 e\right\}$ such that $T$ is a $\beta$-quasi contractive mapping. Moreover, assume that one of the following conditions holds;

- $\quad \varphi$ is continuous;
- $\quad \beta>\max \left\{b, d^{\prime}\right\}$.

Then $S$ and $T$ have a unique common fixed point in $X$.
Proof. This is an immediate consequence of our main Theorem 3.2 since $A=$ $B=X$ and every set is approximately compact with itself. Moreover, the notion of generalized proximal weakly $\beta$-quasi contractive on the self mapping case is exactly a weak $\beta$-quasi contractive one.
Corollary 4.5. Let $(X, d)$ be a complete metric space, and let $S, T: X \rightarrow X$ be a quasi-contraction, that is,

$$
\begin{equation*}
d(S x, T y) \leq q \max \{d(x, y), d(x, S x), d(y, T y), d(y, S x), d(x, T y)\} \tag{4.3}
\end{equation*}
$$

for all $x, y \in X$.

Proof. Using our main Theorem 3.2, since $A=B=X$ and every set is approximately compact with its self, the function $\phi(t)=q t$, which is continuous and belongs to the set $\Phi_{1}$.

## 5. Conclusion

Improvements to some best proximity point theorems are proposed. This has been achieved by introducing a suitable mapping called generalized proximal weakly $\beta$ quasi contractive. These are non-self-mappings involving $\beta$-comparision functions. As an application, we establish the existence and uniqueness of well-known fixed point results for the case of self-mappings on complete metric spaces. We confirm our results by a suitable example.

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