GANITA, Vol.73(2), 2023, 59-74

RICCI SOLITON ON SASAKIAN MANIFOLD WITH QUARTER-SYMMETRIC NON-METRIC CONNECTION

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Abstract

In the present article, the Ricci soliton on Sasakian manifold endowed with quarter-symmetric non-metric connection is investigated. On the Sasakian manifold, we have discussed several curvature properties with given connection. In continuation, we have shown that Ricci solitons on Sasakian manifolds endowed with quarter-symmetric non-metric connection is an η-Einstein manifold. We obtain some intriguing findings satisfying the conditions $\widetilde{\mathcal{P}}.\widetilde{\mathcal{S}} = 0$, $\widetilde{\mathcal{R}}.\widetilde{\mathcal{S}} = 0$ and $\widetilde{\mathcal{S}}.\widetilde{\mathcal{R}} = 0$. The condition of Ricci-recurrent, φ-recurrent and Einstein semi-symmetric have also been obtained in this article. Finally, we demonstrate our results with a 3-dimensional example.

2010 *Mathematics subject classification:* primary 53C05; secondary 53D10, 53E20.

Keywords and phrases: Sasakian manifold, Ricci soliton, Einstein manifold, Einstein semi-symmetric.

1. Introduction

The notion of Ricci soliton is a natural generalization of Einstein manifolds. In 1982, Hamilton [9] developed the concept of Ricci flow to identify a canonical metric on a smooth manifold. Ricci flow has emerged as a powerful tool for the study of Riemannian manifolds and specially for manifolds with positive curvature. As an evolution equation for metrics on a Riemannian manifold, the Ricci flow is defined as follows:

$$\frac{\partial g_1}{\partial t} = -2\mathcal{S}. \tag{1.1}$$

A Ricci soliton is a Ricci flow that only changes by one parameter group of diffeomorphism and its scaling. Perelman [21, 22] used the Ricci flow to validate the Poincare conjecture. Ricci soliton $(g_1, \mathcal{V}, \lambda)$ on a Riemannian manifold (Ω^{2n+1}, g_1) is defined as

$$\mathcal{L}_{\mathcal{V}} g_1 + 2\mathcal{S} + 2\lambda g_1 = 0, \tag{1.2}$$

where $\mathcal{L}_{\mathcal{V}}$ is Lie derivative along $\mathcal{V}$ on Ω, $\mathcal{S}$ is the Ricci tensor and λ is a constant. The Ricci soliton is called shrinking, steady and expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ respectively. Sharma [25] studied the Ricci soliton in contact geometry. Ghosh, Sharma and Cho [8] studied gradient Ricci soliton of a non-Sasakian (K, μ)-contact manifolds. Later on, several authors studied Ricci soliton in various contact manifolds. Some of them are Ricci soliton in β-Kenmotsu manifold [15], η-Ricci soliton on 3-dimensional f-Kenmotsu manifolds [14], Ricci soliton on trans-Sasakian manifolds [23] and many others [1, 2, 11, 13, 16, 27].

On the other hand, Shigio Sasaki [26] introduced the Sasakian manifold in 1960. Furthermore, it was investigated by numerous authors [12, 19, 20, 24] in various contests.

A linear connection $\widetilde{\nabla}$ in an $(2n+1)$-dimensional differentiable manifold is called a quarter-symmetric connection [3] if it satisfy

$$\widetilde{\mathcal{T}}(\mathcal{X},\mathcal{Y}) = \bar{\eta}(\mathcal{Y})\bar{\varphi}\mathcal{X} - \bar{\eta}(\mathcal{X})\bar{\varphi}\mathcal{Y}. \tag{1.3}$$

If quarter-symmetric linear connection $\widetilde{\nabla}$ satisfies the condition $(\widetilde{\nabla}_{\mathcal{X}} g_1)(\mathcal{Y},\mathcal{Z}) \neq 0$, then $\widetilde{\nabla}$ is called quarter-symmetric non-metric connection [4, 6, 7].

Motivated by the above studies, we studied Ricci soliton on Sasakian manifolds endowed with quarter-symmetric non-metric connection. The present research article is organized as follows: After the introduction, Section 2 deals with some fundamental definitions of Sasakian manifolds. In section 3, we define a quarter-symmetric non-metric connection and some curvature properties with quarter-symmetric non-metric connection. We study Ricci soliton with quarter-symmetric non-metric connection in section 4. Section 5 is dedicated to Ricci soliton on Sasakian manifold satisfying $\widetilde{\mathcal{P}}.\widetilde{\mathcal{S}} = 0$ with quarter-symmetric non-metric connection. In section 6, we establish some important results on Ricci soliton in Sasakian manifolds satisfying $\widetilde{\mathcal{R}}.\widetilde{\mathcal{S}} = 0$ and $\widetilde{\mathcal{S}}.\widetilde{\mathcal{R}} = 0$. Some basic results and theorems on Ricci-recurrent and φ-recurrent Sasakian manifolds with quarter-symmetric non-metric connection are studied in section 7. In section 8, we investigated Einstein semi-symmetric with Ricci soliton in Sasakian manifold with the defined connection. Lastly in section 9, we demonstrated our findings with a 3-dimensional example.

2. Preliminaries

Let $\Omega^{(2n+1)}$ is a differentiable manifold with an almost contact structure $(\bar{\varphi},\bar{\zeta},\bar{\eta},g_1)$ satisfying

$$\begin{aligned}\bar{\varphi}^2\mathcal{X} = -\mathcal{X} + \bar{\eta}(\mathcal{X})\bar{\zeta}, \quad \bar{\eta}(\bar{\zeta}) = 1, \quad \bar{\varphi}\bar{\zeta} = 0, \quad \bar{\eta}(\bar{\varphi}\mathcal{X}) = 0,\\ g_1(\bar{\varphi}\mathcal{X},\mathcal{Y}) = -g_1(\mathcal{X},\bar{\varphi}\mathcal{Y}), \quad g_1(\mathcal{X},\bar{\zeta}) = \bar{\eta}(\mathcal{X}),\\ g_1(\bar{\varphi}\mathcal{X},\bar{\varphi}\mathcal{Y}) = g_1(\mathcal{X},\mathcal{Y}) - \bar{\eta}(\mathcal{X})\bar{\eta}(\mathcal{Y}),\end{aligned} \tag{2.1}$$

for any vector fields $\mathcal{X},\mathcal{Y} \in \Gamma\Omega$, where $\bar{\varphi}$ is $(1,1)$-tensor field, $\bar{\zeta}$ is a vector field, $\bar{\eta}$ is 1-form and g_1 is the Riemannian metric.
An almost contact metric manifold $\Omega^{(2n+1)}$ is called Sasakian manifold if

$$(\nabla_{\mathcal{X}}\bar{\varphi})\mathcal{Y} = g_1(\mathcal{X},\mathcal{Y})\bar{\zeta} - \bar{\eta}(\mathcal{Y})\mathcal{X}, \tag{2.2}$$

where ∇ is Levi-Civita connection on Riemannian metric. From (2.1), we have

$$\nabla_{\mathcal{X}}\bar{\zeta} = -\bar{\varphi}\mathcal{X}, \tag{2.3}$$

$$(\nabla_{\mathcal{X}}\bar{\eta})(\mathcal{Y}) = g_1(\mathcal{X}, \bar{\varphi}\mathcal{Y}). \tag{2.4}$$

Sasakian manifold $\Omega^{(2n+1)}$ also holds the following relations:

$$\mathcal{R}(\mathcal{X}, \mathcal{Y})\bar{\zeta} = \bar{\eta}(\mathcal{Y})\mathcal{X} - \bar{\eta}(\mathcal{X})\mathcal{Y}, \tag{2.5}$$

$$\mathcal{R}(\bar{\zeta}, \mathcal{Y})\mathcal{X} = g_1(\mathcal{X}, \mathcal{Y})\bar{\zeta} - \bar{\eta}(\mathcal{X})\mathcal{Y}, \tag{2.6}$$

$$\mathcal{R}(\bar{\zeta}, \mathcal{X})\bar{\zeta} = \bar{\eta}(\mathcal{X})\bar{\zeta} - \mathcal{X}, \tag{2.7}$$

$$\mathcal{S}(\mathcal{X}, \bar{\zeta}) = 2n\bar{\eta}(\mathcal{X}), \tag{2.8}$$

$$\mathcal{Q}\bar{\zeta} = 2n\bar{\zeta}, \tag{2.9}$$

$$\mathcal{S}(\bar{\varphi}\mathcal{X}, \bar{\varphi}\mathcal{Y}) = \mathcal{S}(\mathcal{X}, \mathcal{Y}) - 2n\bar{\eta}(\mathcal{X})\bar{\eta}(\mathcal{Y}). \tag{2.10}$$

Definition 2.1. An almost contact metric manifold $\Omega^{(2n+1)}$ is known as an η-Einstein manifolds if there exists the real valued function λ_1, λ_2, such that

$$\mathcal{S}(\mathcal{X}, \mathcal{Y}) = \lambda_1 g(\mathcal{X}, \mathcal{Y}) + \lambda_2\eta(\mathcal{X})\eta(\mathcal{Y}). \tag{2.11}$$

For $\lambda_2 = 0$, the manifold $\Omega^{(2n+1)}$ is known an Einstein manifold.

Definition 2.2. For a Riemannian manifold $(\Omega^{(2n+1)}, g_1)$, the projective curvature tensor [5] is defined as

$$\mathcal{P}(\mathcal{X}, \mathcal{Y})\mathcal{Z} = \mathcal{R}(\mathcal{X}, \mathcal{Y})\mathcal{Z} - \frac{1}{2n}[\mathcal{S}(\mathcal{Y}, \mathcal{Z})\mathcal{X} - \mathcal{S}(\mathcal{X}, \mathcal{Z})\mathcal{Y}]. \tag{2.12}$$

3. Quarter-symmetric non-metric connection

Let $\Omega^{(2n+1)}$ be a Sasakian manifold and ∇ be the Levi-Civita connection on $\Omega^{(2n+1)}$. The quarter-symmetric non-metric connection $\widetilde{\nabla}$ is defined as [17]

$$\widetilde{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{Y} + \frac{1}{2}\bar{\eta}(\mathcal{Y})\bar{\varphi}\mathcal{X} - \frac{1}{2}\bar{\eta}(\mathcal{X})\bar{\varphi}\mathcal{Y} \tag{3.1}$$

satisfying

$$\widetilde{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = \bar{\eta}(\mathcal{Y})\bar{\varphi}\mathcal{X} - \bar{\eta}(\mathcal{X})\bar{\varphi}\mathcal{Y} \tag{3.2}$$

and

$$(\widetilde{\nabla}_{\mathcal{X}}g_1)(\mathcal{Y}, \mathcal{Z}) = -\frac{1}{2}\left[\bar{\eta}(\mathcal{Y})g_1(\bar{\varphi}\mathcal{X}, \mathcal{Z}) + \bar{\eta}(\mathcal{Z})g_1(\bar{\varphi}\mathcal{X}, \mathcal{Y})\right]. \tag{3.3}$$

Also, we find

$$(\widetilde{\nabla}_{\mathcal{X}}\bar{\varphi})\mathcal{Y} = (\nabla_{\mathcal{X}}\bar{\varphi})\mathcal{Y} + \frac{1}{2}\bar{\eta}(\mathcal{Y})\mathcal{X} - \frac{1}{2}\bar{\eta}(\mathcal{X})\bar{\eta}(\mathcal{Y})\bar{\zeta}, \tag{3.4}$$

$$\widetilde{\nabla}_{\mathcal{X}}\bar{\zeta} = \nabla_{\mathcal{X}}\bar{\zeta} + \frac{1}{2}\bar{\varphi}\mathcal{X}, \tag{3.5}$$

$$(\widetilde{\nabla}_{\mathcal{X}}\bar{\eta})\mathcal{Y} = (\nabla_{\mathcal{X}}\bar{\eta})\mathcal{Y}. \tag{3.6}$$

Now putting $\mathcal{X} = \bar{\zeta}$ in (3.3), we find

$$(\widetilde{\nabla}_{\bar{\zeta}}g_1)(\mathcal{Y}, \mathcal{Z}) = 0. \tag{3.7}$$

Thus, we have the following:

Theorem 3.1. *Co-variant differentiation of Riemannian metric g_1 with respect to $\bar{\zeta}$ vanish identically in a contact metric manifold equipped with quarter-symmetric non-metric connection $\widetilde{\nabla}$.*

The curvature tensor $\widetilde{\mathcal{R}}$ endowed with connection $\widetilde{\nabla}$ is defined as

$$\widetilde{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{Z} = \widetilde{\nabla}_{\mathcal{X}}\widetilde{\nabla}_{\mathcal{Y}}\mathcal{Z} - \widetilde{\nabla}_{\mathcal{Y}}\widetilde{\nabla}_{\mathcal{X}}\mathcal{Z} - \widetilde{\nabla}_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z}. \tag{3.8}$$

Using (3.1) in (3.8), we get

$$\begin{aligned} \widetilde{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{Z} \;=\;& \mathcal{R}(\mathcal{X}, \mathcal{Y})\mathcal{Z} + \frac{1}{2}[(\nabla_{\mathcal{X}}\bar{\eta})\mathcal{Z}.\bar{\varphi}\mathcal{Y} - (\nabla_{\mathcal{X}}\bar{\eta})\mathcal{Y}.\bar{\varphi}\mathcal{Z} \\ & -(\nabla_{\mathcal{Y}}\bar{\eta})\mathcal{Z}.\bar{\varphi}\mathcal{X} + (\nabla_{\mathcal{Y}}\bar{\eta})\mathcal{X}.\bar{\varphi}\mathcal{Z} + \bar{\eta}(\mathcal{Z})(\nabla_{\mathcal{X}}\bar{\varphi})\mathcal{Y} \\ & -\bar{\eta}(\mathcal{Y})(\nabla_{\mathcal{X}}\bar{\varphi})\mathcal{Z} - \bar{\eta}(\mathcal{Z})(\nabla_{\mathcal{Y}}\bar{\varphi})\mathcal{X} + \bar{\eta}(\mathcal{X})(\nabla_{\mathcal{Y}}\bar{\varphi})\mathcal{Z}] \\ & +\frac{1}{4}\left[\bar{\eta}(\mathcal{X})\bar{\eta}(\mathcal{Z})\mathcal{Y} - \bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{Z})\mathcal{X}\right]. \end{aligned} \tag{3.9}$$

Using (2.2) and (2.4), we have

$$\begin{aligned} \widetilde{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{Z} \;=\;& \mathcal{R}(\mathcal{X}, \mathcal{Y})\mathcal{Z} + \frac{1}{2}[2g_1(\mathcal{Y}, \bar{\varphi}\mathcal{X})\bar{\varphi}\mathcal{Z} + g_1(\mathcal{X}, \bar{\varphi}\mathcal{Z})\bar{\varphi}\mathcal{Y} \\ & -g_1(\mathcal{Y}, \bar{\varphi}\mathcal{Z})\bar{\varphi}\mathcal{X} + \bar{\eta}(\mathcal{X})g_1(\mathcal{Y}, \mathcal{Z})\bar{\zeta} - \bar{\eta}(\mathcal{Y})g_1(\mathcal{X}, \mathcal{Z})\bar{\zeta}] \\ & +\frac{1}{4}\left[\bar{\eta}(\mathcal{X})\bar{\eta}(\mathcal{Z})\mathcal{Y} - \bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{Z})\mathcal{X}\right]. \end{aligned} \tag{3.10}$$

Contracting (3.10) along $\mathcal{X}$, we have

$$\widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{Z}) \;=\; \mathcal{S}(\mathcal{Y}, \mathcal{Z}) - \frac{n}{2}\bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{Z}). \tag{3.11}$$

The above equation yields

$$\widetilde{\mathcal{Q}}\mathcal{Y} \;=\; \mathcal{Q}\mathcal{Y} - \frac{n}{2}\bar{\eta}(\mathcal{Y}). \tag{3.12}$$

Again contracting (3.11), we have

$$\widetilde{\tau} \;=\; \tau - \frac{n}{2}. \tag{3.13}$$

Throughout the study, $\mathcal{R}; \widetilde{\mathcal{R}}$, $\mathcal{S}; \widetilde{\mathcal{S}}$, $\mathcal{Q}; \widetilde{\mathcal{Q}}$ and $\tau; \widetilde{\tau}$ are used for the curvature tensor, Ricci tensor, Ricci operator and scalar curvature with Levi-Civita connection ∇ and quarter-symmetric non-metric connection $\widetilde{\nabla}$ respectively.

Replacing $\mathcal{X}$ by $\bar{\zeta}$ in (3.10) and using (2.1), we get

$$\widetilde{\mathcal{R}}(\bar{\zeta},\mathcal{Y})\mathcal{Z} = \frac{3}{2}g_1(\mathcal{Y},\mathcal{Z})\bar{\zeta} - \frac{3}{4}\left[\bar{\eta}(\mathcal{Z})\mathcal{Y} + \bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{Z})\bar{\zeta}\right]. \tag{3.14}$$

Again replacing $\mathcal{Z}$ by $\bar{\zeta}$ in (3.10) and using (2.1), we get

$$\begin{aligned} \widetilde{\mathcal{R}}(\mathcal{X},\mathcal{Y})\bar{\zeta} &= \frac{3}{4}\mathcal{R}(\mathcal{X},\mathcal{Y})\bar{\zeta} \\ &= \frac{3}{4}\left[\bar{\eta}(\mathcal{Y})\mathcal{X} - \bar{\eta}(\mathcal{X})\mathcal{Y}\right]. \end{aligned} \tag{3.15}$$

In view of (2.1), (3.10) and $g_1(\mathcal{R}(\mathcal{X},\mathcal{Y},\mathcal{Z}),\mathcal{U}) = -g_1(\mathcal{R}(\mathcal{X},\mathcal{Y},\mathcal{U}),\mathcal{Z})$, we have

$$\bar{\eta}(\widetilde{\mathcal{R}}(\mathcal{X},\mathcal{Y})\mathcal{Z}) = \frac{3}{4}\left[g_1(\mathcal{Y},\mathcal{Z})\bar{\eta}(\mathcal{X}) - g_1(\mathcal{X},\mathcal{Z})\bar{\eta}(\mathcal{Y})\right]. \tag{3.16}$$

Contracting (3.15), we get

$$\widetilde{\mathcal{S}}(\mathcal{Y},\bar{\zeta}) = \frac{3n}{2}\bar{\eta}(\mathcal{Y}). \tag{3.17}$$

If Riemannian curvature tensor equipped with $\widetilde{\nabla}$ vanish, then as a result of contraction, the equation (3.10) yields

$$\mathcal{S}(\mathcal{Y},\mathcal{Z}) = \frac{n}{2}\bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{Z}). \tag{3.18}$$

Again contracting (3.18), we get

$$\tau = \frac{n}{2}. \tag{3.19}$$

Which leads to the following theorem:

THEOREM 3.2. *The scalar curvature is constant if Riemannian curvature tensor endowed with $\widetilde{\nabla}$ vanishes identically.*

4. Ricci soliton on $(\Omega, \bar{\varphi}, \bar{\zeta}, \bar{\eta}, g_1)$ with quarter-symmetric non-metric connection

Ricci soliton $(g_1, \bar{\zeta}, \lambda)$ with respect to $\widetilde{\nabla}$ [11] has given as

$$(\widetilde{\mathcal{L}}_{\bar{\zeta}}g_1)(\mathcal{X},\mathcal{Y}) + 2\widetilde{\mathcal{S}}(\mathcal{X},\mathcal{Y}) + 2\lambda g_1(\mathcal{X},\mathcal{Y}) = 0. \tag{4.1}$$

Using (2.1), (2.3) and (3.1), we have

$$\begin{aligned} (\widetilde{\mathcal{L}}_{\bar{\zeta}}g_1)(\mathcal{X},\mathcal{Y}) &= g_1(\widetilde{\nabla}_{\mathcal{X}}\bar{\zeta},\mathcal{Y}) + g_1(\mathcal{X},\widetilde{\nabla}_{\mathcal{Y}}\bar{\zeta}) \\ &= 0. \end{aligned} \tag{4.2}$$

Using (4.2) in (4.1) leads to

$$\widetilde{\mathcal{S}}(\mathcal{X},\mathcal{Y}) = -\lambda g_1(\mathcal{X},\mathcal{Y}) \tag{4.3}$$

and

$$\widetilde{Q}\mathcal{X} = -\lambda\mathcal{X}. \tag{4.4}$$

Replacing $\mathcal{Y}$ by $\bar{\zeta}$ in (4.3), we get

$$\widetilde{\mathcal{S}}(\mathcal{X},\xi) = -\lambda\bar{\eta}(\mathcal{X}). \tag{4.5}$$

Again contracting (4.3), we have

$$\widetilde{\tau} = -\lambda(2n+1). \tag{4.6}$$

In view (3.11) and (4.3) one can easily write

$$\mathcal{S}(\mathcal{X},\mathcal{Y}) = \lambda g_1(\mathcal{X},\mathcal{Y}) + \frac{n}{2}\bar{\eta}(\mathcal{X})\bar{\eta}(\mathcal{Y}). \tag{4.7}$$

Thus, we have the following theorem:

Theorem 4.1. *If $(g_1,\bar{\zeta},\lambda)$ is a Ricci soliton on manifold $\Omega^{(2n+1)}$ equipped with $\widetilde{\nabla}$, then $\Omega^{(2n+1)}$ is an η-Einstein manifold with ∇.*

Replacing $\mathcal{Y}$ by $\bar{\zeta}$ in (4.7), we get

$$\mathcal{S}(\mathcal{X},\bar{\zeta}) = -\left(\lambda - \frac{n}{2}\right)\bar{\eta}(\mathcal{X}). \tag{4.8}$$

From (2.8) and (4.8), we find

$$\lambda = -\frac{3n}{2} < 0.$$

Thus, we have the following corollary:

Corollary 4.2. *A triplet $(g_1,\bar{\zeta},\lambda)$ on Sasakian manifold $\Omega^{(2n+1)}$ endowed with $\widetilde{\nabla}$ is always shrinking.*

Let $(g_1,\mathcal{V},\lambda)$ be a Ricci soliton on Sasakian manifold $\Omega^{(2n+1)}$ with connection $\widetilde{\nabla}$ such that $\mathcal{V}$ is pointwise collinear with $\bar{\zeta}$ i.e. $\mathcal{V} = b\bar{\zeta}$, where b is a function, then

$$\begin{aligned} &bg_1(\widetilde{\nabla}_{\mathcal{X}}\bar{\zeta},\mathcal{Y}) + (\mathcal{X}b)\bar{\eta}(\mathcal{Y}) + bg_1(\mathcal{X},\widetilde{\nabla}_{\mathcal{Y}}\bar{\zeta}) \\ &+(\mathcal{Y}b)\bar{\eta}(\mathcal{X}) + 2\widetilde{\mathcal{S}}(\mathcal{X},\mathcal{Y}) + 2\lambda g_1(\mathcal{X},\mathcal{Y}) = 0. \end{aligned} \tag{4.9}$$

On replacing $\mathcal{Y}$ by $\bar{\zeta}$ in (4.9) and using (2.1), (2.3), (3.1) and (3.17), we get

$$(\mathcal{X}b) + (\bar{\zeta}b)\bar{\eta}(\mathcal{X}) + (2\lambda + 3n)\bar{\eta}(\mathcal{X}) = 0. \tag{4.10}$$

Putting $\mathcal{X} = \bar{\zeta}$ in (4.10), we have

$$\bar{\zeta}b + \lambda + \frac{3n}{2} = 0. \tag{4.11}$$

Equations (4.11) and (4.10) yields

$$db = -\left[\lambda + \frac{3n}{2}\right]\bar{\eta}. \tag{4.12}$$

Applying d on (4.12), we find

$$-\left[\lambda + \frac{3n}{2}\right] d\bar{\eta} \quad = \quad 0. \tag{4.13}$$

Since $d\bar{\eta} \neq 0$, we have

$$\lambda \quad = \quad -\frac{3n}{2} < 0. \tag{4.14}$$

From (4.12) and (4.14), we have

$$db = 0. \tag{4.15}$$

Hence $'b'$ is constant.
As a result, we have the following theorem:

THEOREM 4.3. *A $(g, \bar{\zeta}, \lambda)$ on Sasakian manifold equipped with $\widetilde{\nabla}$ such that $\mathcal{V}$ is pointwise collinear with $\bar{\zeta}$, then $\mathcal{V}$ is constant multiple of $\bar{\zeta}$ and soliton is shrinking.*

5. Ricci soliton on Sasakian manifold endowed with $\widetilde{\nabla}$ satisfying $\widetilde{\mathcal{P}}.\widetilde{\mathcal{S}} = 0$

Let us consider that the manifold $\Omega^{(2n+1)}$ equipped with quarter-symmetric non-metric connection $\widetilde{\nabla}$ satisfying the condition

$$(\widetilde{\mathcal{P}}(\mathcal{X}, \mathcal{Y}).\widetilde{\mathcal{S}})(\mathcal{Z}, \mathcal{U}) \quad = \quad 0, \tag{5.1}$$

then, we have

$$\widetilde{\mathcal{S}}(\widetilde{\mathcal{P}}(\mathcal{X}, \mathcal{Y})\mathcal{Z}, \mathcal{U}) + \widetilde{\mathcal{S}}(\mathcal{Z}, \widetilde{\mathcal{P}}(\mathcal{X}, \mathcal{Y})\mathcal{U}) \quad = \quad 0. \tag{5.2}$$

Replacing $\mathcal{X}$ by $\bar{\zeta}$ in (5.2), we have

$$\widetilde{\mathcal{S}}(\widetilde{\mathcal{P}}(\bar{\zeta}, \mathcal{Y})\mathcal{Z}, \mathcal{U}) + \widetilde{\mathcal{S}}(\mathcal{Z}, \widetilde{\mathcal{P}}(\bar{\zeta}, \mathcal{Y})\mathcal{U}) \quad = \quad 0. \tag{5.3}$$

Putting $\mathcal{X} = \bar{\zeta}$ in (2.12), we have

$$\widetilde{\mathcal{P}}(\bar{\zeta}, \mathcal{Y})\mathcal{Z} = \widetilde{\mathcal{R}}(\bar{\zeta}, \mathcal{Y})\mathcal{Z} - \frac{1}{2n}\left[\widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{Z})\bar{\zeta} - \widetilde{\mathcal{S}}(\bar{\zeta}, \mathcal{Z})\mathcal{Y}\right]. \tag{5.4}$$

Using (3.14), (3.17), (5.4) in (5.3), we get

$$\begin{aligned} &\frac{9n}{4} g_1(\mathcal{Y}, \mathcal{Z})\bar{\eta}(\mathcal{U}) + \frac{9n}{4} g_1(\mathcal{Y}, \mathcal{U})\bar{\eta}(\mathcal{Z}) - \frac{3}{4}\bar{\eta}(\mathcal{Z})\widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{U}) \\ &\qquad -\frac{3}{4}\bar{\eta}(\mathcal{U})\widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{Z}) - \frac{9n}{4}\bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{Z})\bar{\eta}(\mathcal{U}) = 0. \end{aligned} \tag{5.5}$$

Now putting $\mathcal{U} = \bar{\zeta}$ in (5.5) and then using (2.1) and (3.17), we obtain

$$\widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{Z}) = 3n g_1(\mathcal{Y}, \mathcal{Z}) - \frac{3n}{2}\bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{Z}). \tag{5.6}$$

Now from (4.1), (4.2) and (5.6), we get

$$\begin{aligned} & g_1(\widetilde{\nabla}_{\mathcal{X}}\bar{\zeta}, \mathcal{Y}) + g_1(\mathcal{X}, \widetilde{\nabla}_{\mathcal{Y}}\bar{\zeta}) \\ & + 2[(3n + \lambda)g_1(\mathcal{X}, \mathcal{Y}) - \frac{3n}{2}\bar{\eta}(\mathcal{X})\bar{\eta}(\mathcal{Y})] = 0. \end{aligned} \tag{5.7}$$

Putting $\mathcal{X} = \mathcal{Y} = \bar{\zeta}$ in (5.7) and then using (2.1) and (3.5), we obtain $g_1(\nabla_{\bar{\zeta}}\bar{\zeta}, \bar{\zeta}) = -\left(\lambda + \frac{3n}{2}\right)$. Since $g_1(\nabla_{\mathcal{X}}\bar{\zeta}, \bar{\zeta}) = 0$ for any vector field $\mathcal{X}$ on $\Omega^{(2n+1)}$ and $\bar{\zeta}$ has a constant term, then we have

$$\lambda = -\frac{3n}{2}. \tag{5.8}$$

Consequently (5.7) reduces to

$$g_1(\widetilde{\nabla}_{\mathcal{X}}\bar{\zeta}, \mathcal{Y}) + g_1(\mathcal{X}, \widetilde{\nabla}_{\mathcal{Y}}\bar{\zeta}) + 3n[g_1(\mathcal{X}, \mathcal{Y}) - \bar{\eta}(\mathcal{X})\bar{\eta}(\mathcal{Y})] = 0. \tag{5.9}$$

Putting $\mathcal{X} = \bar{\zeta}$ in (5.9) and then using (2.1), (2.3) and (3.5), we have

$$g_1(\nabla_{\bar{\zeta}}\bar{\zeta}, \mathcal{Y}) = 0, \tag{5.10}$$

$\forall\ \mathcal{Y}$ on $\Omega^{(2n+1)}$. Hence, one can write

$$\nabla_{\bar{\zeta}}\bar{\zeta} = 0, \tag{5.11}$$

i.e. $\bar{\zeta}$ is a geodesic vector field. Hence we have the following theorem:

THEOREM 5.1. *If a Ricci soliton on Sasakian manifold endowed with $\widetilde{\nabla}$ satisfy $\widetilde{\mathcal{P}}.\widetilde{\mathcal{S}} = 0$, then*

1. *$(\Omega^{(2n+1)}, g_1)$ is an η-Einstein manifold.*
2. *The Ricci soliton is shrinking.*
3. *$\bar{\zeta}$ is a geodesic vector field.*

6. Ricci soliton on Sasakian manifold endowed with $\widetilde{\nabla}$ satisfying $\widetilde{\mathcal{R}}.\widetilde{\mathcal{S}} = 0$ and $\widetilde{\mathcal{S}}.\widetilde{\mathcal{R}} = 0$

Let us consider $\widetilde{\mathcal{R}}.\widetilde{\mathcal{S}} = 0$, then

$$\widetilde{\mathcal{S}}(\widetilde{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{Z}, \mathcal{U}) + \widetilde{\mathcal{S}}(\mathcal{Z}, \widetilde{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{U}) = 0. \tag{6.1}$$

Putting $\mathcal{X} = \bar{\zeta}$ in (6.1), we have

$$\widetilde{\mathcal{S}}(\widetilde{\mathcal{R}}\left(\bar{\zeta}, \mathcal{Y}\right)\mathcal{Z}, \mathcal{U}) + \widetilde{\mathcal{S}}(\mathcal{Z}, \widetilde{\mathcal{R}}\left(\bar{\zeta}, \mathcal{Y}\right)\mathcal{U}) = 0. \tag{6.2}$$

Using (3.14) and (3.17) in (6.2), we have

$$\begin{aligned} 3ng_1(\mathcal{Y}, \mathcal{Z})\bar{\eta}(\mathcal{U}) + 3ng_1(\mathcal{Y}, \mathcal{U})\bar{\eta}(\mathcal{Z}) - \bar{\eta}(\mathcal{Z})\widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{U}) \\ -\bar{\eta}(\mathcal{U})\widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{Z}) - 3n\bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{Z})\bar{\eta}(\mathcal{U}) = 0. \end{aligned} \tag{6.3}$$

Again putting $\mathcal{Z} = \bar{\zeta}$ in (6.3) and using (2.1), (3.17), we obtain

$$\widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{U}) = 3ng_1(\mathcal{Y}, \mathcal{U}) - \frac{3n}{2}\bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{U}). \tag{6.4}$$

Thus, we have the following theorem:

THEOREM 6.1. *A manifold $\Omega^{(2n+1)}$ endowed with $\widetilde{\nabla}$, which satisfy $\widetilde{\mathcal{R}}.\widetilde{\mathcal{S}} = 0$, is an η-Einstein manifold.*

Putting $\mathcal{Y} = \mathcal{U} = \bar{\zeta}$ in (6.4) and using (4.5), then we have the following corollary:

COROLLARY 6.2. *If a triplet $(g_1, \bar{\zeta}, \lambda)$ of Ricci soliton on Sasakian manifold with $\widetilde{\nabla}$ satisfy $\widetilde{\mathcal{R}}.\widetilde{\mathcal{S}} = 0$, then the Ricci soliton is shrinking.*

Now under the condition $\widetilde{\mathcal{S}}.\widetilde{\mathcal{R}} = 0$, we have

$$\begin{aligned}(\mathcal{X} \wedge_{\widetilde{\mathcal{S}}} \mathcal{Y})\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{Z} + \widetilde{\mathcal{R}}((\mathcal{X} \wedge_{\widetilde{\mathcal{S}}} \mathcal{Y})\mathcal{U},\mathcal{V})\mathcal{Z}\\ +\widetilde{\mathcal{R}}(\mathcal{U},(\mathcal{X} \wedge_{\widetilde{\mathcal{S}}} \mathcal{Y})\mathcal{V})\mathcal{Z} + \widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})(\mathcal{X} \wedge_{\widetilde{\mathcal{S}}} \mathcal{Y})\mathcal{Z} = 0,\end{aligned} \tag{6.5}$$

where the endomorphism $(\mathcal{X} \wedge_{\widetilde{\mathcal{S}}} \mathcal{Y})$ is defined as

$$(\mathcal{X} \wedge_{\widetilde{\mathcal{S}}} \mathcal{Y})\mathcal{Z} \quad = \quad \widetilde{\mathcal{S}}(\mathcal{Y},\mathcal{Z})\mathcal{X} - \widetilde{\mathcal{S}}(\mathcal{X},\mathcal{Z})\mathcal{Y}. \tag{6.6}$$

Taking $\mathcal{Y} = \bar{\zeta}$ in (6.5), we have

$$\begin{aligned}(\mathcal{X} \wedge_{\widetilde{\mathcal{S}}} \bar{\zeta})\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{Z} + \widetilde{\mathcal{R}}((\mathcal{X} \wedge_{\widetilde{\mathcal{S}}} \bar{\zeta})\mathcal{U},\mathcal{V})\mathcal{Z}\\ \widetilde{\mathcal{R}}(\mathcal{U},(\mathcal{X} \wedge_{\widetilde{\mathcal{S}}} \bar{\zeta})\mathcal{V})\mathcal{Z} + \widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})(\mathcal{X} \wedge_{\widetilde{\mathcal{S}}} \bar{\zeta})\mathcal{Z} = 0.\end{aligned} \tag{6.7}$$

In view of (6.6), (6.7), we have

$$\begin{aligned}\widetilde{\mathcal{S}}(\bar{\zeta},\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{Z})\mathcal{X} - \widetilde{\mathcal{S}}(\mathcal{X},\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{Z})\bar{\zeta} + \widetilde{\mathcal{S}}(\bar{\zeta},\mathcal{U})\widetilde{\mathcal{R}}(\mathcal{X},\mathcal{V})\mathcal{Z}\\ -\widetilde{\mathcal{S}}(\mathcal{X},\mathcal{U})\widetilde{\mathcal{R}}(\bar{\zeta},\mathcal{V})\mathcal{Z} + \widetilde{\mathcal{S}}(\bar{\zeta},\mathcal{V})\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{X})\mathcal{Z} - \widetilde{\mathcal{S}}(\mathcal{X},\mathcal{V})\widetilde{\mathcal{R}}(\mathcal{U},\bar{\zeta})\\ +\widetilde{\mathcal{S}}(\bar{\zeta},\mathcal{Z})\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{X} - \widetilde{\mathcal{S}}(\mathcal{X},\mathcal{Z})\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\bar{\zeta} = 0.\end{aligned} \tag{6.8}$$

The equations (3.17) and (6.8) lead to

$$\begin{aligned}\frac{3n}{2}[\bar{\eta}(\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{Z})\mathcal{X} + \bar{\eta}(\mathcal{U})\widetilde{\mathcal{R}}(\mathcal{X},\mathcal{V})\mathcal{Z} + \bar{\eta}(\mathcal{V})\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{X})\mathcal{Z}\\ +\bar{\eta}(\mathcal{Z})\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{X}] - \widetilde{\mathcal{S}}(\mathcal{X},\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{Z})\bar{\zeta} - \widetilde{\mathcal{S}}(\mathcal{X},\mathcal{U})\widetilde{\mathcal{R}}(\bar{\zeta},\mathcal{V})\mathcal{Z}\\ -\widetilde{\mathcal{S}}(\mathcal{X},\mathcal{V})\widetilde{\mathcal{R}}(\mathcal{U},\bar{\zeta})\mathcal{Z} - \widetilde{\mathcal{S}}\mathcal{X},\mathcal{Z})(\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\bar{\zeta} = 0.\end{aligned} \tag{6.9}$$

Taking inner product with $\bar{\zeta}$ results

$$\begin{aligned}\frac{3n}{2}[\bar{\eta}(\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{Z})\bar{\eta}(\mathcal{X}) + \bar{\eta}(\mathcal{U})\bar{\eta}(\widetilde{\mathcal{R}}(\mathcal{X},\mathcal{V})\mathcal{Z})\\ +\bar{\eta}(\mathcal{V})\bar{\eta}(\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{X})\mathcal{Z}) + \bar{\eta}(\mathcal{Z})\eta(\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{X})]\\ -\widetilde{\mathcal{S}}(\mathcal{X},\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\mathcal{Z}) - \widetilde{\mathcal{S}}(\mathcal{X},\mathcal{U})\bar{\eta}(\widetilde{\mathcal{R}}(\bar{\zeta},\mathcal{V})\mathcal{Z})\\ -\widetilde{\mathcal{S}}(\mathcal{X},\mathcal{V})\bar{\eta}(\widetilde{\mathcal{R}}(\mathcal{U},\bar{\zeta})\mathcal{Z}) - \widetilde{\mathcal{S}}(\mathcal{X},\mathcal{Z})\bar{\eta}(\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{V})\bar{\zeta}) = 0.\end{aligned} \tag{6.10}$$

Putting $\mathcal{U} = \mathcal{Z} = \bar{\zeta}$ and then using (3.14), (3.15), (3.16) and (3.17), we get

$$\widetilde{\mathcal{S}}(\mathcal{X},\mathcal{V}) = -\frac{3n}{2}g_1(\mathcal{X},\mathcal{V}) + \frac{9n}{2}\bar{\eta}(\mathcal{X})\bar{\eta}(\mathcal{V}). \tag{6.11}$$

So, we have the following theorem:

Theorem 6.3. *A Sasakian manifold endowed with* $\widetilde{\nabla}$ *together with* $\widetilde{\mathcal{S}}.\widetilde{\mathcal{R}} = 0$*, is an* η*-Einstein manifold.*

Replacing $\mathcal{V}$ by $\bar{\zeta}$ in (6.11) and using (4.5), we have

$$\lambda = -3n \tag{6.12}$$

Corollary 6.4. *If a Ricci soliton* $(g_1, \bar{\zeta}, \lambda)$ *on Sasakian manifold with* $\widetilde{\nabla}$ *satisfy* $\widetilde{\mathcal{S}}.\widetilde{\mathcal{R}} = 0$*, then the Ricci soliton is shrinking.*

7. Ricci soliton on ricci-recurrent and φ-recurrent Sasakian manifold with quarter-symmetric non-metric connection

Suppose a manifold $\Omega^{(2n+1)}$ equipped with $\widetilde{\nabla}$ is Ricci-recurrent with respect to $\widetilde{\nabla}$ i.e. $\Omega^{(2n+1)}$ possesses a Ricci tensor $\widetilde{\mathcal{S}}$ satisfies

$$(\widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{S}})(\mathcal{Y}, \mathcal{Z}) \quad = \quad \mathcal{A}(\mathcal{U})\widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{Z}). \tag{7.1}$$

Setting $\mathcal{Z} = \bar{\zeta}$ in (7.1), we have

$$(\widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{S}})(\mathcal{Y}, \bar{\zeta}) \quad = \quad \mathcal{A}(\mathcal{U})\widetilde{\mathcal{S}}(\mathcal{Y}, \bar{\zeta}). \tag{7.2}$$

Now using (3.17) in (7.2), we get

$$(\widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{S}})(\mathcal{Y}, \bar{\zeta}) \quad = \quad \frac{3n}{2}\bar{\eta}(\mathcal{Y})\mathcal{A}(\mathcal{U}). \tag{7.3}$$

It is well known that

$$(\widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{S}})(\mathcal{Y}, \bar{\zeta}) = \widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{S}}(\mathcal{Y}, \bar{\zeta}) - \widetilde{\mathcal{S}}(\widetilde{\nabla}_{\mathcal{U}}\mathcal{Y}, \bar{\zeta}) - \widetilde{\mathcal{S}}(\mathcal{Y}, \widetilde{\nabla}_{\mathcal{U}}\bar{\zeta}). \tag{7.4}$$

Using (2.3), (3.6) and (3.17) in (7.3), we get

$$(\widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{S}})(\mathcal{Y}, \bar{\zeta}) = \frac{3n}{2}g_1(\mathcal{U}, \bar{\varphi}\mathcal{Y}) + \frac{1}{2}\widetilde{\mathcal{S}}(\mathcal{Y}, \bar{\varphi}\mathcal{U}). \tag{7.5}$$

From (7.3) and (7.5), we have

$$\widetilde{\mathcal{S}}(\mathcal{Y}, \bar{\varphi}\mathcal{U}) = 3ng_1(\mathcal{Y}, \bar{\varphi}\mathcal{U}) + 3n\bar{\eta}(\mathcal{Y})\mathcal{A}(\mathcal{U}). \tag{7.6}$$

Putting $\mathcal{Y} = \bar{\varphi}\mathcal{Y}$ in (7.6), we obtain

$$\mathcal{S}(\mathcal{Y}, \mathcal{U}) = 3ng_1(\mathcal{Y}, \mathcal{U}) - \frac{n}{2}\bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{U}). \tag{7.7}$$

Putting $\mathcal{U} = \bar{\zeta}$ in (7.7) and using (2.1), (3.17), we get

$$-(\lambda - \frac{n}{2})\bar{\eta}(\mathcal{Y}) = 3n\bar{\eta}(\mathcal{Y}) - \frac{n}{2}\bar{\eta}(\mathcal{Y}). \tag{7.8}$$

Consequently, we have

$$\lambda = -2n. \tag{7.9}$$

Thus, we have the following theorem:

THEOREM 7.1. *The Ricci soliton* $(g_1, \bar{\zeta}, \lambda)$ *on ricci-recurrent Sasakian manifold admitting* $\widetilde{\nabla}$ *is shrinking and the manifold* $(\Omega^{(2n+1)}, g_1)$ *is an* η*-Einstein manifold.*

Now, we discuss the φ-recurrent Sasakian manifold $\Omega^{(2n+1)}$ equipped with $\widetilde{\nabla}$:

$$\bar{\varphi}^2((\widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{R}})(\mathcal{X}, \mathcal{Y})\mathcal{Z}) = \mathcal{A}(\mathcal{U})\widetilde{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{Z}. \tag{7.10}$$

By virtue of (2.1) and (7.10), we have

$$\begin{aligned} -g_1((\widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{R}})(\mathcal{X}, \mathcal{Y})\mathcal{Z}, \mathcal{Z}) + \bar{\eta}((\widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{R}})(\mathcal{X}, \mathcal{Y})\mathcal{Z})\bar{\eta}(\mathcal{Z}) \\ = \mathcal{A}(\mathcal{U})g_1(\widetilde{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{Z}, \mathcal{Z}). \end{aligned} \tag{7.11}$$

Contracting (7.11) with $\mathcal{X}$ and $\mathcal{Z}$, we find

$$-(\widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{S}})(\mathcal{Y}, \mathcal{Z}) = \mathcal{A}(\mathcal{U})\widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{Z}). \tag{7.12}$$

Replacing $\mathcal{Z}$ by $\bar{\zeta}$ in (7.12) leads to

$$-(\widetilde{\nabla}_{\mathcal{U}}\widetilde{\mathcal{S}})(\mathcal{Y}, \bar{\zeta}) = \mathcal{A}(\mathcal{U})\widetilde{\mathcal{S}}(\mathcal{Y}, \bar{\zeta}). \tag{7.13}$$

In view of (3.17), (7.5) and (7.13), we have

$$\widetilde{\mathcal{S}}(\mathcal{Y}, \bar{\varphi}\mathcal{U}) = 3ng_1(\mathcal{Y}, \bar{\varphi}\mathcal{U}) - 3n\bar{\eta}(\mathcal{Y})\mathcal{A}(\mathcal{U}). \tag{7.14}$$

In light of $\mathcal{Y} = \bar{\varphi}\mathcal{Y}$ the above equation leads to

$$\mathcal{S}(\mathcal{Y}, \mathcal{U}) = 3ng_1(\mathcal{Y}, \mathcal{U}) - \frac{n}{2}\bar{\eta}(\mathcal{Y})\eta(\mathcal{U}). \tag{7.15}$$

Thus, we have the following theorem:

THEOREM 7.2. *A* φ*-recurrent Sasakian manifold* $\Omega^{(2n+1)}$ *endowed with* $\widetilde{\nabla}$ *is an* η*-Einstein manifold.*

8. Solitons on Einstein Semi-symmetric Sasakian manifold with quarter-symmetric non-metric connection

Let us consider Einstein semi-symmetric Sasakian manifold $\Omega^{(2n+1)}$ endowed with $\widetilde{\nabla}$

$$(\widetilde{\mathcal{R}}(\mathcal{X}, \mathcal{Y}).\widetilde{E})(\mathcal{Z}, \mathcal{U}) = 0. \tag{8.1}$$

Above equation can be rewritten as

$$\widetilde{E}(\widetilde{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{Z}, \mathcal{U}) + \widetilde{E}(\mathcal{Z}, \widetilde{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\mathcal{U}) = 0. \tag{8.2}$$

In light of

$$\widetilde{E}(\mathcal{Y}, \mathcal{Z}) = \widetilde{\mathcal{S}}(\mathcal{Y}, \mathcal{Z}) - \frac{\widetilde{\tau}}{n}g_1(\mathcal{Y}, \mathcal{Z}), \tag{8.3}$$

the equation (8.2) can be rewritten as

$$\begin{aligned}&\widetilde{\mathcal{S}}(\widetilde{\mathcal{R}}(\mathcal{X},\mathcal{Y})\mathcal{Z},\mathcal{U})+\widetilde{\mathcal{S}}(\mathcal{Z},\widetilde{\mathcal{R}}(\mathcal{X},\mathcal{Y})\mathcal{U})\\ &=\frac{\widetilde{\tau}}{n}[g_1(\widetilde{\mathcal{R}}(\mathcal{X},\mathcal{Y})\mathcal{Z},\mathcal{U})+g_1(\mathcal{Z},\widetilde{\mathcal{R}}(\mathcal{X},\mathcal{Y})\mathcal{U})].\end{aligned} \tag{8.4}$$

Putting $\mathcal{X}=\bar{\zeta}$ in (8.4) leads to

$$\begin{aligned}&\widetilde{\mathcal{S}}(\widetilde{\mathcal{R}}\left(\bar{\zeta},\mathcal{Y}\right)\mathcal{Z},\mathcal{U})+\widetilde{\mathcal{S}}(\mathcal{Z},\widetilde{\mathcal{R}}\left(\bar{\zeta},\mathcal{Y}\right)\mathcal{U})\\ &=\frac{\widetilde{\tau}}{n}[g_1(\widetilde{\mathcal{R}}(\bar{\zeta},\mathcal{Y})\mathcal{Z},\mathcal{U})+g_1(\mathcal{Z},\widetilde{\mathcal{R}}(\bar{\zeta},\mathcal{Y})\mathcal{U})].\end{aligned} \tag{8.5}$$

In light of (3.14) and (3.17) the above equation (8.5) can be written as

$$\begin{aligned}&\frac{9n}{4}g_1(\mathcal{Y},\mathcal{Z})\bar{\eta}(\mathcal{U})-\frac{3}{4}\bar{\eta}(\mathcal{Z})\widetilde{\mathcal{S}}(\mathcal{Y},\mathcal{U})-\frac{9}{4}\bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{U})\bar{\eta}(\mathcal{Z})\\ &+\frac{9n}{4}g_1(\mathcal{Y},\mathcal{U})\bar{\eta}(\mathcal{Z})-\frac{3}{4}\bar{\eta}(\mathcal{U})\widetilde{\mathcal{S}}(\mathcal{Y},\mathcal{Z})\\ &=\frac{\widetilde{\tau}}{n}[\frac{3}{4}g_1(\mathcal{Y},\mathcal{Z})\bar{\eta}(\mathcal{U})+\frac{3}{4}g_1(\mathcal{Y},\mathcal{U})\bar{\eta}(\mathcal{Z})-\frac{3}{2}\bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{Z})\bar{\eta}(\mathcal{U})].\end{aligned} \tag{8.6}$$

Putting $\mathcal{Z}=\bar{\zeta}$ in (8.6) and taking reference of (3.13), (3.17) and (3.19), we obtain

$$\widetilde{\mathcal{S}}(\mathcal{Y},\mathcal{U})=3ng_1(\mathcal{Y},\mathcal{U})-\frac{3n}{2}\bar{\eta}(\mathcal{Y})\bar{\eta}(\mathcal{U}). \tag{8.7}$$

Replacing $\mathcal{U}=\bar{\zeta}$ in (8.7) and using (4.5), we have

$$\lambda=-\frac{3n}{2}. \tag{8.8}$$

This leads to the following theorem:

THEOREM 8.1. *Every Einstein semi-symmetric Sasakian manifold $\Omega^{(2n+1)}$ is an η-Einstein manifold and the triplet $(g_1,\bar{\zeta},\lambda)$ is always shrinking.*

9. Example

Let us take a 3-dimensional manifold $\Omega^{2n+1}=[(x,y,z)\in\mathbb{R}^3|z\neq 0]$; where (x,y,z) are the cartesian coordinates in $\mathbb{R}^3$. Choosing the vector fields

$$v_1=x(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})-2y\frac{\partial}{\partial z},\quad v_2=\frac{\partial}{\partial y},\quad v_3=\frac{\partial}{\partial z}=\bar{\zeta},$$

where v_1,v_2 and v_3 are linearly independent. Let us define a Riemannian metric g_1 as

$$\begin{aligned}&g_1(v_1,v_2)=g_1(v_2,v_3)=g_1(v_3,v_1)=0,\\ &g_1(v_1,v_1)=g_1(v_2,v_2)=g_1(v_3,v_3)=1,\end{aligned} \tag{9.1}$$

and $\bar{\varphi}$ is defined as

$$\bar{\varphi}(v_1) = v_2, \bar{\varphi}(v_2) = -v_1, \bar{\varphi}(v_3) = 0. \tag{9.2}$$

The definition of Lie bracket provides

$$[v_1, v_2] = 2v_3, \qquad [v_1, v_3] = 0, \qquad [v_2, v_3] = 0. \tag{9.3}$$

One can also find

$$\begin{aligned} 2g_1(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) &= \mathcal{X}g_1(\mathcal{Y}, \mathcal{Z}) + \mathcal{Y}g_1(\mathcal{Z}, \mathcal{X}) - \mathcal{Z}g_1(\mathcal{X}, \mathcal{Y}) \\ &+ g_1([\mathcal{X}, \mathcal{Y}], \mathcal{Z}) - g_1([\mathcal{Y}, \mathcal{Z}], \mathcal{X}) + g_1([\mathcal{Z}, \mathcal{X}], \mathcal{Y}). \end{aligned} \tag{9.4}$$

In light of Koszul's formula, we get

$$\begin{aligned} &\nabla_{v_1} v_1 = 0, \qquad \nabla_{v_1} v_2 = v_3, \quad \nabla_{v_1} v_3 = -v_2, \\ &\nabla_{v_2} v_1 = -v_3, \quad \nabla_{v_2} v_2 = 0, \quad \nabla_{v_2} v_3 = v_1, \\ &\nabla_{v_3} v_1 = -v_2, \quad \nabla_{v_3} v_2 = v_1, \quad \nabla_{v_3} v_3 = 0. \end{aligned} \tag{9.5}$$

Let us take a vector field $\mathcal{X} = \mathcal{X}^1 v_1 + \mathcal{X}^2 v_2 + \mathcal{X}^3 v_3$ and the structure vector field $\bar{\zeta} = v_3$, then we have

$$\begin{aligned} \nabla_{\mathcal{X}}\bar{\zeta} &= \nabla_{\mathcal{X}^1 v_1 + \mathcal{X}^2 v_2 + \mathcal{X}^3 v_3} v_3 \\ &= \mathcal{X}^1 \nabla_{v_1} v_3 + \mathcal{X}^2 \nabla_{v_2} v_3 + \mathcal{X}^3 \nabla_{v_3} v_3 \\ &= -\mathcal{X}^1 v_2 + \mathcal{X}^2 v_1 \end{aligned} \tag{9.6}$$

and

$$\begin{aligned} \nabla_{\mathcal{X}}\bar{\zeta} &= -\bar{\varphi}\mathcal{X} \\ &= -\bar{\varphi}(\mathcal{X}^1 v_1 + \mathcal{X}^2 v_2 + \mathcal{X}^3 v_3) \\ &= -[\mathcal{X}^1(\bar{\varphi}v_1) + \mathcal{X}^2(\bar{\varphi}v_2) + \mathcal{X}^3(\bar{\varphi}v_3)] \\ &= -\mathcal{X}^1 v_2 + \mathcal{X}^2 v_1, \end{aligned} \tag{9.7}$$

where $\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3$ are the scalars. From (9.6) and (9.7), the structure $\left(\bar{\varphi}, \bar{\zeta}, \bar{\eta}, g_1\right)$ is Sasakian structure. Hence the manifold $\Omega^3\left(\bar{\varphi}, \bar{\zeta}, \bar{\eta}, g_1\right)$ is Sasakian manifold. In reference of equations (2.1), (2.3), (3.1), (9.2) and (9.5), we have

$$\begin{aligned} &\widetilde{\nabla}_{v_1} v_1 = 0, \qquad \widetilde{\nabla}_{v_1} v_2 = v_3, \qquad \widetilde{\nabla}_{v_1} v_3 = -\frac{1}{2}v_2, \\ &\widetilde{\nabla}_{v_2} v_1 = -v_3, \quad \widetilde{\nabla}_{v_2} v_2 = 0, \qquad \widetilde{\nabla}_{v_2} v_3 = \frac{1}{2}v_1, \\ &\widetilde{\nabla}_{v_3} v_1 = -\frac{3}{2}v_2, \quad \widetilde{\nabla}_{v_3} v_2 = \frac{3}{2}v_1, \quad \widetilde{\nabla}_{v_3} v_3 = 0. \end{aligned} \tag{9.8}$$

From equations (3.2) and (3.3), we have

$$\widetilde{\mathcal{T}}(v_1, v_3) = \bar{\eta}(v_3)\bar{\varphi}v_1 - \bar{\eta}(v_1)\bar{\varphi}v_3 = v_3 \neq 0$$

and

$$\begin{aligned}(\widetilde{\nabla}_{v_1} g_1)(v_2, v_3) &= -\frac{1}{2}[\bar{\eta}(v_2)g_1(\bar{\varphi}v_1, v_3) + \bar{\eta}(v_3)g_1(\bar{\varphi}v_1, v_2)] \\ &= -\frac{1}{2} \neq 0.\end{aligned}$$

Consequently, a quarter-symmetric non-metric connection defined in (3.1) is verified. Also,

$$\begin{aligned}\widetilde{\nabla}_{\mathcal{X}}\bar{\zeta} &= \widetilde{\nabla}_{\mathcal{X}^1 v_1 + \mathcal{X}^2 v_2 + \mathcal{X}^3 v_3} v_3 \\ &= \mathcal{X}^1 \widetilde{\nabla}_{v_1} v_3 + \mathcal{X}^2 \widetilde{\nabla}_{v_2} v_3 + \mathcal{X}^3 \widetilde{\nabla}_{v_3} v_3 \\ &= -\frac{1}{2}\mathcal{X}^1 v_2 - \frac{1}{2}\mathcal{X}^2 v_1. \end{aligned} \tag{9.9}$$

The equation (3.5) can be verified in light of equations (9.7) and (9.9).
The components of $\mathcal{R}$ of ∇ are defined as

$$\begin{aligned}&\mathcal{R}(v_1, v_2)v_2 = -3v_1, \quad \mathcal{R}(v_1, v_3)v_3 = v_1, \quad \mathcal{R}(v_2, v_3)v_3 = v_2, \\ &\mathcal{R}(v_1, v_2)v_3 = 0, \quad \mathcal{R}(v_3, v_2)v_2 = v_3, \quad \mathcal{R}(v_2, v_1)v_1 = -3v_2, \\ &\mathcal{R}(v_1, v_3)v_2 = 0, \quad \mathcal{R}(v_3, v_1)v_1 = v_3, \quad \mathcal{R}(v_2, v_1)v_3 = 0,\end{aligned} \tag{9.10}$$

Hence we can verify the equations (2.5), (2.6) and (2.7).
Similarly, the component of curvature tensor $\widetilde{\mathcal{R}}$ of connection $\widetilde{\nabla}$ can be written as:

$$\begin{aligned}&\widetilde{\mathcal{R}}(v_1, v_2)v_2 = -\frac{7}{2}v_1, \quad \widetilde{\mathcal{R}}(v_1, v_3)v_3 = \frac{3}{4}v_1, \quad \widetilde{\mathcal{R}}(v_2, v_3)v_3 = \frac{3}{4}v_2, \\ &\widetilde{\mathcal{R}}(v_1, v_2)v_3 = 0, \quad \widetilde{\mathcal{R}}(v_3, v_2)v_2 = \frac{3}{2}v_3, \quad \widetilde{\mathcal{R}}(v_2, v_1)v_1 = -\frac{7}{2}v_2, \\ &\widetilde{\mathcal{R}}(v_1, v_3)v_2 = 0, \quad \widetilde{\mathcal{R}}(v_3, v_1)v_1 = \frac{3}{2}v_3, \quad \widetilde{\mathcal{R}}(v_2, v_1)v_3 = 0,\end{aligned} \tag{9.11}$$

Thus (3.10), (3.14), (3.15) and (3.16) can be varified easily.
In view of (9.10) and (9.11), the Ricci tensor equipped with the connection ∇ and $\widetilde{\nabla}$ respectively are

$$\mathcal{S}(v_1, v_1) = \mathcal{S}(v_2, v_2) = -2, \mathcal{S}(v_3, v_3) = 2 \tag{9.12}$$

and

$$\widetilde{\mathcal{S}}(v_1, v_1) = \widetilde{\mathcal{S}}(v_2, v_2) = -2, \widetilde{\mathcal{S}}(v_3, v_3) = \frac{3}{2}. \tag{9.13}$$

In view of (9.12) and (9.13), the scalar curvatures with ∇ and $\widetilde{\nabla}$ respectively are defined as under:

$$\tau = \sum_{i=1}^{3} \mathcal{S}(e_i, e_i) = \mathcal{S}(\nu_1, \nu_1) + \mathcal{S}(\nu_2, \nu_2) + \mathcal{S}(\nu_3, \nu_3) = -2,$$

$$\widetilde{\tau} = \sum_{i=1}^{3} \widetilde{\mathcal{S}}(e_i, e_i) = \widetilde{\mathcal{S}}(\nu_1, \nu_1) + \widetilde{\mathcal{S}}(\nu_2, \nu_2) + \widetilde{\mathcal{S}}(\nu_3, \nu_3) = -\frac{3}{2}.$$

Conflicts of Interest: The authors declare no conflict of interest.

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