

CONFORMAL YAMABE SOLITONS AND CONFORMAL QUASI-YAMABE SOLITONS ON HYPERBOLIC SASAKIAN MANIFOLDS

JHANTU DAS , KALYAN HALDER and ARINDAM BHATTACHARYYA

Abstract

The current article is devoted to hyperbolic Sasakian manifolds admitting conformal Yamabe soliton and conformal quasi-Yamabe soliton. First, we have discussed the properties of hyperbolic Sasakian manifold admitting conformal Yamabe soliton with various types of soliton vector fields under certain conditions. Then it is proven that if a Ricci semi-symmetric hyperbolic Sasakian manifold admits a conformal Yamabe soliton, then the manifold is of constant sectional curvature 1 and the soliton vector field associated with the soliton is conformal. In case of pointwise collinear soliton vector field, the conformal quasi-Yamabe soliton reduces to conformal Yamabe soliton on hyperbolic Sasakian manifolds. Finally, we have given an example of 3-dimensional hyperbolic Sasakian manifold and verified our results.

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1. Introduction

In [25], R. S. Hamilton first time initiated the Yamabe solitons as a self-similar solutions to the Yamabe flow that moves by one parameter family of diffeomorphisms ϕ_t generated by a fixed smooth vector field X (see more details [6, 13, 14, 37]). The Yamabe flow on a Riemannian or pseudo-Riemannian manifold (M, g) of dimension n , ($n \geq 3$) is defined as the evolution equation of the metric $g = g(t)$ as follows

$$\frac{\partial}{\partial t}(g(t)) = -r(g(t)),$$

where r is the scalar curvature of the manifold M . It should be noted that in two-dimension the Yamabe flow is equivalent to the Ricci flow [24] whereas the Yamabe flow and the Ricci flow do not agree for dimensions strictly greater than two, as the Yamabe flow preserve the metric's conformal class but the Ricci flow does not in general.

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Therefore, a metric g of a complete Riemannian or pseudo-Riemannian manifold (M, g) is said to be Yamabe soliton if there exists a smooth vector field X on M such that

$$\frac{1}{2}\mathcal{L}_X g = (r - \tau)g, \quad (1.1)$$

where τ is a real constant and \mathcal{L}_X is the Lie derivative in the direction of X . The Yamabe soliton is said to be shrinking or expanding according as $\tau < 0$ or $\tau > 0$, respectively whereas steady if $\tau = 0$. Over the years, several authors have investigated Yamabe solitons in various context (see [8, 11, 26, 30, 35, 40, 43–45]) and many others.

In 2013, E. Barbosa et al. [2] initiated the concept of almost Yamabe soliton as a generalization of the classical Yamabe soliton (1.1) by setting τ to be a smooth function on M . In [36], Seko-Maeta completely classified almost Yamabe solitons in the context of hypersurfaces in Euclidean spaces.

In 2018, B. Y. Chen et al. [9] initiated the concept of quasi-Yamabe soliton as a generalization of Yamabe soliton. A Riemannian or pseudo-Riemannian manifold (M, g) is said to be a quasi-Yamabe soliton if for some real constant τ , there exist a real valued smooth function μ and a smooth vector field X on M satisfying the equation

$$\frac{1}{2}\mathcal{L}_X g = (r - \tau)g + \mu X^* \otimes X^*, \quad (1.2)$$

where X^* is the dual 1-form of X . If $\mu = 0$, then quasi-Yamabe soliton becomes Yamabe soliton (1.1).

In 2005, A. E. Fischer [20] started the study of conformal Ricci flow, which is a modified version of Ricci flow that modifies the volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow equations on a smooth closed connected oriented manifold M , are given by

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg, \quad r(g) = -1, \quad (1.3)$$

where p is a non-dynamical (time-dependent) scalar field and $r(g)$ is the scalar curvature of M . The term $-pg$ acts as the constraint force to maintain the scalar curvature constraint in equation. Note that these evolution equations are analogous to famous Navier-Stokes equations in fluid mechanics with divergence free constraint. The non-dynamical scalar field p is also called the conformal pressure.

In 2015, N. Basu et al. [3] initiated the concept of conformal Ricci soliton, which is a generalization of the Ricci soliton and the equation is as follows

$$\mathcal{L}_X g + 2S = [2\tau - (p + \frac{2}{n})]g, \quad (1.4)$$

where τ is a real constant and p is the conformal pressure. This equation also satisfies the conformal Ricci flow equation (1.3). Furthermore, the conformal Ricci soliton is

said to be gradient [21] if $X = \text{grad}(f)$, for some C^∞ -function f on M . For the case of conformal gradient Ricci soliton, we have

$$\nabla^2 f + S = [\tau - \frac{1}{2}(p + \frac{2}{n})]g, \quad (1.5)$$

where ∇^2 stands for the Hessian operator.

Recently, Roy et al. [19] established the notion of conformal Yamabe soliton as follows:

DEFINITION 1.1. A Riemannian or pseudo-Riemannian manifold (M, g) of dimension $(2n + 1)$ is called a conformal Yamabe soliton if there exist a real constant τ and a smooth vector field X such that

$$\mathcal{L}_X g + [2\tau - 2r - (p + \frac{2}{2n+1})]g = 0, \quad (1.6)$$

where p is the conformal pressure.

In view of Eq. (1.2) and Eq. (1.4), we introduce the notion of conformal quasi-Yamabe soliton as:

DEFINITION 1.2. A Riemannian or pseudo-Riemannian manifold (M, g) of dimension $(2n + 1)$ is said to admit conformal quasi-Yamabe soliton if

$$\mathcal{L}_X g + [2\tau - 2r - (p + \frac{2}{2n+1})]g = 2\mu X^* \otimes X^*, \quad (1.7)$$

where X^* is the dual 1-form of X . This notion is denoted by (g, X, τ, μ) . For $\mu = 0$, the conformal quasi-Yamabe soliton (g, X, τ, μ) reduces to the conformal Yamabe soliton (g, X, τ) . A conformal quasi-Yamabe soliton is said to be shrinking or expanding according as $\tau < 0$ or $\tau > 0$, respectively whereas steady if $\tau = 0$.

In the last few years, many authors have studied on quasi-Yamabe solitons and their generalizations in different contact metric manifolds [10, 18, 22, 23, 27, 29, 31, 41]. In [9], the authors proved that an Euclidean hypersurface is totally umbilical if and only if it admits a Yamabe soliton with the tangential component of the position vector field as the soliton vector field. Also, in [5], the authors studied almost quasi-Yamabe solitons on warped product manifolds and established a Bochner-type formula for a gradient almost quasi-Yamabe soliton.

On the other hand, in 1976, Upadhyay and Dube [39] introduced a new almost contact structure, namely, an almost contact hyperbolic (f, g, η, ξ) -structure. An odd dimensional smooth manifold of differentiability class C^∞ equipped with an almost hyperbolic contact (f, g, η, ξ) -structure is called an almost hyperbolic contact manifold. Further, it was worked by number of authors [1, 4, 28, 34]. A smooth non-vanishing vector field $X \in T_p(M)$ is said to be time-like (resp., null, space-like, and non-space-like) if it satisfies $g_p(X, X) < 0$ (resp., $= 0$, > 0 , and ≤ 0) [12, 32], where $T_p(M)$ is a tangent vector space of M at $p \in M$. The Ricci curvature tensor S and the scalar

curvature tensor r of an almost hyperbolic contact metric manifold M^{2n+1} endowed with semi-Riemannian metric g are, respectively, defined as follows:

$$S(\mathcal{U}, \mathcal{V}) = \sum_{j=1}^{2n} g(e_j, e_j)g(R(e_j, \mathcal{U})\mathcal{V}, e_j) - g(R(\xi, \mathcal{U})\mathcal{V}, \xi), \quad (1.8)$$

$$r = \sum_{j=1}^{2n} g(e_j, e_j)S(e_j, e_j) - S(\xi, \xi) \quad (1.9)$$

for all $\mathcal{U}, \mathcal{V} \in \chi(M^{2n+1})$, the Lie algebra of smooth vector fields on M^{2n+1} , where $\{e_1, e_2, \dots, e_{2n}, e_{2n+1} = \xi\}$ is a local orthonormal basis of the tangent vector space at each point of M^{2n+1} and ξ is a unit time-like smooth vector field on M^{2n+1} .

Motivated by the above studies, the current article explores the study of hyperbolic Sasakian manifold admitting conformal Yamabe soliton and conformal quasi-Yamabe soliton. The paper organized as follows: After brief introduction, in section 2, we concerned with preliminary. In Section 3 of this paper we have studied hyperbolic Sasakian manifold admitting a conformal Yamabe soliton (g, ξ, τ) . Also, we have studied conformal Yamabe soliton (g, X, τ) on Ricci semi-symmetric hyperbolic Sasakian manifold. In section 4, we deal with the study of conformal quasi-Yamabe solitons on hyperbolic Sasakian manifolds. Here we proved that on hyperbolic Sasakian manifold, the conformal quasi-Yamabe soliton reduces to conformal Yamabe soliton under certain condition of soliton vector field. Finally, in Section 5, we give an example of hyperbolic Sasakian manifold satisfying both conformal Yamabe soliton and conformal quasi-Yamabe soliton and also our results.

2. Preliminaries

A $(2n+1)$ -dimensional smooth manifold M^{2n+1} endowed with an almost hyperbolic contact (φ, ξ, η) -structure is called an almost hyperbolic contact manifold [39]. Then for all $U \in \chi(M^{2n+1})$ we have,

$$\varphi^2(U) = U + \eta(U)\xi, \quad \eta(\varphi U) = 0, \quad (2.1)$$

where φ is a $(1, 1)$ -tensor field, η is a 1-form, ξ is a time-like smooth vector field on M^{2n+1} . Also, from Eq. (2.1) we have,

$$\varphi\xi = 0, \quad \eta(\xi) = -1, \quad \text{rank}(\varphi) = 2n. \quad (2.2)$$

An almost hyperbolic contact manifold M^{2n+1} is said to be an almost hyperbolic contact metric manifold if the semi-Riemannian metric g of M^{2n+1} satisfies the following conditions

$$g(U, \xi) = \eta(U), \quad g(\varphi U, \varphi V) = -g(U, V) - \eta(U)\eta(V), \quad (2.3)$$

for all $U, V \in \chi(M^{2n+1})$.

From (2.3) it can easily be seen that

$$g(\varphi U, V) + g(U, \varphi V) = 0 \quad (2.4)$$

for all $U, V \in \chi(M^{2n+1})$.

A $(2n + 1)$ -dimensional almost hyperbolic contact metric manifold M is called a hyperbolic Sasakian manifold [7] if it satisfies the following condition

$$(\nabla_U \varphi)(V) = g(U, V)\xi - \eta(V)U, \quad (2.5)$$

where ∇ stands for the Levi-Civita connection of g . From (2.5) it is clear that

$$\nabla_U \xi = -\varphi U \quad (2.6)$$

and

$$(\nabla_U \eta)V = -g(\varphi U, V) = g(U, \varphi V). \quad (2.7)$$

In a $(2n + 1)$ -dimensional hyperbolic Sasakian manifolds the following relations are satisfied [7]:

$$R(U, V)\xi = \eta(V)U - \eta(U)V, \quad (2.8)$$

$$\eta(R(U, V)W) = \eta(U)g(V, W) - \eta(V)g(U, W), \quad (2.9)$$

$$R(\xi, U)V = g(U, V)\xi - \eta(V)U, \quad (2.10)$$

$$S(U, \xi) = 2n\eta(U), \quad (2.11)$$

$$Q\xi = 2n\xi \quad (2.12)$$

for all $U, V, W \in \chi(M^{2n+1})$ and where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator respectively.

DEFINITION 2.1. A smooth vector field X on a $(2n + 1)$ -dimensional semi-Riemannian manifold (M^{2n+1}, g) is said to be a conformal vector field if there exists a smooth function ρ on M^{2n+1} such that

$$\mathcal{L}_X g = 2\rho g. \quad (2.13)$$

In particular, if $\rho = 0$, then X is said to be a Killing vector field and if $\rho = \text{constant}$, then X becomes homothetic vector field [15–17, 42].

DEFINITION 2.2. A smooth vector field X on a contact metric manifold is said to be an infinitesimal contact transformation [38], if there exists a smooth function ω on M^{2n+1} such that

$$\mathcal{L}_X \eta = \omega \eta, \quad (2.14)$$

where $\mathcal{L}_X \eta$ is the Lie derivative of η by X . In particular, the infinitesimal contact transformation with a vanishing smooth function ω reduces to the strict infinitesimal contact transformation.

3. Hyperbolic Sasakian manifolds admitting conformal Ricci solitons

THEOREM 3.1. *If a $(2n+1)$ -dimensional hyperbolic Sasakian manifold admits a conformal Yamabe soliton (g, ξ, τ) , then the scalar curvature of the manifold is $\tau - (\frac{p}{2} + \frac{1}{2n+1})$.*

PROOF. Let M be a $(2n+1)$ -dimensional hyperbolic Sasakian manifold admitting conformal Yamabe soliton (g, ξ, τ) . Then we have from (1.6) that

$$(\mathcal{L}_\xi g)(U, V) + [2\tau - 2r - (p + \frac{2}{2n+1})]g(U, V) = 0, \quad (3.1)$$

for all $U, V \in \chi(M)$. Now from the definition of Lie derivatives we have

$$(\mathcal{L}_\xi g)(U, V) = g(-\varphi U, V) + g(U, -\varphi V) = 0. \quad (3.2)$$

Therefore, the Eq. (3.1) reduces to

$$[2\tau - 2r - (p + \frac{2}{2n+1})]g(U, V) = 0 \quad (3.3)$$

which implies that

$$r = \tau - (\frac{p}{2} + \frac{1}{2n+1}). \quad (3.4)$$

Thus, the scalar curvature is $\tau - (\frac{p}{2} + \frac{1}{2n+1})$. This completes the proof. \square

COROLLARY 3.2. *If a $(2n+1)$ -dimensional hyperbolic Sasakian manifold admits a conformal Yamabe soliton (g, ξ, τ) , then the soliton is shrinking if $r + (\frac{p}{2} + \frac{1}{2n+1}) < 0$, steady if $r + (\frac{p}{2} + \frac{1}{2n+1}) = 0$, expanding if $r + (\frac{p}{2} + \frac{1}{2n+1}) > 0$.*

COROLLARY 3.3. *If a $(2n+1)$ -dimensional hyperbolic flat Sasakian manifold admits a conformal Yamabe soliton (g, ξ, τ) , then the soliton is shrinking if $(\frac{p}{2} + \frac{1}{2n+1}) < 0$, steady if $(\frac{p}{2} + \frac{1}{2n+1}) = 0$, expanding if $(\frac{p}{2} + \frac{1}{2n+1}) > 0$.*

PROOF. As the manifold is flat, i.e., $r = 0$, then Eq. (3.4) becomes

$$\tau = (\frac{p}{2} + \frac{1}{2n+1}). \quad (3.5)$$

Hence the soliton is shrinking if $\tau < 0$, steady if $\tau = 0$ and expanding if $\tau > 0$. This completes the proof. \square

Next we prove the following theorem.

THEOREM 3.4. *If the metric of a $(2n+1)$ -dimensional hyperbolic Sasakian manifold is a conformal Yamabe soliton (g, X, τ) , then the soliton vector field X is solenoidal if and only if the scalar curvature is $\tau - (\frac{p}{2} + \frac{1}{2n+1})$.*

PROOF. From Eq. (1.6), we can write

$$(\mathbb{L}_X g)(U, V) + [2\tau - 2r - (p + \frac{2}{2n+1})]g(U, V) = 0 \quad (3.6)$$

for all $U, V \in \chi(M)$. Let an orthonormal basis $\{e_i : i = 1, 2, \dots, 2n+1\}$ of the tangent vector space at each point of the manifold. Then taking $U = e_i, V = e_i$ in the above Eq. (3.6) and summing over i ($i = 1, 2, \dots, 2n+1$), we get

$$\operatorname{div}(X) + [2\tau - 2r - (p + \frac{2}{2n+1})](2n+1) = 0. \quad (3.7)$$

Let X be a solenoidal vector field. Then we have $\operatorname{div}(X) = 0$ and so from the above Eq. (3.7) we have $r = \tau - (\frac{p}{2} + \frac{1}{2n+1})$. Again if $r = \tau - (\frac{p}{2} + \frac{1}{2n+1})$, then it follows from Eq. (3.7) that $\operatorname{div}(X) = 0$, which means X is solenoidal. This completes the proof. \square

COROLLARY 3.5. *Let the metric g of a $(2n+1)$ -dimensional hyperbolic Sasakian manifold M admit a conformal Yamabe soliton (g, X, τ) , where $X = \operatorname{grad}(f)$, for some C^∞ -function on M . Then the Laplacian equation satisfied by f is*

$$\Delta(f) = [(p + \frac{2}{2n+1}) + 2r - 2\tau](2n+1). \quad (3.8)$$

PROOF. Considering the soliton vector field X as gradient type, i.e., $X = \operatorname{grad}(f)$, for some C^∞ -function on M . Then Eq. (3.7) gives

$$\Delta(f) = [(p + \frac{2}{2n+1}) + 2r - 2\tau](2n+1). \quad (3.9)$$

Hence the proof. \square

Next, we consider that M is a $(2n+1)$ -dimensional hyperbolic Sasakian manifold admits a conformal Yamabe soliton (g, X, τ) and the soliton vector field X is pointwise collinear with ξ . In this regard our next result is in the following:

THEOREM 3.6. *Let a $(2n+1)$ -dimensional hyperbolic Sasakian manifold M admit a conformal Yamabe soliton (g, X, τ) with soliton vector field X which is pointwise collinear with ξ . Then X is a constant multiple of ξ and the scalar curvature $r = \tau - (\frac{p}{2} + \frac{1}{2n+1})$.*

PROOF. Let us consider the soliton vector field X is pointwise collinear with ξ , then we can find a non-zero smooth function α on M such that $X = \alpha\xi$. Then from (1.6) we derive

$$\alpha(\mathbb{L}_{\xi} g)(U, V) + U(\alpha)\eta(V) + V(\alpha)\eta(U) + [2\tau - 2r - (p + \frac{2}{2n+1})]g(U, V) = 0, \quad (3.10)$$

which by virtue of Eq. (2.6) becomes

$$U(\alpha)\eta(V) + V(\alpha)\eta(U) + [2\tau - 2r - (p + \frac{2}{2n+1})]g(U, V) = 0. \quad (3.11)$$

Replacing V by ξ in Eq. (3.11) gives

$$U(\alpha) = [\xi(\alpha) + \{2\tau - 2r - (p + \frac{2}{2n+1})\}]\eta(U). \quad (3.12)$$

Again replacing U by ξ in Eq. (3.12) we have

$$\xi(\alpha) = -\{\tau - r - (\frac{p}{2} + \frac{1}{2n+1})\}. \quad (3.13)$$

Substituting the value of $\xi(\alpha)$ in Eq. (3.12) we infer

$$d(\alpha) = \{\tau - r - (\frac{p}{2} + \frac{1}{2n+1})\}\eta, \quad (3.14)$$

where d stands for the exterior derivative operator. Applying d on both sides of Eq. (3.14) and using *Poincaré lemma* $d^2 \equiv 0$, we get

$$\{\tau - r - (\frac{p}{2} + \frac{1}{2n+1})\}d\eta = 0. \quad (3.15)$$

Since $d\eta \neq 0$, we infer

$$\tau = r + (\frac{p}{2} + \frac{1}{2n+1}). \quad (3.16)$$

Substituting the value of τ in Eq. (3.14) gives us $d(\alpha) = 0$ i.e., $\alpha = \text{constant}$. Thus X is a constant multiple of ξ . Then it follows the first part of the statement. Also from (3.16), we can see say that the scalar curvature $r = \tau - (\frac{p}{2} + \frac{1}{2n+1})$. This proves the second part of the theorem and hence completes the proof. \square

Next, we prove the following theorem.

THEOREM 3.7. *If a $(2n + 1)$ -dimensional hyperbolic Sasakian manifold M admits a conformal Yamabe soliton (g, X, τ) satisfies the condition $R(U, V) \cdot S = 0$, then the manifold M is of constant sectional curvature 1 and the soliton vector field X is a conformal vector field.*

PROOF. Let us consider a $(2n + 1)$ -dimensional hyperbolic Sasakian manifold M with conformal Yamabe soliton (g, X, τ) satisfying the condition $R(U, V) \cdot S = 0$. Then we have

$$(\mathcal{L}_X g)(U, V) + [2\tau - 2r - (p + \frac{2}{2n+1})]g(U, V) = 0 \quad (3.17)$$

and

$$S(R(U, V)W, Z) + S(W, R(U, V)Z) = 0 \quad (3.18)$$

for all $U, V, Z, W \in \chi(M)$. Replacing Z by ξ in Eq. (3.18) and using Eqs. (2.8) and (2.11) we get

$$2n\eta(R(U, V)W) + \eta(V)S(W, U) - \eta(U)S(W, V) = 0. \quad (3.19)$$

Letting $U = \xi$ in Eq. (3.19) and using use of Eqs. (2.10) and (2.11) we obtain

$$S(W, V) = 2ng(W, V). \quad (3.20)$$

This shows that the manifold M is Einstein. Tracing the Eq. (3.20) yields

$$r = 2n(2n + 1). \quad (3.21)$$

We know that, if the scalar curvature r of a m -dimensional manifold is constant then it is equal to $m(m - 1)$ times the sectional curvature. Therefore, from (3.21), we can conclude that the manifold M is of constant sectional curvature 1.

In view of Eq. (3.21), the Eq. (3.17) reduces to

$$(\mathcal{L}_X g)(U, V) = 2\rho g(U, V), \quad (3.22)$$

where $\rho = \{(\frac{p}{2} + \frac{1}{2n}) - \tau\}$. In the sense of definition (2.1), the soliton vector field X is a conformal vector field. \square

If X is Killing, then Eq. (3.22) gives us $\tau = (\frac{p}{2} + \frac{1}{2n})$. Again if $\tau = (\frac{p}{2} + \frac{1}{2n})$, then Eq. (3.22) gives $(\mathcal{L}_X g)(U, V) = 0$, for all $U, V \in \chi(M)$, which implies X is a Killing vector field. Thus we conclude the following:

COROLLARY 3.8. *If a $(2n + 1)$ -dimensional Ricci semi-symmetric hyperbolic Sasakian manifold M admits a conformal Yamabe soliton (g, X, τ) , then the soliton vector field X is Killing if and only if $\tau = (\frac{p}{2} + \frac{1}{2n})$.*

Now setting $X = \xi$, in Eq. (3.22) and using (2.6) we get $\rho g(U, V) = 0$, for all $U, V \in \chi(M)$. It follows that $\rho = 0$, that is $\tau = (\frac{p}{2} + \frac{1}{2n})$. This leads to the following:

COROLLARY 3.9. *If a $(2n + 1)$ -dimensional Ricci semi-symmetric hyperbolic Sasakian manifold M admits a conformal Yamabe soliton (g, ξ, τ) , then the soliton is shrinking if $(p + \frac{1}{n}) < 0$, steady if $(p + \frac{1}{n}) = 0$, expanding if $(p + \frac{1}{n}) > 0$.*

4. Hyperbolic Sasakian manifolds admitting conformal quasi-Yamabe solitons

In this section we consider a conformal quasi-Yamabe soliton on a $(2n + 1)$ -dimensional hyperbolic Sasakian manifold with the soliton vector field X pointwise collinear with timelike smooth vector field ξ . In this regard, our main theorem is as follows:

THEOREM 4.1. *Let a $(2n + 1)$ -dimensional hyperbolic Sasakian manifold M admit a conformal quasi-Yamabe soliton (g, X, τ, μ) whose soliton vector field X is pointwise collinear with the timelike smooth vector field ξ , then we can state the following:*

- (i) *The scalar curvature of M is $\tau - (\frac{p}{2} + \frac{1}{2n+1})$.*
- (ii) *The soliton reduces to the conformal Yamabe soliton (g, X, τ) .*
- (iii) *The soliton vector field X becomes a constant multiple of ξ .*
- (iv) *The soliton vector field X is a strict infinitesimal contact transformation.*

PROOF. Let us assume that a $(2n + 1)$ -dimensional hyperbolic Sasakian manifold M admits a conformal quasi-Yamabe soliton (g, X, τ, μ) such that X is pointwise collinear

with ξ , i.e., $X = c\xi$, for some non-zero smooth function $c : M \rightarrow \mathbb{R}$. Then from Eq. (1.7) we derive

$$\begin{aligned} (\mathcal{L}_{c\xi}g)(U, V) &= c(\mathcal{L}_\xi g)(U, V) + U(c)\eta(V) + V(c)\eta(U) \\ &= 2\{r - \tau + (\frac{p}{2} + \frac{1}{2n+1})\}g(U, V) + 2\mu c^2\eta(U)\eta(V). \end{aligned} \quad (4.1)$$

In view of Eq. (2.6), the Eq. (4.1) reduces to

$$U(c)\eta(V) + V(c)\eta(U) = 2\{r - \tau + (\frac{p}{2} + \frac{1}{2n+1})\}g(U, V) + 2\mu c^2\eta(U)\eta(V). \quad (4.2)$$

By taking $V = \xi$ in Eq. (4.2) and using Eq. (2.2), we get

$$U(c) = [\xi(c) - 2\{r - \tau + (\frac{p}{2} + \frac{1}{2n+1})\} + 2\mu c^2]\eta(U). \quad (4.3)$$

Again plugging $U = \xi$ in the above Eq. (4.3) we obtain by virtue of Eq. (2.2) that

$$\xi(c) = \{r - \tau + (\frac{p}{2} + \frac{1}{2n+1})\} - \mu c^2. \quad (4.4)$$

Next, we shall consider a local orthonormal basis $\{e_i : i = 1, 2, \dots, 2n+1\}$ of the tangent vector space at each point of the manifold M . Then taking $U = V = e_i$ in Eq. (4.2) and summing over i we obtain

$$\xi(c) = \{r - \tau + (\frac{p}{2} + \frac{1}{2n+1})\}(2n+1) - \mu c^2. \quad (4.5)$$

Finally, comparing the Eq. (4.5) with the Eq. (4.4) we lead

$$r = \tau - (\frac{p}{2} + \frac{1}{2n+1}). \quad (4.6)$$

Therefore, the scalar curvature of M is $\tau - (\frac{p}{2} + \frac{1}{2n+1})$. This complete the part-(i) of the theorem. Taking reference of Eqs. (4.6) and (4.4), the Eq. (4.3) reduces to

$$d(c) = \mu c^2\eta. \quad (4.7)$$

Applying d on both sides of (4.7) and utilizing *Poincaré lemma* $d^2 \equiv 0$, we conclude that $\mu c^2 = 0$ and therefore $\mu = 0$. Thus, the conformal quasi-Yamabe soliton reduces to the conformal Yamabe soliton, i.e., part-(ii) of the theorem. On substituting $\mu = 0$ in Eq. (4.7) leads to $d(c) = 0$, which implies that $c = \text{constant}$. This complete the part-(iii) of the theorem.

Again taking $V = \xi$ in Eq. (4.1) we obtain by virtue of Eq. (2.2) that

$$(\mathcal{L}_X g)(U, \xi) = 2[\{r - \tau + (\frac{p}{2} + \frac{1}{2n+1})\} - \mu c^2]\eta(U). \quad (4.8)$$

On putting $\mu = 0$ in (4.8) and taking reference of Eq. (4.6) yield

$$(\mathcal{L}_X \eta)(U) = 0 \quad (4.9)$$

for all $U \in \chi(M)$. Thus in view of definition (2.2), we can say that X is a strict infinitesimal contact transformation, i.e., part-(iv) of the theorem and hence completes the proof. \square

In view of Eq. (4.6), we can state the following:

COROLLARY 4.2. *If a $(2n + 1)$ -dimensional hyperbolic Sasakian manifold M admit a conformal quasi-Yamabe soliton (g, X, τ, μ) whose soliton vector field X pointwise collinear with the timelike smooth vector field ξ , then the soliton is*

- (i) *shrinking if $r + (\frac{p}{2} + \frac{1}{2n+1}) < 0$,*
- (ii) *steady if $r + (\frac{p}{2} + \frac{1}{2n+1}) = 0$ and*
- (iii) *expanding if $r + (\frac{p}{2} + \frac{1}{2n+1}) > 0$.*

COROLLARY 4.3. *If a $(2n + 1)$ -dimensional hyperbolic flat Sasakian manifold M admit a conformal quasi-Yamabe soliton (g, X, τ, μ) whose soliton vector field X pointwise collinear with the timelike smooth vector field ξ , then the soliton is shrinking if $(\frac{p}{2} + \frac{1}{2n+1}) < 0$, steady if $(\frac{p}{2} + \frac{1}{2n+1}) = 0$, expanding $(\frac{p}{2} + \frac{1}{2n+1}) > 0$.*

5. Example

Let us consider a three dimensional manifold $M^3 = \{(u, v, w) \in \mathbb{R}^3 : w \neq 0\}$ with the standard coordinate system (u, v, w) of \mathbb{R}^3 . Let us consider the smooth vector fields $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ of M^3 be such that

$$[\mathcal{E}_1, \mathcal{E}_2] = 2\mathcal{E}_3, \quad [\mathcal{E}_2, \mathcal{E}_3] = 0, \quad [\mathcal{E}_1, \mathcal{E}_3] = 0.$$

Let us define the semi-Riemannian metric g on M^3 by

$$g(\mathcal{E}_i, \mathcal{E}_j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and a $(1, 1)$ tensor field φ on M^3 by

$$\varphi(\mathcal{E}_1) = \mathcal{E}_2, \quad \varphi(\mathcal{E}_2) = \mathcal{E}_1, \quad \varphi(\mathcal{E}_3) = 0.$$

Now considering $\mathcal{E}_3 = \xi$, let η be the 1-form on M^3 , defined by

$$g(U, \mathcal{E}_3) = \eta(U), \quad \forall U \in \chi(M^3)$$

Then it can be observed that $\eta(\xi) = -1$. Using the linearity property of φ and g we obtain

$$\varphi^2 U = U + \eta(U)\xi, \quad g(\varphi U, \varphi V) = -g(U, V) - \eta(U)\eta(V), \quad \forall U, V \in \chi(M^3)$$

Hence the structure (g, φ, ξ, η) defines an almost hyperbolic contact metric structure on M^3 . Let ∇ be a Riemannian connection with respect to semi-Riemannian metric g . Now using the well-known Koszul's formula given by

$$\begin{aligned} 2g(\nabla_U V, W) = & U g(V, W) + V g(W, U) - W g(U, V) - g(U, [V, W]) \\ & - g(V, [U, W]) + g(W, [U, V]) \end{aligned}$$

, we calculate the following:

$$\begin{aligned}\nabla_{\mathcal{E}_1}\mathcal{E}_1 &= 0, & \nabla_{\mathcal{E}_1}\mathcal{E}_2 &= \mathcal{E}_3, & \nabla_{\mathcal{E}_1}\mathcal{E}_3 &= -\mathcal{E}_2, \\ \nabla_{\mathcal{E}_2}\mathcal{E}_1 &= -\mathcal{E}_3, & \nabla_{\mathcal{E}_2}\mathcal{E}_2 &= 0, & \nabla_{\mathcal{E}_2}\mathcal{E}_3 &= -\mathcal{E}_1, \\ \nabla_{\mathcal{E}_3}\mathcal{E}_1 &= -\mathcal{E}_2, & \nabla_{\mathcal{E}_3}\mathcal{E}_2 &= -\mathcal{E}_1, & \nabla_{\mathcal{E}_3}\mathcal{E}_3 &= 0.\end{aligned}$$

In view of the above results it is clear that the manifold M^3 satisfies

$$\nabla_U\xi = -\varphi U, \quad \forall U \in \chi(M)$$

Hence $M^3(g, \varphi, \xi, \eta)$ is a 3-dimensional hyperbolic Sasakian manifold.

Thus from the above computations and using the well-known formula

$$R(U, V)W = \nabla_U\nabla_VW - \nabla_V\nabla_UW - \nabla_{[U, V]}W,$$

the non-vanishing components of the curvature tensor R as follows:

$$\begin{aligned}R(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_1 &= 3\mathcal{E}_2, & R(\mathcal{E}_2, \mathcal{E}_1)\mathcal{E}_1 &= -3\mathcal{E}_2, & R(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_1 &= -\mathcal{E}_3, \\ R(\mathcal{E}_3, \mathcal{E}_1)\mathcal{E}_1 &= \mathcal{E}_3, & R(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_2 &= 3\mathcal{E}_1, & R(\mathcal{E}_2, \mathcal{E}_1)\mathcal{E}_2 &= -3\mathcal{E}_1, \\ R(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_2 &= \mathcal{E}_3, & R(\mathcal{E}_3, \mathcal{E}_2)\mathcal{E}_2 &= -\mathcal{E}_3, & R(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_3 &= -\mathcal{E}_1, \\ R(\mathcal{E}_3, \mathcal{E}_1)\mathcal{E}_3 &= \mathcal{E}_1, & R(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_3 &= -\mathcal{E}_2, & R(\mathcal{E}_3, \mathcal{E}_2)\mathcal{E}_3 &= \mathcal{E}_2.\end{aligned}$$

Using the well-known formula $S(U, V) = \sum_{i=1}^3 g(\mathcal{E}_i, \mathcal{E}_i)g(R(\mathcal{E}_i, U)V, \mathcal{E}_i)$, the non-vanishing components of the Ricci tensor S can be easily be calculated as

$$S(\mathcal{E}_1, \mathcal{E}_1) = -2, \quad S(\mathcal{E}_2, \mathcal{E}_2) = 2, \quad S(\mathcal{E}_3, \mathcal{E}_3) = -2.$$

Again, the scalar curvature r of the given hyperbolic Sasakian manifold can be calculated as under:

$$r = \sum_{i=1}^3 g(\mathcal{E}_i, \mathcal{E}_i)S(\mathcal{E}_i, \mathcal{E}_i) = S(\mathcal{E}_1, \mathcal{E}_1) - S(\mathcal{E}_2, \mathcal{E}_2) - S(\mathcal{E}_3, \mathcal{E}_3) = -2.$$

Now from the Eq. (1.6) we deduce $\tau = (\frac{p}{2} - \frac{5}{3})$. Therefore, (g, ξ, τ) is a conformal Yamabe soliton on $M^3(g, \varphi, \xi, \eta)$ for $\tau = (\frac{p}{2} - \frac{5}{3})$. Moreover, we can see that the manifold M^3 is of constant scalar curvature $r = -2 = (\frac{p}{2} - \frac{5}{3}) - (\frac{p}{2} + \frac{1}{3}) = \tau - (\frac{p}{2} + \frac{1}{3})$, which verifies Theorem (3.1). Again, if we take $X = \xi$ and $r = -2$, then from the Eq. (1.7) we deduce $\tau = (\frac{p}{2} - \frac{5}{3})$ and $\mu = 0$. Hence (g, ξ, τ, μ) defines a conformal quasi-Yamabe soliton on $M^3(g, \varphi, \xi, \eta)$, which reduces to a conformal Yamabe soliton for $\mu = 0$. Hence, the statement of Theorem (4.1) is verified.

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Jhantu Das, Department of Mathematics, Sidho-Kanho-Birsha University, Purulia-723104, West Bengal, India
e-mail: dasjhantu54@gmail.com

Kalyan Halder, Department of Mathematics, Sidho-Kanho-Birsha University, Purulia-723104, West Bengal, India
e-mail: drkalyanhalder@gmail.com

Arindam Bhattacharyya, Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India
e-mail: bhattachar1968@yahoo.co.in