GANITA,Vol.73(2), 2023, 89-104

CHARACTERIZATION OF THE PERFECT FLUID LORENTZIAN α-PARA KENMOTSU SPACETIMES

RAJENDRA PRASAD, ABHINAV VERMA and VINDHYACHAL SINGH YADAV ✉

Abstract

In this article, we explore the characteristics of the Ricci solitons (briefly, *RS*) together with Yamabe solitons (briefly, *YS*) on the perfect fluid Lorentzian α-para Kenmotsu spacetimes (briefly, (L α-PK)$_{ST}$). Certain conclusions corresponding to applications of such spacetimes in general relativity and cosmology are obtained. We have given an example to verify the results in the following different sections.

2010 *Mathematics subject classification:* Primary 53C44, 53C50; Secondary 53B30, 53C25.
Keywords and phrases: Perfect fluid spacetime, Einstein field equation, Ricci Soliton, Yamabe Soliton, Energy momentum tensor..

1. Introduction

The Ricci flow concept was given by Hamilton in 1982 [7]. It is an outstanding tool to analyse the structure of a manifold. It is a process, which deforms the metric of a Riemannian manifold M by removing the irregularities. The following equation defines the Ricci flow,

$$\frac{\partial g}{\partial t} + 2Ric = 0, \tag{1.1}$$

where, g, Ric and t are Riemannian metric, Ricci tensor and time, respectively. We suppose that $\phi_t : M \to M, t \in \mathcal{R}$ is a family of diffeomorphisms, which is 1-parameter group of transformations, then it gives rise to a vector field called the infinitesimal generator and integral curves. In the space of metrics of $\phi_t : M \to M$, *RS* are static points in Ricci flow. $g(0)$ is the initial metric of ϕ_t and $g(t)$ is the pullback of $g(0)$. *RS* on a Riemannian manifold (M, g) is a special solution to the Ricci flow and is a natural generalization of Einstein metric, which is defined as triple (g, V, λ) with g, a Riemannian metric, V, a vector field and λ, a real scalar s.t.

$$\frac{1}{2}\pounds_V g + S + \lambda g = 0. \tag{1.2}$$

In the above equation, S denotes a Ricci tensor, $\pounds_V g$ represents Lie-derivative of g w.r.t. V on M and $\lambda \in \mathcal{R}$. Here, *RS* is expanding, steady and shrinking, according as $\lambda > 0, \lambda = 0$ and $\lambda < 0$, respectively.

A *YS* on (M, g) is described by the relation

$$\frac{1}{2}(\pounds_V g) = (\kappa - \lambda)g, \tag{1.3}$$

where, (M, g) is a Riemannian (or semi-Riemannian) manifold, V is vector field. In relation (1.3), λ and κ denote soliton constant and scalar curvature, respectively [2]. If $\lambda < 0, \lambda = 0$ or $\lambda > 0$, then *YS* is expanding, steady or shrinking respectively. Hamilton [7] gave the concept of Yamabe flow as a device to construct Yamabe metric on the compact Riemannian manifolds. A time dependent metric $g(\_, t)$ on M is said to evolve by the Yamabe flow, provided the metric g satisfies,

$$\frac{\partial g(t)}{\partial t} + \kappa g(t) = 0, g(0) = g_0, \tag{1.4}$$

on M. A *YS* is a special solution of the Yamabe flow, which moves by 1-parameter family of diffeomorphisms ϕ_t generated by a fixed vector field V on M [5]. Point-wise elliptic gradient estimate for the Yamabe flow on a locally conformally flat compact Riemannian manifold is found by Ye [19]. Hui et al. [10] considered Kenmotsu manifolds and found some geometrical outcomes of the *YS*. In case of the Ricci flow, the *YS* or the singularities of the Yamabe flow emerged naturally.
Actually, importance of Yamabe flow lies in the fact that it is a natural geometric deformation to the metric of constant scalar curvature. Yamabe flow corresponds to the rapid diffusion case of porous medium equation in mathematical physics. Like *RS*, *YS* is also a special solution of Yamabe flow, which moves by 1-parameter family of diffeomorphisms ϕ_t generated by a fixed vector field V on M and a homothetic, that is, $g(\_, t) = \varsigma(t)\phi_*(t)g_0$.
For a *YS*, if $V = Df$ is true for a smooth function f on M, then the relation (1.3) turns into $Hessf = (\kappa - \lambda)g$, here $Hessf$ represents the Hessian of f and D represents the gradient operator of g on M. Here, we call f, the potential function of *YS* and Df, a gradient of *YS*.

Definition 1.1. A vector field X on an almost contact Riemannian manifold M is an infinitesimal transformation [2], provided there exists a smooth function γ on M s.t.

$$(\pounds_X \eta)(Y) = \gamma\eta(Y), \tag{1.5}$$

$\forall$ smooth vector fields X, Y on M. Let $\gamma = 0$, subsequently X is said to be strict infinitesimal transformations on M.

Definition 1.2. A vector field V on semi-Riemannian manifold (M, g) of dimension-n is called conformal vector field, provided

$$\frac{1}{2}\pounds_V g = \psi g$$

satisfies for ψ, where ψ is smooth fuction [18]. Conformal vector field V on (M, g) holds relations given below

$$(\pounds_V S)(X, Y) = -(n-2)g(\nabla_X D\psi, Y) + (\triangle\psi)g(X, Y) \tag{1.6}$$

and

$$\frac{1}{2}\pounds_V \kappa = -\psi\kappa + (n-1)\triangle\psi \tag{1.7}$$

$\forall$ vector fields X, Y on M, here, D represents the gradient operator, where as $\triangle$ is Laplacian operator.

We construct this article in the manner given ahead: In the first section, introduction is given, while section 2 covers preliminaries. Section 3 contains RS on Lorentzian α-para Kenmotsu manifold (M, ϕ, ξ, η, g). In section 4, we discuss about the perfect fluid on LP-Kenmotsu spacetimes and section 5 contains YS on the (L α-PK)$_{ST}$. In the last section, an example is given to verify the results obtained in different sections.

2. Preliminaries

An n-dimensional (n may be even or odd) smooth manifold M is said to be Lorentzian almost paracontact manifold, provided M is equipped with a $(1, 1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a (0, 2) type Lorentzian metric g. Let g_m: $T_m\ M \times T_m\ M \to \mathcal{R}$ be an inner product of signature (-,+,+,....., +), here m is a point in M, $T_m M$ represents tangent space of smooth manifold M at m and $\mathcal{R}$ is real number space. Some basic results, given below hold:

$$\phi^2(X) = X + \eta(X)\xi, \qquad \eta(\xi) = -1, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \qquad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

$\forall\ X, Y$ on M, and structure (ϕ, ξ, η, g) is said to be Lorentzian almost paracontact structure. An n-dimensional smooth manifold M endowed with structure (ϕ, ξ, η, g) is said to be Lorentzian almost paracontact manifold [6, 14]. Results given below hold [14] for Lorentzian almost paracontact manifold,

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \Omega(X, Y) = \Omega(Y, X), \tag{2.3}$$

here, $\Omega(X, Y) = g(X, \phi Y)$.

Definition 2.1. A Lorentzian almost paracontact manifold M is said to be Lorentzian para-Kenmotsu manifold provided

$$(\nabla_X\phi)(Y) = -g(\phi X, Y)\xi - \eta(Y)\phi X$$

$\forall\ X$, Y [8, 9, 16].

Hence, the following:

Definition 2.2. A Lorentzian para-Kenmotsu manifold is said to be Lorentzian α-para Kenmotsu manifold, provided

$$(\nabla_Z\Omega)(X,Y) + \alpha\eta(X)\Omega(Y,Z) + \alpha\eta(Y)\Omega(X,Z) = 0, \tag{2.4}$$

$\forall\ X, Y, Z$ on M, where α is a non-zero smooth function and

$$\Omega(\phi X, Y) = -\frac{1}{\alpha}(\nabla_X\eta)(Y).$$

We define,

$$\overline{\Omega}(X,Y) = \Omega(\phi X, Y),$$

then, we have,

$$\overline{\Omega}(X,Y) = -\frac{1}{\alpha}(\nabla_X\eta)(Y) \tag{2.5}$$

and,

$$\overline{\Omega}(X,Y) = \overline{\Omega}(Y,X),$$

where, ∇ is covariant differential operator.

From equation (2.4), we get,

$$(\nabla_X\phi)(Y) = -\alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X. \tag{2.6}$$

Putting $Y = \xi$ in the above equaton, we get,

$$(\nabla_X\phi)(\xi) = -\alpha g(\phi X, \xi)\xi - \alpha\eta(\xi)\phi X.$$

From the above equation, we have,

$$\nabla_X(\phi\xi) - \phi(\nabla_X\xi) = -\alpha\eta(\phi X)\xi - \alpha\eta(\xi)\phi X.$$

Using equations (2.1) and (2.3), we obtain,

$$-\phi(\nabla_X\xi) = \alpha\phi X.$$

Operating ϕ on both sides of the above relation and using relation (2.1), it yields

$$\nabla_X\xi + \eta(\nabla_X\xi)\xi = -\alpha(X + \eta(X)\xi).$$

Relation (2.1) implies $\eta(\nabla_X\xi) = 0$. Using this relation in the above equation, we get

$$\nabla_X\xi = -\alpha X - \alpha\eta(X)\xi. \tag{2.7}$$

Also,

$$(\nabla_X\eta)(Y) = \nabla_X\eta(Y) - \eta(\nabla_X Y) = g(Y, \nabla_X\xi). \tag{2.8}$$

Relations (2.7) and (2.8) together yield

$$(\nabla_X\eta)(Y) = -(\alpha)[g(X,Y) + \eta(X)\eta(Y)]. \tag{2.9}$$

In particular, if α satisfies (2.9) together with the following relation

$$\nabla_X \alpha = d\alpha(X) = \sigma\eta(X), \tag{2.10}$$

then ξ is said to be concircular vector field. Here, σ is a smooth function and η is 1-form.
For, Lorentzian α para-Kenmotsu manifold $M(\phi, \xi, \eta, g)$, following results hold good,

$$\eta(R(X,Y)Z) = (\alpha^2 + \sigma)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.11}$$

$$S(X,\xi) = (n-1)(\alpha^2 + \sigma)\eta(X), \tag{2.12}$$

$$R(X,Y)\xi = (\alpha^2 + \sigma)[\eta(Y)X - \eta(X)Y], \tag{2.13}$$

$$R(\xi,Y)X = (\alpha^2 + \sigma)[g(X,Y)\xi - \eta(X)Y], \tag{2.14}$$

$$(\nabla_X \phi)(Y) = -\alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X \tag{2.15}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)(\alpha^2 + \sigma)\eta(X)\eta(Y), \tag{2.16}$$

$\forall\ X, Y, Z$ on M. Here, R and S represent the curvature tensor and Ricci tensor of the manifold (M, g), respectively [12].
Semi-Riemannian geometry, applied in the theory of relativity, was discussed by O'Neill in 1983 [15]. Kaigorodov has explored the curvature structure of the spacetime [13]. Raychaudhary et al.[17] have extended the above ideas of the gerneral theory of spacetime. Chaki and Roy have explored the spacetimes along the covariant constant energy momentum tensor (briefly, *EMT*) [3].

3. Ricci solitons on Lorentzian α-para Kenmotsu manifold

The concept of *RS* is introduced by R.Hamilton in eighties of twentieth century [7]. So, the relation (1.2) is given by

$$\frac{1}{2}(\pounds_V g)(X,Y) + S(X,Y) + \lambda g(X,Y) = 0, \tag{3.1}$$

where, $\pounds_V$ is the Lie derivative operator w.r.t. vector field V and λ is a real constant. For vector field V, there are two conditions: $V \perp \xi$ and $V \in \text{Span}\ \{\xi\}$. Here, we consider only the case $V = \xi$. In view of relations (2.2) and (2.7), equation (3.1) reduces to,

$$S(X,Y) = (\alpha - \lambda)g(X,Y) + \alpha\eta(X)\eta(Y), \tag{3.2}$$

$$S(X,\xi) = S(\xi,X) = -\lambda\eta(X), S(\xi,\xi) = \lambda. \tag{3.3}$$

$$QX = (\alpha - \lambda)X + \alpha\eta(X)\xi, \tag{3.4}$$

$$\kappa = -\lambda n + (n-1)\alpha, Q\xi = -\lambda\xi, \tag{3.5}$$

$$\lambda = -(n-1)(\alpha^2 + \sigma), \tag{3.6}$$

here, $\alpha^2 + \sigma \neq 0$ and κ is the scalar curvature of M. To reach our aim, we examine some proposition and lemma.

Definition 3.1. A tensor β of second order is said to be a second order parallel tensor provided $\nabla\beta = 0$, here, ∇ denotes the operator of covariant differentiation w.r.t. metric g.

Lemma 3.2. *On a Lorentzian α-para Kenmotsu manifold M, a second order symmetric parallel tensor is a constant multiple of associated metric g.*

Proof. We assume that β is a (0, 2) type symmetric tensor, then by definition, it is parallel, if $\nabla\beta = 0$. This provides

$$\beta(R(W, X)Y, Z) + \beta(Y, R(W, X)Z) = 0, \tag{3.7}$$

$\forall\ X, Y, Z, W$ on M. Putting $W = Y = Z = \xi$ in the above relation, we have

$$\beta(R(\xi, X)\xi, \xi) = 0,$$

here, β is symmetric.
On the Lorentzian α-para Kenmotsu manifold,

$$R(\xi, X)\xi = (\alpha^2 + \sigma)(X + \eta(X)\xi).$$

The above relation together with $\beta(R(\xi, X)\xi, \xi) = 0$ gives,

$$(\alpha^2 + \sigma)(g(X, \xi)\beta(\xi, \xi) + \beta(X, \xi)) = 0,$$

here, $\alpha^2 + \sigma \neq 0$. Therefore,

$$g(X, \xi)\beta(\xi, \xi) + \beta(X, \xi) = 0. \tag{3.8}$$

By differentiating covariantly w.r.t. Y, relation (3.8) gives

$$\begin{aligned}(\nabla_Y g)(X, \xi)\beta(\xi, \xi) + g(\nabla_Y X, \xi)\beta(\xi, \xi) + g(X, \nabla_Y\xi)\beta(\xi, \xi) +\\ g(X, \xi)(\nabla_Y\beta)(\xi, \xi) + 2g(X, \xi)\beta(\nabla_Y\xi, \xi)\\ + (\nabla_Y\beta)(X, \xi) + \beta(\nabla_Y X, \xi) + \beta(X, \nabla_Y\xi) = 0.\end{aligned} \tag{3.9}$$

Replacing X = $\nabla_Y X$ in (3.8), we obtain,

$$g(\nabla_Y X, \xi)\beta(\xi, \xi) + \beta(\nabla_Y X, \xi) = 0. \tag{3.10}$$

Using $\nabla_Y\xi = \text{-}\alpha Y - \alpha\eta(Y)\xi$ and equation (3.9) in the above relation, we get

$$\beta(X, Y) = -g(X, Y)\beta(\xi, \xi). \tag{3.11}$$

Differentiating covariantaly the above equation w.r.t. Z on M, we conclude that $\beta(\xi, \xi)$ is constant. Hence, the lemma 3.2 is proved. □

Proposition 3.3. *Let Lorentzian α-para Kenmotsu manifold M be of dimension n. If it admits a skew-symmetric tensor ϕ and $(0, 2)$ type symmetric tensor field h, then the structure (ϕ, ξ, η, g) has RS (g, ξ, λ), provided h is parallel w.r.t. ∇ on M.*

Proof. Suppose that,

$$h(X,Y) = (\pounds_{\xi} g)(X,Y) + 2S(X,Y). \tag{3.12}$$

With the help of equation (2.7) and equation (3.2), equation (3.12) reduces to

$$h(X,Y) = -2\lambda g(X,Y). \tag{3.13}$$

Putting $X = Y = \xi$ in (3.13) and comparing with (3.6) , we obtain

$$h(\xi,\xi) = 2\lambda = -2(n-1)(\alpha^2 + \sigma) \neq 0. \tag{3.14}$$

The foregoing lemma and equation (3.13) complete the statement of proposition. □

Corollary 3.4. *An n-dimensional Lorentzian α-para Kenmotsu manifold $M(\phi,\xi,\eta,g)$, $n > 1$, endowed with RS (g,ξ,λ) is an η-Einstein manifold and the RS (g,ξ,λ) is expanding and shrinking provided $\alpha^2 < \sigma$ and $\alpha^2 > \sigma$, respectively.*

Specifically, if $\alpha = 1$, the relation (3.14) provides $\lambda = -(n-1) < 0$, if $n > 1$. So, the following corollary can be stated:

Corollary 3.5. *An RS (g,ξ,λ) on a Lorentzian para-Kenmotsu manifold of dimension n, $n > 1$, is always shrinking.*

4. Perfect Fluid LP-Kenmotsu Spacetimes

Under this section, we study (L α-PK)$_{ST}$, i.e., Loentzian α-para Kenmotsu manifold of dimension-4, here α is a constant coefficient. Because, α being constant, therefore relation (2.10) gives σ= 0. Hence relation (2.12) yields

$$S(X,\xi) = 3\alpha^2\eta(X). \tag{4.1}$$

Above equation implies that $3\alpha^2$ is eigen value of Ricci tensor of (L α-PK)$_{ST}$. Let d be length of Ricci tensor, subsequently

$$d^2 = \sum_{i=1}^{n} S(Qe_i, e_i), \tag{4.2}$$

here, Q is symmetric endomorphism of tangent space at a point corresponding to Ricci tensor S. Let $\{e_i, e_n = \xi\}_{i=1}^{n-1}$ be an orthonormal basis of tangent space at each point of the manifold. Putting $X = Y = e_i$, $1 \leq i \leq 4$, in (3.2) and take summation from 1 to 4, we have,

$$\sum_{i=1}^{4} \epsilon_i S(e_i,e_i) = (\alpha-\lambda)\sum_{i=1}^{4} \epsilon_i g(e_i,e_i) + \alpha \sum_{i=1}^{4} \epsilon_i \eta(e_i)\eta(e_i),$$

where $\epsilon_i = g(e_i,e_i)$. By the definition of scalar curvature,

$$\kappa = 4c_1 + c_2, \tag{4.3}$$

where $c_1 = (\alpha-\lambda)$ and $c_2 = -\alpha$. Again from (3.3) and (4.1), we obtain

$$S(\xi,\xi) = -c_1 - c_2 = \lambda = -3\alpha^2. \tag{4.4}$$

In view of (3.2), (4.2) and (4.4), we find

$$d^2 = 3c_1^2 + (c_1 + c_2)^2. \tag{4.5}$$

Due to d^2 being constant, therefore $£_X d^2 = 0$. Since a compact Riemannian manifold of dimension-3 or more having scalar curvature constant admits an infinitesimal non-isometric conformal transformation X s.t. $£_X d^2 = 0$, subsequently, this is Einstein manifold [18]. In view of the preceding and above relations, we express the proposition in following way.

Proposition 4.1. *If (L α-PK)$_{ST}$ together with RS (g, ξ, λ) admits an infinitesimal non-isometric conformal transformation, then manifold is Einstein manifold, where as soliton is always shrinking.*

Taking into consideration the relation (4.3) together with propostion 4.1, we express corollaries, written below

Corollary 4.2. *An η-Einstein (L α-PK)$_{ST}$ along with an infinitesimal non-isometric conformal transformation doesn't admit a proper RS (g, ξ, λ).*

Corollary 4.3. *An (L α-PK)$_{ST}$ endowed with RS (g, ξ, λ) has a constant scalar curvature.*

We assume that the (L α-PK)$_{ST}$ is perfect fluid. Subsequently on it, Einstein's field equation (briefly, *EFE*) with cosmological term Λ is mentioned below

$$G(X, Y) = \tau T(X, Y) - \Lambda g(X, Y). \tag{4.6}$$

The Einstien tensor G(X,Y) is given by

$$G(X, Y) = S(X, Y) - \frac{\kappa}{2} g(X, Y).$$

From the above relation and the equation (4.6), we have

$$S(X, Y) - \frac{\kappa}{2} g(X, Y) = \tau T(X, Y) - \Lambda g(X, Y), \tag{4.7}$$

$\forall\ X, Y$. Here, τ reperesents gravitational constant and T is $(0, 2)$-type *EMT*. *EMT* is defined to be a perfect fluid [15], provided

$$T(X, Y) = (\rho + p)B(X)B(Y) + pg(X, Y), \tag{4.8}$$

where, ρ and p denote the energy density function and isotropic pressure function of the fluid, respectively. In the above equation, $B \neq 0$ is 1-form s.t. $g(X, U) = B(X)$, $\forall$ X and the flow vector field of the fluid is represented by U. In case, ρ and p are nil identically, subsequently the matter of the fluid is not pure and dust. If Reeb vector field ξ is a flow vector field of the fluid in (L α-PK)$_{ST}$, so, *EMT* has the relation

$$T(X, Y) = (\rho + p)\eta(X)\eta(Y) + pg(X, Y). \tag{4.9}$$

Using relation (3.2), (4.3), (4.4) and (4.7), we have

$$T(X, Y) = \frac{1}{\tau}[(\Lambda - \frac{1}{2}\alpha - 3\alpha^2)g(X, Y) + \alpha\eta(X)\eta(Y)]. \tag{4.10}$$

Thus, we conclude the following proposition:

Proposition 4.4. *If a perfect fluid (L α-PK)$_{ST}$ satisfies EFE with a cosmological term Λ, then relation (4.10) defines EMT of the space.*

Relations (4.4), (4.9) together with (4.10) give

$$\lambda = -\{\tau\rho + \Lambda - \frac{3}{2}\alpha\}.$$

So, the *RS* (g, ξ, λ) on a perfect fluid (L α-PK)$_{ST}$ expands, steady or shrinks, according as $\tau\rho + \Lambda < \frac{3}{2}\alpha$, $\tau\rho + \Lambda = \frac{3}{2}\alpha$ or $\tau\rho + \Lambda > \frac{3}{2}\alpha$, respectively.

Corollary 4.5. *Let perfect fluid (L α-PK)$_{ST}$ endowed with RS (g, ξ, λ) satisfies the EFE with cosmological term Λ, subsequently RS (g, ξ, λ) expands, steady or shrinks according as $\tau\rho + \Lambda < \frac{3}{2}\alpha$, $\tau\rho + \Lambda = \frac{3}{2}\alpha$ or $\tau\rho + \Lambda > \frac{3}{2}\alpha$, respectively.*

Proposition 4.6. *If a perfect fluid (L α-PK)$_{ST}$ satisfies EFE with cosmological term, subsequently (L α-PK)$_{ST}$ is quasi-Einstein. Furthermore, the perfect fluid (L α-PK)$_{ST}$ to be dust iff Lie-derivative of EMT along ξ vanishes.*

Proof. Relation (4.9) as well as (4.10) give,

$$(\rho + p - \frac{\alpha}{\tau})\eta(X)\eta(Y) = (\frac{2\Lambda - \alpha - 6\alpha^2 - 2\tau p}{2\tau})g(X, Y), \tag{4.11}$$

On putting $X = e_i$ and taking sumation $i = 1, 2, 3, 4$ the above equation gives

$$\Lambda = \frac{1}{4}\{(3p - \rho)\tau + 3\alpha(4\alpha + 1)\}. \tag{4.12}$$

Putting $X = Y = \xi$, relation (4.11) gives,

$$\Lambda = -\tau\rho + 3\alpha^2 + \frac{3}{2}\alpha. \tag{4.13}$$

Equations (4.12) and (4.13) together give

$$(\rho + p) = \frac{\alpha}{\tau}, \tau \neq 0. \tag{4.14}$$

In view of (4.9) and (4.14), the equation (4.7) gives

$$S = (\frac{\kappa}{2} - \Lambda + p\tau)g + \alpha\eta \otimes \eta.$$

Putting $X = Y = e_i$, $1 \leq i \leq 4$, in the above relation, it yields

$$\kappa = 4\Lambda + \alpha - 4p\tau. \tag{4.15}$$

From last two results, we find

$$S = (\Lambda + \frac{\alpha}{2} - p\tau)g + \alpha\eta \otimes \eta, \tag{4.16}$$

which indicates that perfect fluid (L α-PK)$_{ST}$ is quasi-Einstein [4, 11].
Futhermore, relation (4.9), together with (4.14) gives,

$$\tau T = p\tau g + \alpha\eta \otimes \eta, \tag{4.17}$$

if $\alpha \neq 0$, $\tau \neq 0$.
Lie derivative of (4.17) along ξ yields

$$\begin{aligned}\tau(\pounds_\xi T)(X, Y) = p\tau(\pounds_\xi g)(X, Y) + \alpha\{(\pounds_\xi g)(X, \xi)g(Y, \xi)\\ + g(X, \pounds_\xi \xi)g(Y, \xi) + g(X, \xi)(\pounds_\xi g)(Y, \xi) + g(X, \xi)g(Y, \pounds_\xi \xi)\}\end{aligned}$$

$\forall$ X,Y.
Clearly, $(\pounds_\xi g)(X, Y) = -2\alpha\{g(X, Y) + \eta(X)\eta(Y)\}$ and $\pounds_\xi \xi = 0$. From here, the foregoing equation takes the shape

$$\tau(\pounds_\xi T)(X, Y) = p\tau(\pounds_\xi g)(X, Y).$$

For every $\tau \neq 0$, from the above equation, we obtain

$$(\pounds_\xi T)(X, Y) = p(\pounds_\xi g)(X, Y). \tag{4.18}$$

Generally, $g(\phi X, \phi Y) \neq 0$, so ξ is not a Killing vector field on perfect fluid (L α-PK)$_{ST}$, in otherwords, $\pounds_\xi g \neq 0$. Hence, relation (4.18) gives $(\pounds_\xi T)(X, Y) = 0$ iff $p = 0$, if $\tau \neq 0$.
Thus, proposition 4.6 is proved. □

Corollary 4.7. *Let EFE with cosmological term Λ be satisfied by the perfect fluid (L α-PK)$_{ST}$. If Lie-derivative of EMT w.r.t. ξ vanishes, then expansion scalar together with acceleration vector becomes zero.*

Proof. We mention energy density and force equation in the followiong way,

$$\xi\rho = -(p + \rho)div\xi$$

and

$$(\rho + p)\nabla_\xi \xi = -grad\ p - (\xi p)\xi,$$

seperately, [15]. Relations (4.14), (4.18) and above two relations imply $div\xi = 0$, $\nabla_\xi \xi = 0$, where, $div\xi$ and $\nabla_\xi \xi$ denote expansion scalar as well as acceleration vector of the perfect fluid, respectively. So, corollary 4.7 is proved. □

Proposition 4.8. *Suppose that perfect fluid (L α-PK)$_{ST}$ satisfies the EFE with a cosmological term. If Lie-derivative of EMT along ξ is zero, then*

$$(\nabla_X S)(Y, Z) = \tau(\nabla_X T)(Y, Z).$$

Proof. With the help of relations (2.1), (4.1), (4.14) with (4.18), relation (4.16) reduces to,

$$S(X,Y) = (\Lambda + \frac{\alpha}{2})g(X,Y) + \alpha\eta(X)\eta(Y).$$

Replacing Y by ξ in the above equation, we have $\Lambda = 3\alpha^2 + \frac{\alpha}{2}$. Thus above relation turns into

$$S(X,Y) = \alpha(1+3\alpha)g(X,Y) + \alpha\eta(X)\eta(Y). \tag{4.19}$$

Now differentiating covariantly the above equation w.r.t. X, we get

$$(\nabla_X S(Y,Z) = -\alpha^2\{\eta(Z)g(X,Y) + \eta(Y)g(X,Z) + 2\eta(X)\eta(Y)\eta(Z)\}.$$

Furthermore, differentiating covariantly the relation (4.17) w.r.t. X, we obatain

$$(\nabla_X T)(Y,Z) = -\rho\alpha\{\eta(Z)g(X,Y) + \eta(Y)g(X,Z) + 2\eta(X)\eta(Y)\eta(Z)\}.$$

From above two relations, together with relations (4.14), (4.18), we obtain

$$\tau(\nabla_X T)(Y,Z)) = (\nabla_X S)(Y,Z). \tag{4.20}$$

□

Definition 4.9. Tensor U of type $(0,2)$ on a pesudo-Riemannian manifold is called cyclic parallel and Codazzi tensors, provided

$$(\nabla_X U)(Y,Z) + (\nabla_Y U)(Z,X) + (\nabla_Z U)(X,Y)) = 0$$

and

$$(\nabla_X U)(Y,Z) = (\nabla_Y U)(X,Z)$$

$\forall\ X, Y, Z$.

In the light of aforementioned definition as well as proposition 4.8, we state corollaries in the manner decscribed below:

Corollary 4.10. *Let a perfect fluid (L α-PK)$_{ST}$ satisfies EFE with a cosmological term. If Lie-derivative of EMT w.r.t. ξ vanishes, then Ricci tensor is of Codazzi type iff EMT is too Codazzi type.*

Corollary 4.11. *Suppose that perfect fluid (L α-PK)$_{ST}$ satisfies EFE with a cosmological term and Lie-derivative of EMT w.r.t. ξ vanishes. Then necessary and sufficient condition for the Ricci tensor to-be cyclic parallel is that EMT is cyclic parallel.*

5. Yamabe solitions on the Lorentzian α-para Kenmotsu spacetimes

The characteristics of *YS* on the (L α-PK)$_{ST}$ will be discussed in this part. At present, we justify the presence of *YS* (g, ξ, λ) for $V = \xi$ on an (L α-PK)$_{ST}$ in the following proposition:

Proposition 5.1. *A YS (g, ξ, λ) on (L α-PK)$_{ST}$, where α is constant coefficient, doesn't exist.*

Proof. We shall prove the proposition by contradiction. Assume that the (L α-PK)$_{ST}$ admit the *YS* (g, ξ, λ), then relation (1.3) for $V = \xi$ provides

$$\frac{1}{2}(\nabla_\xi g)(X, Y) = (\kappa - \lambda)g(X, Y), \tag{5.1}$$

which indicates,

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2(\kappa - \lambda)g(X, Y). \tag{5.2}$$

Taking into consideration the relation (2.7), relation (5.2) gives,

$$-\alpha\{g(X, Y) + \eta(X)\eta(Y)\} = (\kappa - \lambda)g(X, Y). \tag{5.3}$$

After replacing $X = \xi$ in relation (5.3), we obtain $\lambda = \kappa$, this implies to $\pounds_\xi g = 0$. In other words, ξ is a Killing vector field, contradicting our assumption, since normally $(\pounds_\xi g)(X, Y) = -2\alpha\{g(X, Y)+\eta(X)\eta(Y)\} \neq 0$. This completes the proof of the proposition 5.1. □

Proposition 5.2. *Any infinitesimal contact transformation on (L α-PK)$_{ST}$ endowed with YS (g, V, λ) is infinitesimal strict contact transformation.*

Proof. We assume that (M, g) is (L α-PK)$_{ST}$. Consider a Lorentzian para-Kenmotsu manifold of dimension 4 with constant coefficient α. Clearly, relations (4.15), together with (4.16), implies that scalar curvature of spacetime is constant. Consequently, relation (1.6), together with, (1.7) provides

$$(\pounds_V S)(X, Y) = -(n - 2)g(\nabla_X D\kappa, Y) + \Delta\kappa g(X, Y) \tag{5.4}$$

and

$$\psi = \kappa - \lambda.$$

Equation (5.4) gives

$$(\pounds_V S)(X, Y) = 0. \tag{5.5}$$

Replacing, $Y = \xi$ in (5.5), we obtain,

$$(\pounds_V S)(X, \xi) = 0. \tag{5.6}$$

Alternatively,

$$(\pounds_V S)(X, \xi) = \pounds_V(S(X, \xi)) - S(\pounds_V X, \xi) - S(X, \pounds_V \xi).$$

With the help of equations (1.5), (4.1) and (5.6), the preceding equation gives

$$S(X, \pounds_V \xi) = 3\alpha^2(\pounds_V \eta)(X) = 3\alpha^2\gamma\eta(X). \tag{5.7}$$

Putting $X = \xi$ in (5.7), we have

$$S(\xi, \pounds_V \xi) = -3\gamma\alpha^2. \tag{5.8}$$

In view of (4.1) and (5.8), we have,

$$\eta(\pounds_V \xi) = -\gamma. \tag{5.9}$$

Repeatedly, relation (1.5), together with relation (5.9), gives,

$$(\pounds_V \eta)(\xi) = -\gamma, \tag{5.10}$$

which implies,

$$\pounds_V(\eta(\xi)) - \eta(\pounds_V \xi) = 0. \tag{5.11}$$

Relation (5.9) and relation (5.11) give $\gamma = 0$. The definition 1.1, together with the relation (5.11), completes the proof of the proposition 5.2. □

Afterwards, let $YS\ (g, V, \lambda)$ has point-wise collinear potential vector field V with ξ, in other words, $V = \mu\xi$, where, μ is smooth function. So, $V = \mu\xi$ gives,

$$\nabla_X V = \nabla_X(\mu\xi) = (X\mu)\xi - \alpha\mu\phi^2(X). \tag{5.12}$$

Equations (2.1), (2.2), (2.7) and (5.12) yield

$$(\pounds_V g)(X, Y) = (X\mu)\eta(Y) + (Y\mu)\eta(X) - 2\alpha\mu g(X, Y) - 2\alpha\mu\eta(X)\eta(Y). \tag{5.13}$$

From (1.3) and (5.13) taken together, give

$$(X\mu)\eta(Y) + (Y\mu)\eta(X) - 2\alpha\mu g(X, Y) - 2\alpha\mu\eta(X)\eta(Y) = 2(\kappa - \lambda)g(X, Y). \tag{5.14}$$

On contraction, relation (5.14) along X, Y gives

$$\xi\mu = 4(\kappa - \lambda) + 3\alpha\mu. \tag{5.15}$$

Replacing $Y = \xi$ in (5.14) and using (5.15), it gives

$$X\mu = \{2(\kappa - \lambda) + 3\alpha\mu\}\eta(X). \tag{5.16}$$

Putting $X = \xi$ in (5.16), we obtain,

$$\xi\mu = -(2(\kappa - \lambda) + 3\alpha\mu). \tag{5.17}$$

Equation (5.15), together with the relation (5.17), gives $\lambda = \kappa + \alpha\mu$. Equation $\lambda = \kappa + \alpha\mu$, together with the relation (5.16), gives

$$X\mu = \alpha\mu\eta(X),$$

which gives,

$$g(D\mu, X) = \alpha\mu g(X, \xi).$$

In other words,

$$D\mu = \alpha\mu\xi = \alpha V.$$

So, the subsequent conclusions hold.

Proposition 5.3. *Let $YS\ (g, V, \lambda)$ be admitted by (L α-PK)$_{ST}$, so potential vector field V and $D\mu$ are linearly dependent.*

Corollary 5.4. *Let $YS\ (g, V, \lambda)$ be admitted by (L α-PK)$_{ST}$. Also potential vector field V is point-wise collinear with ξ, subsequently, (L α-PK)$_{ST}$ has space of constant curvature.*

6. Illustration

We take a smooth manifold $M = \{(u, v, w, t) \in \mathcal{R}^4$: u, v, w is non zero, t>0$\}$ of dimension-4, here (u, v, w, t) is the standard coordinate in 4-dimensional real space $\mathcal{R}^4$. Consider a set of linearly independent vector fields $\{e_1, e_2, e_3, e_4\}$ at every point of M. We define,

$$e_1 = e^{u+bt}\frac{\partial}{\partial u}, e_2 = e^{v+bt}\frac{\partial}{\partial v}, e_3 = e^{w+bt}\frac{\partial}{\partial w}, e_4 = \frac{\partial}{\partial t},$$

here, $b \neq 0$ constant. Lorentzian metric g on M is established in the following way:

$$g_{ij} = g(e_i, e_j) = \begin{cases} 0 & \text{if } i \neq j \\ -1 & \text{if } i = j = 4 \\ 1 & or\ else. \end{cases}$$

Assuming η is one-form corresponding to g is defined by

$$\eta(X) = g(X, e_4),$$

$\forall\ X \in \chi(TM)$, here $\chi(TM)$ be collection of vector fields on M. We define ϕ as $(1, 1)$-tensor field as follows:

$$\phi(e_1) = e_1, \phi(e_2) = e_2, \phi(e_3) = e_3, \phi(e_4) = 0,$$

from linearity property of ϕ and g, the following results can be easily proved:

$$\eta(e_4) = -1, \phi^2(X) = X + \eta(X)e_4, g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$\forall\ X, Y \in \chi(TM)$. So, when $e_4 = \xi$, structure (ϕ, ξ, η, g) leading to Lorentzian paracontact structure as well as manifold M equipped with Lorentzian paracontact structure is said to be Lorentzian paracontact manifold of dimension-4.
We represent $[X, Y]$ as Lie-derivative of X, Y, defined as $[X, Y] = XY - YX$. The non-zero constituents of Lie bracket are evaluated as below:

$$[e_1, e_4] = -be_1, [e_2, e_4] = -be_2, [e_3, e_4] = -be_3.$$

Let Riemannian connection w.r.t. g be denoted by ∇. So, when $e_4 = \xi$, we have the following results:

$$\nabla_{e_1}e_1 = -be_4, \nabla_{e_1}e_2 = 0, \nabla_{e_1}e_3 = 0, \nabla_{e_1}e_4 = -be_1,$$

$$\nabla_{e_2}e_1 = 0, \nabla_{e_2}e_2 = -be_4, \nabla_{e_2}e_3 = 0, \nabla_{e_2}e_4 = -be_2,$$

$$\nabla_{e_3}e_1 = 0, \nabla_{e_3}e_2 = 0, \nabla_{e_3}e_3 = -be_4, \nabla_{e_3}e_4 = -be_3,$$

$$\nabla_{e_4}e_1 = 0, \nabla_{e_4}e_2 = 0, \nabla_{e_4}e_3 = 0, \nabla_{e_4}e_4 = 0.$$

Assuming $X \in \chi(TM)$, so $X = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$, here $\{e_1, e_2, e_3, e_4\}$ be the basis of $\chi(TM)$. Above relations help verify $\nabla_X e_4 = -b\{X + \eta(X)e_4\}$ for each $X \in \chi(TM)$. Hence M is a Lorentzian para-Kenmotsu manifold of dimension-4 with coefficient

$\alpha = b \neq 0$. From the above relations, the non-vanishing constituents of the curvature tensor are evaluated as follows,

$$R(e_1, e_2)e_1 = -b^2 e_2, R(e_1, e_3)e_1 = -b^2 e_3, R(e_1, e_4)e_1 = -b^2 e_4,$$

$$R(e_1, e_2)e_2 = b^2 e_1, R(e_2, e_3)e_2 = -b^2 e_3, R(e_2, e_4)e_2 = -b^2 e_4,$$

$$R(e_1, e_3)e_3 = b^2 e_1, R(E_2, e_3)e_3 = b^2 e_2, R(e_3, e_4)e_3 = -b^2 e_4,$$

$$R(e_1, e_4)e_4 = -b^2 e_1, R(e_2, e_4)e_4 = -b^2 e_2, R(e_3, e_4)e_4 = -b^2 e_3.$$

From the definition of Ricci tensor S on M we have,
$S(X, Y) = \Sigma_{i=1}^{4} \varepsilon_i g(R(e_i, X)Y, e_i)$, here $\varepsilon_i = g(e_i, e_i)$. So, matrix representation of S is given by

$$S = \begin{bmatrix} 3b^2 & 0 & 0 & 0 \\ 0 & 3b^2 & 0 & 0 \\ 0 & 0 & 3b^2 & 0 \\ 0 & 0 & 0 & -3b^2 \end{bmatrix}.$$

Also, scalar curvature $\kappa = \Sigma_{i=1}^{4} \varepsilon_i S(e_i, e_i) = 12b^2$, this implies that (L α-PK)$_{ST}$ of dimension-4 has constant scalar curvature. From here the relations (2.8), (2.9), (2.11)-(2.16) together with (4.3) hold.

7. Acknowledgements

We would like to thank the reviewers for their constructive comments. We would also like to thank the administration of *BGP* for their editorial advice during the preperation of this work.

8. Conflicts of Interest

The authors declare no conflict of interest regarding the publication of this paper.

References

[1] E. Barbosa, E. Ribeiro Jr, *On conformal solitons of Yamabe flow*, Arch. Math. **101** (2013) 79–89.
[2] D.E. Blair, *Geometry of Contact and Symmetric Manifolds*, Biachauser, Boston (2010).
[3] M.C. Chaki, S. Roy, *Spacetime with covariant constant energy momentum tensor*, Int.J.Theor.Phys. **35** (1996) 1027–1032.
[4] S.K. Chaubey, *Existence of N(k)-quasi Einstein manifolds*, Facta Universitatis (NIS) Ser. Math. Inform. **32** (2017) 369–385.
[5] B. Chow, *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature*, Commun. Pure Appl. Math. **45** (1992) 1003–1014.
[6] U.C. De, A.A. Shaikh, *Lorentzian para-Sasakian Manifolds, in: Complex Manifolds and Contact Manifolds*, Narosa Publishing House Pvt. Ltd. (2009), pp.259–260.
[7] R.S. Hamilton, *The Ricci flow on surfaces*, Contemp. Math. **71** (1988) 237–261.
[8] A. Haseeb and R. Prasad, *Certain results on Lorentzian para-Kenmotsu Manifolds*, Bol. Soc. Paran. Mat.(3s.) **39** (3) (2021) 201–220.

[9] A. Haseeb and R. Prasad, *Some results on Lorentzian Para-Kenmotsu Manifolds*, Bulletin of the Transilvania University of Brasov, **13** (62) No.1-2020 Series III : Mathematics, Informatics, physics, pp.185–198.
[10] S.K. Hui, Y.C. Mandal, *Yamabe solitons on Kenmotsu manifolds*, Commun. Korean Math. Soc. 34 (2019) 321–331.
[11] S.K. Hui, R. Prasad, D. Chakraborty, *Ricci solitons on Kenmotsu Manifolds with respect to quater symmetric non metric ϕ-connection*, Ganita **67** (2) (2017) 195–204.
[12] H. Kachar, *Infinitesimal characterization of Friedmann Universes*, Arch. Math.Basel. **38** (1982) 58–64.
[13] V.R. Kaigorodov, *The curvature structure of spacetime*, Pro. Geom. **14** (1983) 177–202.
[14] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Nat. Sci. **12** (1989) 151–156.
[15] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Pure and Applied Mathematics, Vol. **103**, Academic press, New York (1983).
[16] Pankaj, S.K. Chaubey and R. Prasad, *Three Dimensional Lorentzian para-Kenmotsu manifolds and Yamabe Soliton*, Honam Mathematical J. **43** (4) (2021) 613-626.
[17] A.K. Raychaudhar, S. Benerji, A. Benerji, *General Relativity, Astro-physics and Cosmology*, Springer-Verlag (1992).
[18] K. Yano, *Integral formulas in Riemannian Geometry*, Marcel Dekkr, Inc., New York (1970).
[19] R. Ye, *Global existence and convergence of Yamabe flow*, J. Diff. Geom. **39** (1994) 35–50.

Rajendra Prasad, Department of Mathematics & Astronomy, University of Lucknow, Lucknow, India
e-mail: rp.manpur@rediffmail.com

Abhinav Verma, Department of Mathematics & Astronomy, University of Lucknow, Lucknow, India
e-mail: vabhinav831@gmail.com

Vindhyachal Singh Yadav, Department of Mathematics & Astronomy, University of Lucknow, Lucknow, India
e-mail: vs.yadav4@gmail.com