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COMPARATIVE STUDY OF ANALYTICAL AND NUMERICAL SOLUTIONS FOR SOLUTE TRANSPORT IN TWO-DIMENSIONAL HOMOGENEOUS AND ANISOTROPIC POROUS MEDIA WITH TIME-DEPENDENT VELOCITY FIELD

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Abstract

An analytical model for solute transport in two-dimensional homogeneous and anisotropic porous media with time dependent velocity field is studied and the result is compared with numerical model using explicit method. Mathematical models provide a very good description to observe the contaminant concentration pattern in finite and semi-infinite aquifer. This work deals with a two-dimensional solute transport model for a semi-infinite homogeneous and anisotropic porous formation. The impact of longitudinal, lateral as well as off-diagonal directions with temporally dependent dispersion coefficient is considered. Initial background concentration is assumed as space-dependent concentration. The input concentration is considered as logistic sigmoid function. The analytical solution is obtained, with the help of the Laplace Transform Technique and it is compared with the numerical solution obtained with the help of two-level explicit finite difference methods. The velocity distribution pattern is assumed to be transient in the form of the asymptotic, exponential and sigmoid function.

Keywords and phrases: Groundwater, Contamination, Numerical Methods.

1. Introduction

In groundwater modeling a two-dimensional advection-dispersion equation play vital role to observe the contaminant concentration pattern for small groundwater bodies where the longitudinal as well as lateral components of the aquifer are also taken into account. Due to high number of accuracy and large efficiency, analytical solution assumed to be more reliable as compared to the numerical one. Considering a wide range of Reynolds number in two-dimensional isotropic porous media, an analytical and experimental investigation of the longitudinal and lateral dispersion coefficient was studied by Harleman and Rumer (1963).The analytical solution for chemical transport in two-dimensional aquifer was presented assuming a constant velocity field. The solution was obtained by integrating the solution of a modified dimensional differential equation (Latinopolous et al., 1988). Assuming time-dependent dispersion along uniform flow,the two-dimensional advection diffusion equation was solved analytically for instantaneous point injection and continuous point source. The dispersion coefficient was considered as uniform, linear, asymptotically and exponentially varying temporally-dependent term (Aral and Liao, 1996). The analytical solution of the advection-dispersion equation for unsteady flow from instantaneous sources and for steady flow from continuous sources in one-, two-, and three- spatial dimensions was derived for space dependent dispersion coefficient in infinite media (Hunt, 1998). Using Green's function the analytical solution of one-, two- and three- dimensional solute transport problem with time-dependent dispersion coefficient was obtained subjected to Dirichlet as well as Neumann type boundary conditions (Marinoschi, et al., 1999).

Also, there are numerous three-dimensional analytical solutions published in the literature. Sim and Chrysikopoulos (1998) employed Laplace and Fourier transforms to derive an analytical solution for virus transport in infinite, there-dimensional, homogeneous, water saturated porous media, under continuous or point time-periodic source loading. Subsequently, Sim and Chrysikopoulos (1999) employed Laplace, Fourier and finite Fourier cosine transforms to derive analytical solutions for contaminant transport in homogeneous porous media with either semi-infinite or finite thickness, accounting for continuous as well as periodic source loadings from either a point or an elliptic source geometry. Chen (2007) derived an analytical solution of two-dimensional advection-dispersion equation in cylindrical co-ordinates for non-axisymmetrical solute transport in a tracer test system using a power series technique coupled with the Laplace and finite Fourier cosine transform techniques. Here, the longitudinal and transverse dispersivities were assumed to be a linear function of solute distance. In fact recent studies indicate that dispersivities are positively correlated with model sizes (solute distance) or spatial scales. Dai et al. (2007) presented representing aquifer architecture in macrodispersivity models with an analytical solution of the transition probability matrix. Considering dispersion coefficient as directly proportional to seepage velocity, the analytical solution of two- dimensional solute transport problems were presented with the help of Laplace transform technique. The solution was obtained for both the first and third type boundary conditions considering constant longitudinal and lateral dispersion coefficient (Zhan et al., 2009).

However, sometimes the dispersion coefficient and seepage velocity may vary with time. Keeping in view of this fact, a two-dimensional solute transport problem in a homogeneous finite aquifer was solved by Hankel Transform Technique in which the input source concentration was taken at the far end away from the origin. Initially the aquifer was assumed to be clean and the input concentration was taken as time-dependent exponentially decreasing function (Singh et al., 2010). The analytical solution of two-dimensional advection-dispersion equation subject to first- and third-type inlet boundary conditions was studied in the cylindrical co-ordinate system. The finite Hankel transform technique of second kind and the generalized integral transform technique were used to solve the problem (Chen et al., 2011). A hybrid Laplace transform finite analytic method for solving transport problems with large Peclet and Courant numbers were presented by Wang et al. (2012). Most of the problem solved in the previous studied, only the longitudinal and lateral dispersion coefficient terms were taken into consideration. However, the impacts of off diagonal dispersion coefficient terms were ignored in the previous studies which also affect the contaminant concentration pattern of the aquifer.

Frind and Germain (1986) examined the evolution of narrow, sharply defined contaminant plumes often observed in the field by numerical techniques. The solution was obtained by considering all the components of dispersion tensor as longitudinal, transverse as well as off diagonal. In order to find the accuracy and efficiency in the simulation of plumes the result was compared among principle direction method, alternative direction Galerkin method and conventional finite element method. An analytical solution for the advection-dispersion equation usually assumes that boundary and initial conditions are orthogonal to the principal axes of the dispersion tensor. However, this is not always the case in field studies or modeling scenarios. Using the method of Green's functions, a generalized analytical solution of the three-dimensional advective-dispersive equation in a semi-infinite porous medium was obtained. The solution was derived in an arbitrary Cartesian co-ordinate system subject to arbitrary initial condition and third type boundary condition with constant dispersion coefficients (Ellsworth and Butters, 1993).In various cases, the analytical solution of solute transport has been developed for simple geometrical problem whereas numerical solution may be more applicable and convenient for the complex hydrogeological or boundary conditions Batu(2006). Massabo et al., (2006) solved a two-dimensional advection equation with anisotropic dispersion for a homogeneous semi-infinite aquifer considering the constant dispersion coefficient. The effect of chemical decay or adsorption like reaction inside the liquid phase was also considered.

The analytical solution was obtained using Bessel function expansion for impulsive, continuous and finite pulse type pollutants release. In order to study the pollutant transport characteristics of the Han River (Korea) a two-dimensional advection dispersion model was developed using Streamline-Upwind Petrov-Galarkin method. The solution was obtained with the help of finite element method considering all the components of dispersion tensor in the problem (Lee and Seo, 2007). To predict the depth averaged concentration of solute transport in shallow water, a two-dimensional solute transport equation was solved numerically. The solution was derived for the dispersion diffusion tensor of depth-averaged mixing, whose principal direction coincides with the flow direction. In the diffusion stage, a second-order accurate central scheme was used while in the advection stage, a five-point total variation diminishing modification was made to the standard Mac Cormack scheme (Liang et al., 2010). Kong et al., (2011) developed a high resolution model for solving the two-dimensional advection and anisotropic diffusion problem of solute transport in shallow water. The numerical solution was obtained with the help of finite volume method considering all the components of diffusion coefficient tensor. In these problems, the dispersion coefficient tensors were assumed as constant term, however it may vary with time.Djordjevich and Savovic (2013)solved numerically a two dimensionalsolute transport model equation with variable coefficients in nonhomogeneoussemi-infinite medium. A uniform plus type source concentration was consideration for finding the numerical solution using explicit finite difference method.Chen et al. (2016)developed general analytical technique to solve a solute transport model equation by considering random shape sources in one-, two-, and three-dimensions.Guo et al. (2018)presented asymptotic expansion technique to obtainthe solution of two dimensionalsolute transport model for input point source concentration on the open watersurface in a geological formation. The results also analysed for concentration distribution along the longitudinal as well as transversal direction of flow in a geological formation.Guerrero et al. (2009) discusssed an analytical solution of the solute transport equation under theconsideration of linear sorption byusingintegral transform technique method. The leading equation wasfirst transformed into diffusion equation by using a mathematical substitution and found the solution for aquifer medium.

Moreover, both explicit and implicit forms were considered for transient nature of solute transport and inhomogeneity of the porous media. Soraganvi and Kumar (2009) developed numerical solution of ADE by using finite difference technique and finite volume methods to fictitious the flow and transport in a variably saturated zone. However, the one-dimensional ADE was solved with explored the retardation factor and zero-order production terms by employing Laplace integral transform technique (Das et al. 2017). The exact and approximate solutions of the ADE with spatial dispersion in semi-infinite heterogeneous geological formation were explored by Singh and Das (2015). The solutions were developed for the effect of solute retardation in terms of the linear isotherms and various decay parameters. Also, the better agreement between exact and approximate results for one dimensional ADE was found.Li et al. (2020) point out solute transport model equation for an injection well in mobile immobile confined aquifer with the impact radial dispersion, first-order reaction, linear sorption and first-order mass transfer. The breakthrough curves for concentration distribution also investigated for larger dispersivity and smaller effective porosityvalues for aquifer.

In the present study, a two-dimensional advection dispersion equation having different components of dispersion tensor is considered. At the initial stage, aquifer is assumed to be not clean i.e. there are some initial concentration exists in the aquifer in the form of exponentially decreasing function of space dependent terms. At the origin, the input point source concentration is taken as time-dependent in the form of logistic sigmoid function with Dirichlet type boundary conditions. The logistic sigmoid function is horizontally asymptotic in nature i.e. it increases continuously for and tends to 1 as . In the solute transport modeling context, the input point source concentration can be taken as of this form assuming that input concentration would initially increase with time and after a certain time period it would stabilize at an asymptotic value. The dispersion coefficient tensor is directly proportional to seepage velocity concept is used. The transient velocity in the form of 1) asymptotic function 2) exponential function and 3) algebraic sigmoid function is considered. Banks and Jerasate (1962) considered the problem of unsteady flow in porous media in which they derived linear and exponentially time-dependent form of seepage velocity to study the salinity problems. In the present study, the first two types of velocity expressions i.e., asymptotic and exponentially time-dependent, have been taken from Aral and Liao (1996). The algebraic sigmoid function that starts a progress from small beginning, accelerates in the rainy season and reaches up to a limit over a period of time. It shows the significance of different forms of time-dependent velocity expression considered and studied depending upon the complexity of geological formations i.e., homogeneous and anisotropic aquifer. Also, the effects of spatially variable velocity on contaminant transport is of great interest to environmental engineers and hydrogyologists and it has been explored extensively (Chrysikopoulos et al., 1992). The analytical solution is derived with the help of Laplace transform technique and it is compared with the numerical result obtained with the help of two level explicit finite difference methods.

2. Materials & Methods

Consider a two-dimensional homogeneous semi-infinite aquifer subjected to a time-dependent point source contamination in the form of logistic sigmoid function at the origin. The longitudinal and lateral directions at the origin are taken as x and y axes, respectively. Let $c[ML^{-3}]$ is the contaminant concentration in the aquifer at any time $t[T]$; $u[LT^{-1}]$ and $v[LT^{-1}]$ are and x and y groundwater velocity components, respectively; $D_{xx}[L^2T^{-1}]$, $D_{yy}[L^2T^{-1}]$ are the dispersion coefficients along x and y axes, respectively; $D_{xy}[L^2T^{-1}]$ and $D_{yx}[L^2T^{-1}]$ are the off diagonal dispersion coefficients. Initially the aquifer has some initial background concentration, which is function of space variable, say, $c_i\,exp[-\gamma(x+y\sqrt{\frac{D_{yy}}{D_{xx}}})]$,where γ $[L^{-1}]$ is the decay parameter and $c_i[ML^{-3}]$ is the solute concentration.

The advection-dispersion equation representing the two-dimensional homogeneous semi-infinite aquifer can be written as

$$\frac{\delta c}{\delta t} = \frac{\delta}{\delta x}(D_{xx}\frac{\delta c}{\delta x} + D_{xy}\frac{\delta c}{\delta y}) + \frac{\delta}{\delta y}(D_{yy}\frac{\delta c}{\delta y} + D_{yx}\frac{\delta c}{\delta x}) - u\frac{\delta c}{\delta x} - v\frac{\delta c}{\delta y} \tag{2.1}$$

Let u and v be expressed as

$$u = u_0 f(t) \;\text{ and }\; v = v_0 f(t) \tag{2.2}$$

where $u_0[LT^{-1}]$ and $v_0[LT^{-1}]$ are the initial values of u and v, respectively, and $f(t)$ is assumed to be a sinusoidally varying function or an exponentially decreasing function. The dispersion coefficient terms D_{xx}, D_{yy} and D_{xy} or D_{yx} can be expressed as (Batu, 2006).

$$D_{xx} = \frac{a_L u^2 + a_T v^2}{\sqrt{u^2+v^2}} = \frac{(a_L u_0^2 + a_T v_0^2) f(t)}{\sqrt{u_0^2+v_0^2}} = D_{x0} f(t),$$
$$D_{yy} = \frac{a_L u^2 + a_T v^2}{\sqrt{u^2+v^2}} = \frac{(a_T u_0^2 + a_L v_0^2) f(t)}{\sqrt{u_0^2+v_0^2}} = D_{y0} f(t), \tag{2.3}$$
$$D_{xy} = \frac{(a_L + a_T) uv}{\sqrt{u^2+v^2}} = \frac{(a_L - a_T) u_0 v_0 f(t)}{\sqrt{u_0^2+v_0^2}} = D_{xy0} f(t) = D_{yx}$$

where a_L and a_T are the dispersivity $[L]$ along longitudinal and lateral directions that depends on the distribution of aquifer heterogeneities and scale of the field problem (Bedient et al., 1999); D_{x0}, D_{y0} and D_{xy0} are the initial values of D_{xx}, D_{yy} and D_{xy} or D_{yx}, respectively.

As stated earlier, the initial contaminant concentration is a function of space say $c_i\, exp[-\gamma\,(x + y\sqrt{\frac{D_{yy}}{D_{xx}}})]$ at $t = 0$ and at the origin the time dependent point source concentration is taken in the form of logistic sigmoid function as $\frac{c_0}{[1+exp(-qt)]}$. Here $q[T^{-1}]$ is contaminant decay rate coefficient and $c_0[ML^{-3}]$ is solute concentration. Hence, the initial and boundary conditions are expressed below and physical system is depicted in Fig.1.

$$c(x,y,t) = c_i\, exp[-\gamma\,(x + y\sqrt{\frac{D_{yy}}{D_{xx}}})], x > 0, y > 0, t = 0 \tag{2.4}$$

Using Eq. (2.3), Eq. (2.4) can be expressed as

$$c(x,y,t) = c_i\, exp[-\gamma\,(x + y\sqrt{\frac{D_{y0}}{D_{x0}}})], x > 0, y > 0, t = 0 \tag{2.5}$$

$$c(x,y,t) = \frac{c_0}{[1 + exp(-qt)]}, x > 0, y > 0, t = 0 \tag{2.6}$$

$$\frac{\delta c}{\delta x} = 0, \frac{\delta c}{\delta y} = 0, x \to \infty, y \to \infty, t > 0 \tag{2.7}$$

Using Eqs. (2.2) and (2.3), Eq. (2.1) becomes

$$\frac{1}{f(t)}\frac{\delta c}{\delta t} = D_{x0}\frac{\delta^2 c}{\delta x^2} + D_{y0}\frac{\delta^2 c}{\delta y^2} + 2D_{xy0}\frac{\delta^2 c}{\delta x \delta y} - u_0\frac{\delta c}{\delta x} - v_0\frac{\delta c}{\delta y} \tag{2.8}$$

Introducing a new time variable T^* by the following transformation (Crank, 1975):

$$T^* = \int_0^t f(t)dt \tag{2.9}$$

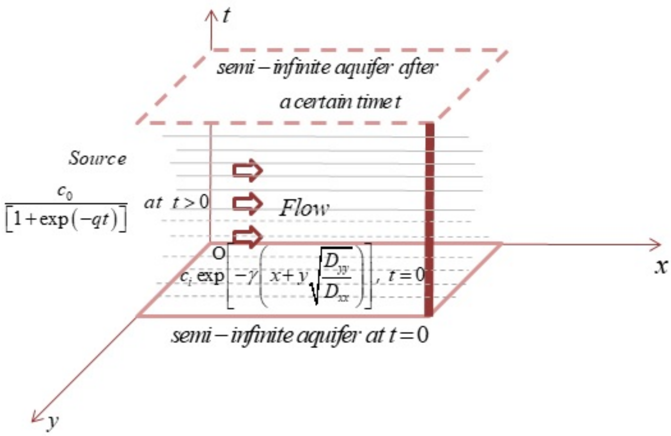

FIGURE 1. Physical Model of the problem

Here, the term $f(t)$ represents time dependent form of velocity expression. Using Eq. (2.9), Eq. (2.8) subject to Eqs. (2.5) - (2.7) reduces to

$$\frac{\delta c}{\delta T^*} = D_{x0}\frac{\delta^2 c}{\delta x^2} + D_{y0}\frac{\delta^2 c}{\delta y^2} + 2D_{xy0}\frac{\delta^2 c}{\delta x \delta y} - u_0\frac{\delta c}{\delta x} - v_0\frac{\delta c}{\delta y} \tag{2.10}$$

$$c(x, y, T^*) = c_i\, exp[-\gamma\,(x + y\sqrt{\frac{D_{y0}}{D_{x0}}})], x > 0, y > 0, T^* = 0 \tag{2.11}$$

$$c(x, y, T^*) = \frac{c_0}{[1 + exp(-qT^*)]}, x > 0, y > 0, T^* > 0 \tag{2.12}$$

$$\frac{\delta c}{\delta x} = 0, \frac{\delta c}{\delta y} = 0, x \to \infty, y \to \infty, T^* > 0 \tag{2.13}$$

3. Analytical Solution

Let us define a new space variable as

$$\xi = x + y\sqrt{\frac{D_{y0}}{D_{x0}}} \;\; or \;\; x + y\sqrt{\frac{v_0}{u_0}} \tag{3.1}$$

Using Eq. (3.1), Eqs. (2.10) - (2.13) can be written as

$$\frac{\delta c}{\delta T^*} = D_1\frac{\delta^2 c}{\delta \xi^2} - U_1\frac{\delta c}{\delta \xi} \tag{3.2}$$

where, $D_1 = D_{x0} + \frac{D_{y0}^2}{D_{x0}} + 2D_{xy0}\sqrt{\frac{D_{y0}}{D_{x0}}}$ and $U_1 = u_0 + v_0\sqrt{\frac{v_0}{u_0}}$
where,

$$c(\xi, T^*) = c_i exp(-\gamma\xi), \;\; \xi > 0, T^* = 0, \tag{3.3}$$

$$c(\xi, T^*) = \frac{c_0}{2}[1 + \frac{qT^*}{2}], \;\; \xi = 0, T^* > 0 \tag{3.4}$$

$$\text{and} \;\; \frac{\delta c}{\delta \xi} = 0, \;\; \xi \to \infty, \;\; T^* > 0 \tag{3.5}$$

In order to reduce the convective term from Eq. (3.2), using the following transformation

$$c(\xi, T^*) = K(\xi, T^*)\, exp\, (\frac{U_1}{2D_1}\xi - \frac{U_1^2}{4D_1}T^*) \tag{3.6}$$

Eqs. (3.2)- (3.5) become

$$\frac{1}{D_1}\frac{\delta K}{\delta T^*} = \frac{\delta^2 K}{\delta \xi^2} \tag{3.7}$$

subject to

$$K(\xi, T^*) = c_i\, exp\, [-(\gamma + \frac{U_1}{2D_1})\xi], \;\; \xi > 0, \;\; T^* = 0 \tag{3.8}$$

$$K(\xi, T^*) = \frac{c_0}{2}[1 + \frac{qT^*}{2}]exp(\frac{U_1^2 T^*}{4D_1}), \;\; \xi = 0, \;\; T^* > 0 \tag{3.9}$$

$$\frac{\delta K}{\delta \xi} = -\frac{U_1 K}{2D_1}, \;\; \xi \to \infty, \;\; T^* > 0 \tag{3.10}$$

Applying Laplace transform to Eqs. (3.7) - (3.10) it give

$$\bar{K}(\xi, p) = c_1\, exp(\sqrt{\frac{p}{D_1}}\xi) + c_2\, exp(-\sqrt{\frac{p}{D_1}}\xi) + \frac{c_i}{p - (\gamma + \frac{U_1}{2D_1})^2 D_1}exp(-\gamma - \frac{U_1}{2D_1})\xi \tag{3.11}$$

where c_1 and c_2 are constants.
Using Eqs. (3.9) and (3.10) in Eq. (3.11), the values of c_1 and c_2 can be obtained as

$$c_1 = 0 \;\; \text{and} \;\; c_2 = \frac{c_0}{2}[\frac{1}{(p - \frac{U_1^2}{4D_1})} + \frac{q}{2}\frac{1}{(p - \frac{U_1^2}{4D_1})^2}] - \frac{c_i}{p - (\gamma + \frac{U_1}{2D_1})^2 D_1} \tag{3.12}$$

Substituting values of c_1 and c_2 from Eq. (3.12), in Eq. (3.11), it can be written as

$$\bar{K}(\xi, p) = [\frac{c_0}{2}\frac{1}{(p - \frac{U_1^2}{4D_1})} + \frac{q}{2}\frac{1}{(p - \frac{U_1^2}{4D_1})^2} - \frac{c_i}{p - (\gamma + \frac{U_1}{2D_1})^2 D_1}]exp(-\sqrt{\frac{p}{D_1}}\xi)$$
$$+ \frac{c_i}{p - (\gamma + \frac{U_1}{2D_1})^2 D_1}]exp(-\gamma - \frac{U_1}{2D_1})\xi \tag{3.13}$$

Now, taking the inverse Laplace transform of Eq. (3.13) the required solution can be written as

$$c(\xi, T^*) = \frac{c_0}{2}c_1(\xi, T^*) + \frac{qc_0}{4}c_2(\xi, T^*) - c_i c_3(\xi, T^*) + c_i c_4(\xi, T^*) \tag{3.14}$$

where,

$$c_1(\xi,T^*)=\frac{1}{2}erfc(\frac{\xi}{2\sqrt{D_1T^*}}-\frac{U_1T^*}{2\sqrt{D_1T^*}})+\frac{1}{2}exp(\frac{U_1\xi}{D_1})erfc(\frac{\xi}{2\sqrt{D_1T^*}}+\frac{U_1T^*}{2\sqrt{D_1T^*}})$$

$$c_2(\xi,T^*)=\frac{1}{2U_1}(U_1T^*-\xi)erfc(\frac{\xi}{2\sqrt{D_1T^*}}-\frac{U_1T^*}{2\sqrt{D_1T^*}})$$
$$+\frac{1}{2U_1}(U_1T^*+\xi)exp(\frac{U_1\xi}{D_1})erfc(\frac{\xi}{2\sqrt{D_1T^*}}+\frac{U_1T^*}{2\sqrt{D_1T^*}})$$

$$c_3(\xi,T^*)=\frac{1}{2}exp(\gamma^2D_1T^*+\gamma U_1T^*-\gamma\xi)erfc(\frac{\xi}{2\sqrt{D_1T^*}}-(\gamma+\frac{U_1}{2D_1})\sqrt{D_1T^*})$$
$$+\frac{1}{2}exp(\gamma^2D_1T^*+\gamma U_1T^*+\gamma\xi+\frac{U_1\xi}{2D_1})erfc(\frac{\xi}{2\sqrt{D_1T^*}}+(\gamma+\frac{U_1}{2D_1})\sqrt{D_1T^*})$$

$$c_4(\xi,T^*)=exp(\gamma^2D_1T^*+\gamma U_1T^*-\gamma\xi)$$

4. Numerical Solution

The numerical solution of Eq. (2.10) with initial and boundary conditions Eqs. (2.11) - (2.13), is obtained with the help of two-level explicit finite difference method. In the present problem, the aquifer is of semi-infinite length. In order to convert the problem of semi-infinite domain, $x\in(0,\infty),\ y\in(0,\infty)$into a finite domain $X\in(0,\infty),\ Y\in(0,\infty)$ following transformation is used

$$X=1-exp(-x) \tag{4.1}$$

$$Y=1-exp(-y) \tag{4.2}$$

Using Eqs. (4.1) and (4.2), Eqs. (2.10)-(2.13) become

$$\begin{aligned}\frac{\delta c}{\delta T^*}&=D_{x0}(1-X)^2\frac{\delta^2c}{\delta X^2}+D_{y0}(1-Y)^2\frac{\delta^2c}{\delta Y^2}+2D_{xy0}(1-X)(1-Y)\frac{\delta^2c}{\delta X\delta Y}\\&-u_0(1-X)\frac{\delta c}{\delta X}-v_0(1-Y)\frac{\delta c}{\delta Y}-D_{x0}(1-X)\frac{\delta c}{\delta X}-D_{y0}(1-Y)\frac{\delta c}{\delta Y}\end{aligned} \tag{4.3}$$

$$c(X,Y,T^*)=c_i\ exp\ [\gamma(log(1-X)+log(1-Y)\sqrt{\frac{D_{y0}}{D_{x0}}})],\ X>0,\ Y>0,\ T^*=0 \tag{4.4}$$

$$c(X,Y,T^*)=\frac{c_0}{[1+exp(-qT^*)]},\ X=0,\ Y=0,\ T^*>0 \tag{4.5}$$

$$\frac{\delta c}{\delta X}=0,\ \frac{\delta c}{\delta Y}=0,\ X=1,\ Y=1,\ T^*>0 \tag{4.6}$$

The X, Y and T^* domains are divided into equal number of subinterval and represented as

$$\begin{aligned} X_i &= X_{i-1} + \Delta X,\ i = 1,2,...,M,\ X_0 = 0,\ \Delta X = 0.1 \\ Y_i &= Y_{i-1} + \Delta Y,\ j = 1,2,...,N,\ Y_0 = 0,\ \Delta Y = 0.1 \\ T_k^* &= T_{k-1}^* + \Delta T^*,\ k = 1,2,...,I,\ T_0^* = 0,\ \Delta T^* = 0.001 \end{aligned} \tag{4.7}$$

The contaminant concentration at a point (X_i, Y_j) at k^{th} sub-interval of time T^* is denoted as $c_{i,j,k}$. The first and second order derivative in Eq. (2.10) is approximated as forward difference approximation and central difference approximation respectively. Using two-level explicit finite difference methods; Eqs. (2.10) to (2.13) can be written as

$$\begin{aligned} c_{i,j,k+1} = {} & c_{i,j,k} + D_{x0}(1 - X_i)^2(c_{i+1,j,k} - 2c_{i,j,k} + c_{i-1,j,k})\frac{\Delta T^*}{\Delta X^2} \\ & + D_{y0}(1 - Y_j)^2)\frac{\Delta T^*}{\Delta Y^2}(c_{i,j+1,k} - 2c_{i,j,k} + c_{i,j-1,k}) \\ & + D_{xy0}(1 - X_i)(1 - Y_i)(c_{i+1,j+1,k} - c_{i-1,j+1,k} + c_{i+1,j-1,k} + c_{i-1,j-1,k})\frac{\Delta T^*}{2\Delta X \Delta Y} \\ & - D_{x0}(1 - X_i)(c_{i+1,j,k} - c_{i-1,j,k})\frac{\Delta T^*}{2\Delta X} - D_{y0}(1 - Y_i)(c_{i,j+1,k} - c_{i,j-1,k})\frac{\Delta T^*}{2\Delta Y} \\ & - u_0(1 - X_i)(c_{i+1,j,k} - c_{i-1,j,k})\frac{\Delta T^*}{2\Delta X} - v_0(1 - Y_j)(c_{i,j+1,k} - c_{i,j-1,k})\frac{\Delta T^*}{2\Delta Y} \end{aligned} \tag{4.8}$$

$$c_{i,j,0} = c_i\ exp\ [\gamma(log(1 - X_i) + log(1 - Y_j)\sqrt{\frac{D_{y0}}{D_{x0}}})], i > 0, j > 0 \tag{4.9}$$

$$c_{0,0,k} = \frac{1}{2}[1 + \frac{QT^*}{2}],\ k > 0 \tag{4.10}$$

$$c_{M,j,k} = c_{M-1,j,k}\ \ and\ \ c_{i,N,k} = c_{i,N-1,k} \quad i = M, j = N, k > 0 \tag{4.11}$$

The limitation of an explicit scheme is that there is a certain stability criterion associated with it, so that the size of time step cannot exceed a certain value. For the present problem, the stability analysis has been done to improve the accuracy of the numerical solution (Bear and Verrujit 1998) and the stability condition for the size of time step is obtained as

$$0 < \Delta T^* \leq \frac{1}{2(\frac{D_{x0}}{(\Delta X)^2} + \frac{D_{y0}}{(\Delta Y)^2} + \frac{D_{xy0}}{(2\Delta X\Delta Y)} + \frac{(u_0+D_{x0})}{2\Delta X} + \frac{(v_0+D_{y0})}{2\Delta Y})} \tag{4.12}$$

which satisfy the results and conditions obtained by Ashtiani and Hosseini (2005).

5. Results and Discussion

We consider three different time-dependent forms of velocity expression in which the first two has been followed by Aral and Liao (1996) and last one is based on the properties of algebraic sigmoid function which include the error function. It starts a progress from small beginning, accelerates in the rainy season and reaches up to a limit over a period of time. These expressions can be written as follows:

1. Exponentially decreasing form of velocity

$$u = u_0 f(t),\ f(t) = 1 - exp(\frac{-mt}{K}) \implies T^* = \frac{1}{m}(mt + K\,(exp(\frac{-mt}{K}) - 1)) \tag{5.1}$$

2. Asymptotic form of velocity

$$u = u_0 f(t),\ f(t) = \frac{mt}{(mt + K)} \implies T^* = \frac{1}{m}(mt - K\frac{mt}{(mt + K)}) \tag{5.2}$$

3. Algebraic Sigmoid form of velocity

$$u = u_0 f(t),\ f(t) = \frac{mt}{\sqrt{(mt)^2 + K^2}} \implies T^* = \frac{1}{m}(\sqrt{(mt)^2 + K^2} - K) \tag{5.3}$$

where K is the arbitrary constant. Considering $K = 0$ in Eq. (5.1) to (5.3); results as $f(t) = 1$. It represents the problem with uniform velocity and dispersion coefficient.

The two-dimensional analytical and numerical results are computed for the input values $a_L = 5\ km$, $a_T = 0.5\ km$, $c_i = 0.01$, $c_0 = 1$, $x = 1\ km$, $y = 1\ km$, $q = 0.0001(/day)$, $\gamma = 0.001(/km)$, $u_0 = 0.001\ km/year$ and $v_0 = 0.0001\ km/year$. Fig. 2 represents the contaminant concentration pattern for uniform velocity and dispersion coefficient i.e. at $K = 0$. The analytical solution is compared with the numerical result. Here, it is observed that the contaminant concentration decreases with distance. The numerical solution follows the same pattern as obtained by the analytical solution. Due to presence of numerical error may be up to 5-10%, the curve slightly deviates from the analytical solution in the middle of the domain.

Fig. 3 represents the contaminant concentration pattern for asymptotic type of velocity expression for different values of dispersivity parameters a_L and a_T . Here, the figure shows that the contaminant concentration increases on increasing the longitudinal and lateral dispersivities at each of the positions which may happen due to off diagonal impact. The contaminant concentration values decreases with distance even we increase the dispersivity parameters. Fig. 4 shows the contaminant concentration pattern for asymptotic type of velocity expression for different values of initial seepage velocities. It represents that the contaminant concentration values increases with increasing the uniform seepage velocity at each of the positions. However, the contaminant concentration values decreases with distance even we increase the seepage velocity.

Fig. 5 represents the concentration profile for different values of arbitrary constant K and it is observed that the contaminant concentration decreases on increasing the value of K at each of the positions. The contaminant concentration values also decreases with distance even we increase the arbitrary constant. The impact of variation in arbitrary constant is significantly observed in this figure.

Fig. 6 shows the concentration profile for asymptotic type of velocity expression for different values of mt. It represents that contaminant concentration decreases with distance and increases with time at each of the positions. The variation in time is significantly observed in concentration pattern.

Fig. 7 shows the concentration profiles for all the three types of transient form of velocity expression i.e., asymptotic, sigmoid and exponential and it is observed that the contaminant concentration values decreases uniformly with distance in each of the velocity expression. The concentration pattern with exponential type of velocity expression decreases more rapidly as compared to the other two. The obtained concentration values are more in asymptotic form of velocity expression than sigmoid and exponential one.

6. Summary and Conclusion

A comparative study of two-dimensional advection dispersion equation in a homogeneous, anisotropic, semi-infinite aquifer is made. Initially the aquifer is not considered as solute free, it is assumed to vary with distance in both longitudinal as well as transverse direction. The input concentration is taken as in the form of logistic sigmoid function at the origin while at the other end the concentration gradients in both the directions are supposed to be zero. The seepage velocity and dispersion coefficient are taken as three different types of time-dependent function 1) exponentially decreasing function 2) asymptotic function 3) algebraic sigmoid function. The contaminant concentration profile is also observed for uniform seepage velocity and dispersion coefficient. The solution is obtained by both analytical and numerical method. To find the analytical solution, Laplace transform technique is used however, for numerical result two-level explicit finite difference method is used. The effect of off diagonal dispersion coefficient is also taken into consideration. The solution of this problem may be applicable to groundwater resource management and assessment of various significant parameters as a preliminary predictive tool. The following conclusions are drawn:

1. The numerical results obtained that follows the same pattern as obtained from the analytical one. This validates the accuracy of analytical method to solve the problem.
2. The contaminant concentration increases on increasing the dispersivity parameters but decreases on increasing the arbitrary constant K.
3. The contaminant concentration decreases gradually with distance in all the cases and increases with time.

4. On comparing all the three types of velocity expressions, the contaminant concentration for exponential form of velocity expression decreases more rapidly as compared to asymptotic and sigmoid type of velocity expressions.
5. The off diagonal impact i.e.,D_{xy} or D_{yx} is significantly observed in contaminant concentration profile even for minimum values of dispersivities from each of the Figs. (2-7).

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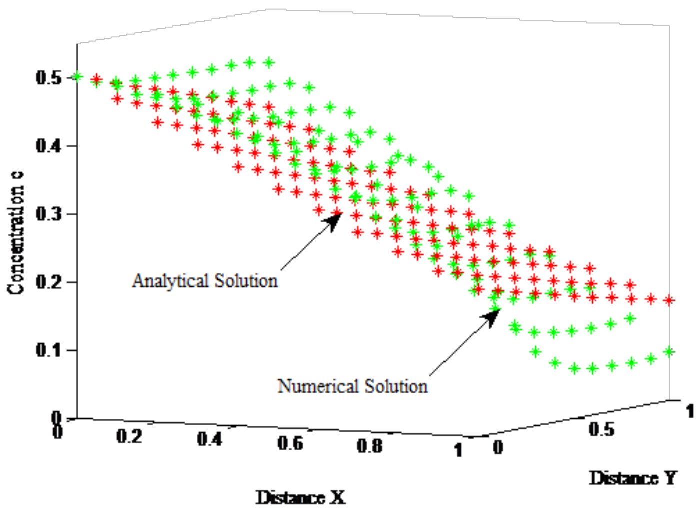

FIGURE 2. Concentration Profile for asymptotic type velocity expression for $K = 0$, $mt = 2$, $a_L = 5$, $a_T = 0.5$, $u_0 = 0.001$, $v_0 = 0.0001$.

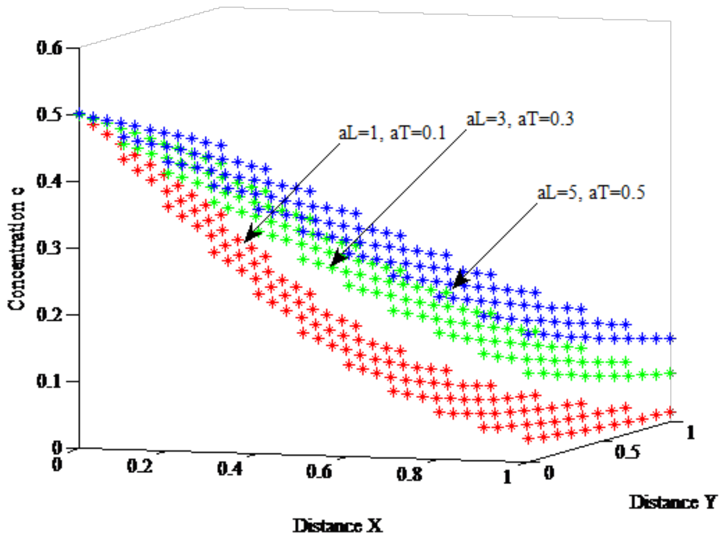

FIGURE 3. Concentration Profile for asymptotic type velocity expression for $K = 10$, $mt = 5$, $u_0 = 0.001$, $v_0 = 0.0001$.

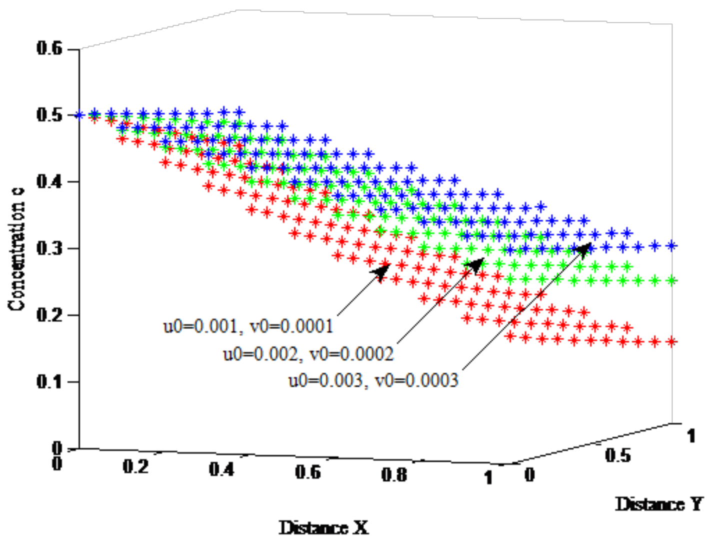

FIGURE 4. Concentration Profile for asymptotic type velocity expression for $K = 10$, $mt = 2$, $a_L = 5$, $a_T = 0.5$.

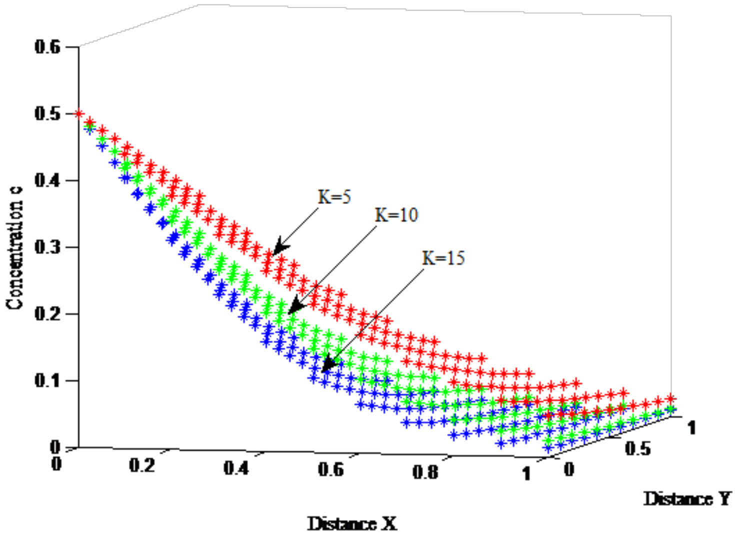

FIGURE 5. Concentration Profile for asymptotic type velocity expression for $mt = 2$, $a_L = 5$, $a_T = 0.5$, $u_0 = 0.001$, $v_0 = 0.0001$.

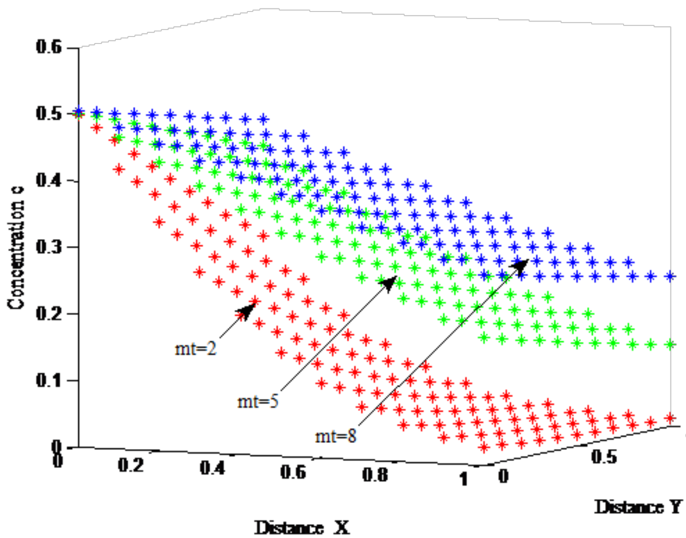

FIGURE 6. Concentration Profile for asymptotic type velocity expression for $K = 5$, $a_L = 5$, $a_T = 0.5$, $u_0 = 0.001$, $v_0 = 0.0001$.

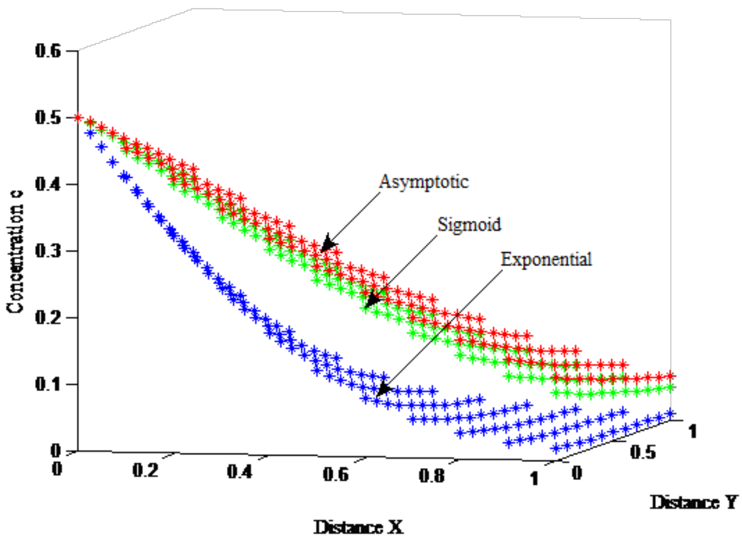

FIGURE 7. Concentration Profile for three different types of velocity expression for $K = 2$, $mt = 2$, $a_L = 5$, $a_T = 0.5$, $u_0 = 0.001$, $v_0 = 0.0001$.

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