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ZALCMAN CONJECTURE AND SECOND ORDER HANKEL DETERMINANT FOR α-SPIRALLIKE STARLIKE FUNCTIONS AND α-SPIRALLIKE BOUNDED TURNING FUNCTIONS CONNECTED WITH k-FIBONACCI SEQUENCE

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Abstract

In this article, we estimate absolute values of second order Hankel determinant, Zalcman conjecture and Fekete-Szegö inequality for α-spirallike starlike functions and α-spirallike bounded turning functions in the unit disc $\mathcal{E}$, subordinate to k-Fibonacci sequence.

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1. Introduction

The class of all regular functions is denoted by $\mathcal{A}$ and it has the following series representation

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots = z + \sum_{n\geq 2}^{\infty} a_n z^n \tag{1.1}$$

in $\mathcal{E} = \{z :| z |< 1\}$ with normalized conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S} \subset \mathcal{A}$, which are simple in $\mathcal{E}$. Let w_1 be a Schwarz function in $\mathcal{E}$ satisfying $w_1(0) = 0$ and $| w_1(z_1) |\leq 1$ such that $h(z_1) = g(w_1(z_1))$ $(z_1 \in \mathcal{E})$. Here h and g are any two analytic functions and we say that "*h is subordinate to g*" $(h \prec g)$. In particular, $h(0) = g(0)$ and $h(\mathcal{E}) \subset g(\mathcal{E})$ when g is univalent in $\mathcal{E}$.

For example: $z_1^{2n} \prec z_1^2$
$h(z_1) = z_1^{2n}$, $g(z_1) = z_1^2$ and $w_1(z_1) = z_1^n$
$h(z_1) = g(w_1(z_1))$
$h(z_1) = g(z_1^n) = (z_1^n)^2 = z_1^{2n}$

Let p be a regular function and it has a series expansion $p(z) = 1 + p_1z + p_2z^2 + \cdots$ in $\mathcal{E}$ satisfying the conditions $p(0) = 1$ and $\Re\mathfrak{e}(p(z)) > 0$. Such set of functions are

represented by $\mathcal{P}$. The functions in the class $\mathcal{P}$ are said to be "Functions with positive real part". The theory of bounded turning functions ($\mathcal{R}$) was developed by Mac Gregor [15] in 1962.
A function $f \in \mathcal{R}$ if and only if $\Re e(f'(z)) > 0$.
Let $\mathcal{S}^* \subset \mathcal{S}$, $\mathcal{S}^*$ be the set of starlike functions, which was defined by Robertson [21].
A function $f \in \mathcal{S}^*$ if and only if

$$\Re e\Big(\frac{zf'(z)}{f(z)}\Big) > 0, \ (z \in \mathcal{E})$$

The generalization of starlikeness leads to the class of spirallike, which gives a useful criterion for univalence. A function $f \in \mathcal{A}$ is called α-spirallike $(-\frac{\pi}{2} < \alpha < \frac{\pi}{2})$ if for each w in $f(\mathcal{E})$ and $t \geq 0$, the logarithmic spiral $(we^{-i\alpha}t)$ is contained in $f(\mathcal{E})$. In 1932, Spacek [26] has studied the class of α-spirallike starlike functions denoted by $SPST(\alpha)$. A function $f \in SPST(\alpha)$ if and only if

$$\Re e\Big[\Big(\frac{zf'(z)}{f(z)}\Big)e^{i\alpha}\Big] > 0, (\forall z \in \mathcal{E}) \tag{1.2}$$

If $f(z)$ has the usual normalization, then near $z = 0$ we have

$$e^{i\alpha}\Big(\frac{zf'(z)}{f(z)}\Big) = cos\alpha + isin\alpha + \sum_{n=1}^{\infty} c_n z^n \tag{1.3}$$

It follows from (1.2) that $cos\alpha \geq 0$ (put $z = 0$) and hence $\mid \alpha \mid \leq \frac{\pi}{2}$. If $\alpha = \pm\frac{\pi}{2}$ then $f(z) = z$, so henceforth we always assume that $\mid \alpha \mid < \frac{\pi}{2}$. If we normalize the expression on right hand side of (1.3) we arrive at

$$\frac{1}{cos\alpha}\Big[e^{i\alpha}\frac{zf'(z)}{f(z)} - isin\alpha\Big] = 1 + \sum_{n=1}^{\infty} p_n z^n = p(z) \tag{1.4}$$

where $p(z) \in \mathcal{P}$. Conversely, if $p(z) \in \mathcal{P}$ then the inequality (1.2) is satisfied in $\mathcal{E}$. When $\alpha = 0$, we get $SPST(0) = \mathcal{S}^*$ (Starlike functions). Falcon and Plaza [6] have studied some basic properties of k-Fibonacci sequence, denoted by $\{F_{k,n}\}_{n=0}^{\infty}$ (k be any non-zero positive real number). For $n \geq 1$, the recurrence relation is given by
$F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ with $F_{k,0} = 0$ & $F_{k,1} = 1$.
Let $F_{k,n}$ be the n^{th} element of k-Fibonacci sequence. Here

$$F_{k,n} = \frac{\lambda_k^n - \tau_k^n}{\sqrt{k^2+4}}, \text{ where } \lambda_k = \frac{k + \sqrt{k^2+4}}{2}, \ \tau_k = \frac{k - \sqrt{k^2+4}}{2}.$$

The class of starlike functions subordinate to k-Fibonacci sequence (SL^k) is the subclass of S^*, which was introduced by Sokol et.al. [8, 18]. A function $f \in SL^k$ if and only if

$$\Big(\frac{zf'(z)}{f(z)}\Big) \prec \tilde{p}_k(z), (\forall z \in \mathcal{E})$$

Here

$$\tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - \tau_k z - \tau_k^2 z^2} \tag{1.5}$$

They have also shown that for $f \in SL^k \Rightarrow | a_n | \leq | \tau_k |^{n-1} F_{k,n}$ equal sign occurs for the function

$$g_k(z) = \frac{z}{1 - k\tau_k z - \tau_k^2 z^2}.$$

Consider the curve $C = \left\{\tilde{p}_k(e^{i\phi});\ \ \phi \in [0, 2\pi)\backslash\{\pi\}\right\}$ is shell shaped and it has the symmetry along the real axis. For $k = \frac{1}{2}$ the curve is given in the following figure

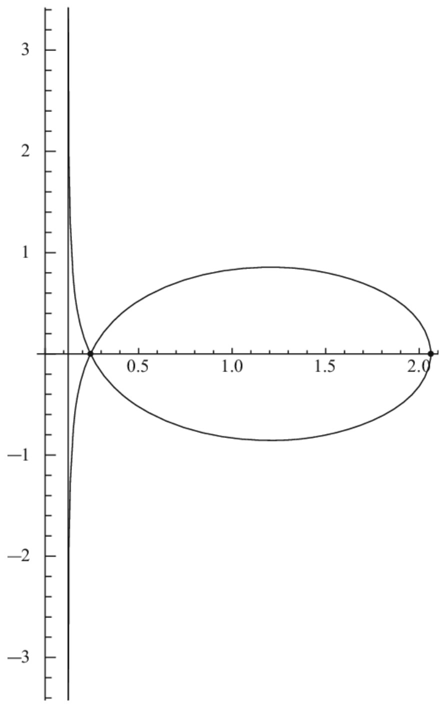

FIGURE 1.

The curve C has a loop when $k \leq 2$, hence it meets the real axis at two points such as $m = k(k^2 + 4)^{-\frac{1}{2}}$ and $n = \frac{1}{2k}(\sqrt{k^2 + 4})$. The curve is like a conchoid, hence it has no loops for $k > 2$. If

$$\tilde{p}_k(z) = \frac{1}{1 - k\tau_k z - \tau_k^2 z^2}\left(1 + \tau_k^2 z^2\right)$$

$$\tilde{p}_k(z) = 1 + \sum_{n=1}^{\infty} p_{k,n} z^n$$

then $p_{k,n} = (F_{k,n-1} + F_{k,n+1})\tau_k^n$. Later Sharma et.al. [24] have estimated the absolute values for the coefficient inequalities for bounded turning functions depicted by k-Fibonacci sequence $(R(\tilde{p}_k(z)))$.
A function $f \in R(\tilde{p}_k(z)) \Leftrightarrow f'(z) \prec \tilde{p}_k(z)$.
"For $n \geq 2$, if f is in S then $\mid a_n^2 - a_{2n-1} \mid \leq (n-1)^2$."
This is known as Zalcman conjecture, which was conjectured by Lawrence Zalcman in 1960. Equality holds good for Keobe function $(K(z)) = \frac{z}{(1-z)^2}$. Zalcman conjecture leads the Bieberbach conjecture. Fekete-Szegö inequality is the special case of Zalcman conjecture for $n = 2$ with $\mu = 1$. Zalcman conjecture is true for the class of typically real functions $(\mathcal{T})$ for $n \geq 3$ and starlike functions $(\forall n \geq 2)$ was proved by Brown and Tsao [3]. For $n \geq 4$, Ma [13] proved the Zalcman conjecture for $\mathcal{K}$ (close to convex functions). Later, in 1999, Ma [14] suggested a generalized Zalcman conjecture for $n, m \geq 2$

If f is in S then $\mid a_m a_n - a_{m+n-1} \mid \leq (m-1)(n-1)$.

The s^{th} Hankel determinant was studied by Pommerenke [19] is given by

$$H_s(t) = \begin{vmatrix} a_n & a_{n+1} & ------- & a_{n+q-1} \\ a_{n+1} & a_{n+2} & ------- & a_{n+q} \\ -- & -- & ------- & --- \\ -- & -- & ------- & --- \\ a_{n+q-1} & a_{n+q} & ------- & a_{n+2q-2} \end{vmatrix} \tag{1.6}$$

Here $s, t \in \mathbb{N}$. The s^{th} Hankel determinant was developed by Noonan and Thomas [16] in 1976. For distinct values of s, t in (1.6), we get distinct Hankel determinants for the function $f \in \mathcal{A}$. $H_s(t)$ has been studied by many authors to quote a few of them Babalola [1], Ehrenborg [5], Layman [11], Noor [17], etc.,
For $s = 2$ and $t = 1$, we obtain the Fekete-Szegö inequality $H_2(1) = (a_3 - a_2^2)$ $(\because a_1 = 1)$ with $\mu = 1$. Several authors [7, 10, 23, 25] have estimated the absolute value of $H_2(1)$ for certain subclasses of regular and simple functions. The Hankel determinant of order two was obtained by substituting $s = 2$ & $t = 2$ in (1.6), denoted by $H_2(2)$.

$$\mid H_2(2) \mid = \mid a_2 a_4 - a_3^2 \mid$$

Hankel determinant of order two was studied by several authors [2, 4, 9, 12].

Inspired by their work, in this article, we are introducing new subclasses of regular functions namely α- spirallike starlike functions and α-spirallike bounded turning functions subordinate to k-Fibonacci sequence, which are denoted by $SPSL^k(\alpha)$, $SPR(\tilde{p}_k(\alpha))$, also obtained the Zalcman conjecture, Fekete-Szegö inequality and $\mid H_2(2) \mid$ for the function f in these classes.
Definition 1: A function $f \in SPSL^k(\alpha)$ if and only if

$$e^{i\alpha}\Big[\frac{zf'(z)}{f(z)}\Big] \prec \tilde{p}_k(z), \ (\forall z \in \mathcal{E})$$

For $\alpha = 0$, we get the class SL^k and this class was studied by Guney et al., [8].
Definition 2: A function $f \in SPR(\tilde{p}_k(\alpha))$ if and only if

$$e^{i\alpha} f'(z) \prec \tilde{p}_k(z), \ (\forall z \in \mathcal{E})$$

For $\alpha = 0$, we get $R(\tilde{p}_k(z))$ and this class was studied by Sharma et. al., [24].

2. Preliminaries

We require the subsequent lemmas.

LEMMA 2.1. *[20] Let $p \in \mathcal{P}$ with the series expansion*

$$p(z) = 1 + c_1 z + c_2 z^2 + \ldots\ldots \ then \ \mid c_n \mid \leq 2, \ (for\ n \geq 1) \tag{2.1}$$

Let $x = \frac{c_1}{2}$, if $\mid c_1 \mid = 2$ then $p(z) \equiv p_1(z) \equiv \frac{1+xz}{1-xz}$.
Conversely, for some $\mid x \mid = 1$, if $p(z) \equiv p_1(z)$ then $c_1 = 2x$. Moreover, we get

$$\mid c_2 - \frac{c_1^2}{2} \mid \leq 2 - \frac{\mid c_1 \mid^2}{2} \tag{2.2}$$

$$if \ \ \mid c_1 \mid < 2 \ \& \ \mid c_2 - \frac{c_1^2}{2} \mid = 2 - \frac{\mid c_1 \mid^2}{2} \ then \ p(z) \equiv p_2(z)$$

here

$$p_2(z) = \frac{1 + w\bar{x}z + (wz + x)z}{1 + \bar{x}wz - z(wz + x)}, \ x = \frac{c_1}{2} \ and \ w = \frac{2c_2 - c_1^2}{4 - \mid c_1 \mid^2},$$

LEMMA 2.2. *[22] Let p be in $\mathcal{P}$ then*

$$\mid c_1 c_2 - c_3 \mid \leq 2 \tag{2.3}$$

LEMMA 2.3. *[25] If $p(z) \prec \tilde{p}_k(z)$ then we have*

$$\mid p_1 \mid \leq \frac{k}{2}(\sqrt{k^2 + 4} - k)$$

$$\mid p_2 \mid \leq \left\{1 + \frac{(k - \sqrt{k^2 + 4})k}{2}\right\}(k^2 + 2)$$

3. MAIN RESULTS

3.1. Second Hankel determinant for $f \in SPSL^k(\alpha)$

THEOREM 3.1. *If f is in $SPSL^k(\alpha)$ and $f(z) = z + \sum_{n \geq 2}^{\infty} a_n z^n$, $(\forall z \in \mathcal{E})$ then*

$$\mid a_2 a_4 - a_3^2 \mid \leq \frac{\tau_k^4}{12} cos^2\alpha \left[cos^2\alpha k^4 + (12 - k^4)\right] \tag{3.1}$$

Proof. : Let $f \in SPSL^k(\alpha)$, defined by

$$\Big(\frac{e^{i\alpha}z}{f(z)}f'(z)\Big) = p(z)$$

If $p \prec \tilde{p}_k$, then from subordination definition, we have $p(z) = \tilde{p}_k(w_1(z))$. Let $h_1 \in \mathcal{P}$ be given as

$$h_1(z) = \frac{1 + w_1(z)}{1 - w_1(z)} = 1 + c_1z + c_2z^2 + \cdots$$

where

$$w_1(z) = \frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \cdots \tag{3.2}$$

Using (3.2), we get

$$\tilde{p}_k(w_1(z)) = 1 + \tilde{p}_{k,1}\left(\frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + ...\right)$$
$$+ \left(\frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + ...\right)^2 \tilde{p}_{k,2} + \cdots$$

$$\tilde{p}_k(w_1(z)) = 1 + \frac{c_1z}{2}\tilde{p}_{k,1} + \frac{z^2}{2}\Big[\Big(c_2 - \frac{c_1^2}{2}\Big)\tilde{p}_{k,1} + \frac{c_1^2}{4}\tilde{p}_{k,2}\Big] + \cdots \tag{3.3}$$

By using the recurrence relation $\tilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\tau_k^n$ one can find the coefficients $\tilde{p}_{k,n}$ of the function $\tilde{p}_k$. Consider

$$\tilde{p}_k(z) = 1 + \sum_{n=1}^{\infty}\tilde{p}_{k,n}z^n = 1 + k\tau_kz + (k^2 + 2)\tau_k^2z^2 + (k^3 + 3k)\tau_k^3z^3 + ... \tag{3.4}$$

If $p(z) = 1 + p_1z + p_2z^2 + \cdots$ then by using the equations (3.3) and (3.4), we get

$$p_1 = \frac{c_1}{2}k\tau_k \tag{3.5}$$

$$p_2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)k\tau_k + \frac{\tau_k^2(k^2 + 2)}{4}c_1^2 \tag{3.6}$$

$$p_3 = \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)k\tau_k + \frac{(k^2 + 2)}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tau_k^2 + \frac{(k^3 + 3k)}{8}c_1^3\tau_k^3 \tag{3.7}$$

Further, from the equation (1.4) we have

$$\Rightarrow \Big[e^{i\alpha}zf'(z) - isin\alpha f(z)\Big] = cos\alpha\Big(p(z)f(z)\Big)$$

$$\Big(e^{i\alpha} - isin\alpha\Big) + \Big(2a_2e^{i\alpha} - ia_2sin\alpha\Big)z + \Big(3a_3e^{i\alpha} - ia_3sin\alpha\Big)z^2 + \Big(4a_4e^{i\alpha} - ia_4sin\alpha\Big)z^3 + \cdots$$
$$= cos\alpha\Big(1 + (a_2 + p_1)z + (p_2 + a_2p_1 + a_3)z^2 + (p_3 + a_2p_2 + a_3p_1 + a_4)z^3 + \cdots\Big)$$

Equating the corresponding coefficients of z in both sides, we get

$$a_2 = e^{-i\alpha} p_1 cos\alpha \tag{3.8}$$

$$a_3 = \frac{e^{-i\alpha}}{2}\Big(p_2 + p_1^2 cos\alpha e^{-i\alpha}\Big)cos\alpha \tag{3.9}$$

$$a_4 = \frac{e^{-i\alpha}cos\alpha}{6}\Big(2p_3 + 3cos\alpha e^{-i\alpha}p_1p_2 + p_1^3 cos^2\alpha e^{-2i\alpha}\Big)cos\alpha \tag{3.10}$$

Utilizing the equations (3.8) to (3.10) and taking modulus, we get

$$| a_2a_4 - a_3^2 |= \frac{cos^2\alpha}{12}\Big|cos^2\alpha p_1^4 - 4p_1p_3 - 3p_2^2\Big| \tag{3.11}$$

Utilizing (3.5) to (3.7) and after some primary calculations, we acquire that

$$| a_2a_4 - a_3^2 |= \frac{\tau_k^2}{12}cos^2\alpha\Big|\Big[\Big(\frac{cos^2\alpha k^4}{16} + \frac{12-k^4}{16}\Big)c_1^4 - \frac{(k^2+2)}{4}c_1^2(c_2 - \frac{c_1^2}{2})\Big]k\tau_k$$
$$+ k^2c_1(c_1c_2 - c_3) + \frac{3}{4}k^2(c_2 - \frac{c_1^2}{2})^2 + \Big(\frac{cos^2\alpha k^4}{16} + \frac{12-k^4-4k^2}{16}\Big)c_1^4\Big| \tag{3.12}$$

We know that

$$\forall n \in \mathbb{N}, \tau_k = \frac{\tau_k^n}{F_{k,n}} - x_{k,n},\ \ x_{k,n} = \frac{F_{k,n-1}}{F_{k,n}},\ \lim_{n\longrightarrow\infty}\Big(\frac{F_{k,n-1}}{F_{k,n}}\Big) =| \tau_k | \tag{3.13}$$

Utilizing (3.13) in (3.12), we get

$$| a_2a_4 - a_3^2 |= \frac{\tau_k^2}{12}cos^2\alpha\Big|\Big[\Big(\frac{cos^2\alpha k^4}{16} + \frac{12-k^4}{16}\Big)c_1^4 - \frac{(k^2+2)}{4}c_1^2(c_2 - \frac{c_1^2}{2})\Big]k\frac{\tau_k^n}{F_{k,n}}$$
$$+ k^2c_1(c_1c_2 - c_3) + \frac{3}{4}k^2c_2(c_2 - \frac{c_1^2}{2}) + \frac{1}{16}\Big[\Big(cos^2\alpha k^4 + (12 - k^4 - 4k^2)\Big)$$
$$- \Big(cos^2\alpha k^4 + (12-k^4)\Big)kx_{k,n}\Big]c_1^4 + \frac{1}{8}\Big((k^2+2)kx_{k,n} - 3k^2\Big)c_1^2(c_2 - \frac{c_1^2}{2})\Big| \tag{3.14}$$

Utilizing triangle inequality, after that using lemmas (2.1) & (2.2) in (3.14), we get

$$| a_2a_4 - a_3^2 |\leq \frac{\tau_k^2}{12}cos^2\alpha\Big|\Big[\Big(\frac{cos^2\alpha k^4}{16} + \frac{12-k^4}{16}\Big)c_1^4 - \frac{(k^2+2)}{4}c_1^2(c_2 - \frac{c_1^2}{2})\Big]\Big|k\frac{| \tau_k |^n}{F_{k,n}}$$
$$+ \frac{\tau_k^2}{12}cos^2\alpha\Big[2k^2 | c_1 | + \frac{3}{2}k^2(2 - \frac{| c_1 |^2}{2}) + \frac{1}{16}\Big|\Big(cos^2\alpha k^4 + (12 - k^4 - 4k^2)\Big)$$
$$- \Big(cos^2\alpha k^4 + (12-k^4)\Big)kx_{k,n}\Big| | c_1 |^4 + \frac{1}{8}\Big|(k^2+2)kx_{k,n} - 3k^2\Big| | c_1 |^2 \Big(2 - \frac{| c_1 |^2}{2}\Big)\Big] \tag{3.15}$$

Let

$$A = \Big|\Big(\frac{cos^2\alpha k^4}{16} + \frac{12-k^4}{16}\Big)c_1^4 - \frac{(k^2+2)}{4}c_1^2(c_2 - \frac{c_1^2}{2})\Big|k\frac{| \tau_k |^n}{F_{k,n}}$$

and

$$B = 2k^2 \mid c_1 \mid + \frac{3}{2}k^2(2 - \frac{\mid c_1 \mid^2}{2}) + \frac{1}{16}\Big|\Big(cos^2\alpha k^4 + (12 - k^4 - 4k^2)\Big) - \Big(cos^2\alpha k^4 + (12 - k^4)\Big)kx_{k,n}\Big| \mid c_1 \mid^4 + \frac{1}{8}\Big|(k^2+2)kx_{k,n} - 3k^2\Big| \mid c_1 \mid^2 \Big(2 - \frac{\mid c_1 \mid^2}{2}\Big)$$

For larger values of n, choose $\mid c_1 \mid = y$ belongs to $[0, 2]$, we get

$$B = \max_{y\in[0,2]} \Big\{2k^2 y + \frac{3}{2}k^2(2 - \frac{y^2}{2}) + \frac{1}{16}\Big[\Big(cos^2\alpha k^4 + (12 - k^4 - 4k^2)\Big) - \Big(cos^2\alpha k^4 + (12 - k^4)\Big)kx_{k,n}\Big]y^4 + \frac{1}{8}\Big((k^2+2)kx_{k,n} - 3k^2\Big)y^2(2 - \frac{y^2}{2})\Big\}$$

Applying (3.13) to A and B, we get

$$B = \lim_{n\longrightarrow\infty} \Big[\max_{y\in[0,2]} \Big\{2k^2 y + \frac{3}{2}k^2(2 - \frac{y^2}{2}) + \frac{1}{16}\Big[\Big(cos^2\alpha k^4 + (12 - k^4 - 4k^2)\Big) - \Big(cos^2\alpha k^4 + (12 - k^4)\Big)kx_{k,n}\Big]y^4 + \frac{1}{8}\Big((k^2+2)kx_{k,n} - 3k^2\Big)y^2(2 - \frac{y^2}{2})\Big\}\Big]$$

At $y = 2$, we acquire $B = \Big(cos^2\alpha k^4 + (12 - k^4)\Big)\tau_k^2$ and the value of A is zero. i.e.,

$$\lim_{n\longrightarrow\infty} \Big|\Big(\frac{cos^2\alpha k^4}{16} + \frac{12 - k^4}{16}\Big)c_1^4 - \frac{(k^2+2)}{4}c_1^2(c_2 - \frac{c_1^2}{2})\Big| k\frac{\mid \tau_k \mid^n}{F_{k,n}} = 0$$

Substituting the values of A and B in (3.15), we get the result given by the equation (3.1). This concludes the proof. □

Corollary 3.1.1: For $\alpha = 0$ in the equation (3.1) reduces to the result of Guney et. al., [8]. That is

$$\mid a_2 a_4 - a_3^2 \mid \leq \tau_k^4.$$

3.2. Fekete-Szegö inequality for $f \in SPSL^k(\alpha)$

Theorem 3.2. *If $f \in SPSL^k(\alpha)$ and $f(z) = z + \sum_{n\geq 2}^{\infty} a_n z^n$, $(\forall z \in \mathcal{E})$ then for any real number μ,*

$$\mid a_3 - \mu a_2^2 \mid \leq \frac{cos\alpha \tau_k^2}{2}\Big[(k^2+2) + cos\alpha(1 + 2 \mid \mu \mid)k^2\Big] \tag{3.16}$$

Proof. : If $f \in SPSL^k(\alpha)$ then for any real number μ, consider $(a_3 - \mu a_2^2)$
Using the equations (3.8), (3.9) and taking modulus, we get

$$\mid a_3 - \mu a_2^2 \mid = \Big|\frac{e^{-i\alpha}}{2}\Big(p_2 + p_1^2 cos\alpha e^{-i\alpha}\Big)cos\alpha - \mu e^{-2i\alpha} p_1^2 cos^2\alpha\Big|$$

Using lemma 2.3 and after some primary calculations, we obtain the required result. This concludes the proof. □

Corollary 3.2.1: For $\mu = 1$ and $\alpha = 0$ in (3.16) gives that $| a_3 - a_2^2 |\leq (2k^2 + 1)\tau_k^2$.

3.2.2 Zalcman conjecture for $SPSL^k(\alpha)$:

Theorem 3.2.2: If $f \in SPSL^k(\alpha)$ and $f(z) = z + \sum_{n\geq 2}^{\infty} a_n z^n$, $(\forall z \in \mathcal{E})$ then

$$| a_2^2 - a_3 |\leq \frac{cos\alpha}{2}\tau_k^2\Big((k^2 + 2) + k^2 cos\alpha\Big)$$

Proof. : If $f \in SPSL^k(\alpha)$ then consider $(a_2^2 - a_3)$.
From the equations (3.8), (3.9) and after that taking modulus, we get

$$| a_2^2 - a_3 |= \frac{e^{-i\alpha}cos\alpha}{2}\Big(| p_2 | + | p_1^2 | e^{-i\alpha} cos\alpha\Big)$$

Using lemma 2.3 and after a simple computation, we get

$$| a_2^2 - a_3 |\leq \frac{cos\alpha}{2}\tau_k^2\Big((k^2 + 2) + k^2 cos\alpha\Big)$$

∗ For $\alpha = 0$, we get $| a_2^2 - a_3 |\leq (k^2 + 1)\tau_k^2$. □

3.3. Hankel determinant of order two for $SPR(\tilde{P}_k(\alpha))$

Theorem 3.3. *If $f \in SPR(\tilde{p}_k(\alpha))$ and $f(z) = z + \sum_{n\geq 2}^{\infty} a_n z^n$, $(\forall z \in \mathcal{E})$ then*

$$| a_2 a_4 - a_3^2 |\leq \frac{\tau_k^4}{72}(5k^2 - k^4 + 32) \tag{3.17}$$

Proof. Let $f \in SPR(\tilde{p}_k(\alpha))$. Define a function p belongs to $\mathcal{P}$ so that

$$f'(z)e^{i\alpha} \prec \tilde{p}_k(z) \Rightarrow f'(z)e^{i\alpha} = p(z)$$
$$e^{i\alpha}\Big(1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots\Big) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

Equating the corresponding coefficients of z in both sides, we get

$$a_2 = \frac{p_1 e^{-i\alpha}}{2} \tag{3.18}$$
$$a_3 = \frac{p_2 e^{-i\alpha}}{3} \tag{3.19}$$
$$a_4 = \frac{p_3 e^{-i\alpha}}{4} \tag{3.20}$$

From the equations (3.18)-(3.20), we get

$$a_2 a_4 - a_3^2 = \frac{e^{-2i\alpha}}{72}\Big(8p_2^2 - 9p_1 p_3\Big)$$

Taking modulus on both sides, after that substituting the values of p_1, p_2, p_3 are given by the equations (3.5) to (3.7) and after a simple computation we get

$$\begin{aligned} | a_2a_4 - a_3^2 | = \frac{\tau_k^2}{72}\Big|\Big[\frac{(5k^2-k^4+32)}{16}c_1^4 - \frac{(k^2+2)}{4}c_1^2(c_2-\frac{c_1^2}{2})\Big]k\tau_k \\ - \frac{9}{4}k^2c_1(c_3-c_1c_2) + 2k^2\Big(c_2-\frac{c_1^2}{2}\Big)^2 + \frac{(32-k^4-4k^2)}{16}c_1^4\Big| \end{aligned} \tag{3.21}$$

Utilizing (3.13) in (3.21), we get

$$\begin{aligned} | a_2a_4 - a_3^2 | = \frac{\tau_k^2}{72}\Big|\Big[\frac{(5k^2-k^4+32)}{16}c_1^4 - \frac{(k^2+2)}{4}c_1^2(c_2-\frac{c_1^2}{2})\Big]k\frac{\tau_k^n}{F_{k,n}} + \frac{9}{4}k^2c_1(c_1c_2-c_3) \\ + 2k^2c_2\Big(c_2-\frac{c_1^2}{2}\Big) + \Big(\frac{(32-k^4-4k^2)}{16} - \frac{(5k^2-k^4+32)kx_{k.n}}{16}\Big)c_1^4 \\ + \Big(\frac{(k^2+2)kx_{k,n}}{4} - k^2\Big)c_1^2(c_2-\frac{c_1^2}{2})\Big| \end{aligned}$$

Using triangle inequality, after that using lemmas (2.1) and (2.2) in the above equation, we have

$$\begin{aligned} | a_2a_4 - a_3^2 | \le \frac{\tau_k^2}{72}\Big|\frac{(5k^2-k^4+32)}{16}c_1^4 - \frac{(k^2+2)}{4}c_1^2(c_2-\frac{c_1^2}{2})\Big|k\frac{|\tau_k|^n}{F_{k,n}} \\ + \frac{\tau_k^2}{72}\Big[\frac{9}{2}k^2|c_1| + 4k^2\Big(2-\frac{|c_1|^2}{2}\Big) + \Big|\frac{(k^2+2)kx_{k,n}}{4} - k^2\Big||c_1|^2\,(2-\frac{|c_1|^2}{2}) \\ + \Big|\frac{(32-k^4-4k^2)}{16} - \frac{(5k^2-k^4+32)kx_{k.n}}{16}\Big||c_1|^4\Big] \end{aligned} \tag{3.22}$$

Let $C = \Big|\frac{(5k^2-k^4+32)}{16}c_1^4 - \frac{(k^2+2)}{4}c_1^2(c_2-\frac{c_1^2}{2})\Big|k\frac{|\tau_k|^n}{F_{k,n}}$ and

$$\begin{aligned} D = \frac{9}{2}k^2|c_1| + 4k^2\Big(2-\frac{|c_1|^2}{2}\Big) + \Big|\frac{(k^2+2)kx_{k,n}}{4} - k^2\Big||c_1|^2\,(2-\frac{|c_1|^2}{2}) \\ + \Big|\frac{(32-k^4-4k^2)}{16} - \frac{(5k^2-k^4+32)kx_{k.n}}{16}\Big||c_1|^4 \end{aligned}$$

For larger values of n, choose $|c_1| = y$ belongs to $[0,2]$, we get

$$\begin{aligned} D = \max_{y\in[0,2]}\Big\{\frac{9}{2}k^2y + 4k^2\Big(2-\frac{y^2}{2}\Big) + \Big[\frac{(k^2+2)kx_{k,n}}{4} - k^2\Big]y^2(2-\frac{y^2}{2}) \\ + \frac{1}{16}\Big((32-k^4-4k^2) - (5k^2-k^4+32)kx_{k.n}\Big)y^4\Big\} \end{aligned}$$

Applying (3.13) to C and D, one can obtain that the value of C is zero. i.e.,

$$\lim_{n\longrightarrow\infty}\Big|\frac{(5k^2-k^4+32)}{16}c_1^4 - \frac{(k^2+2)}{4}c_1^2(c_2-\frac{c_1^2}{2})\Big|k\frac{|\tau_k|^n}{F_{k,n}} = 0$$

and

$$D=\lim_{n\longrightarrow\infty}\Big[\max_{y\in[0,2]}\Big\{\frac{9}{2}k^2y+4k^2\Big(2-\frac{y^2}{2}\Big)+\Big(\frac{(k^2+2)kx_{k,n}}{4}-k^2\Big)y^2(2-\frac{y^2}{2})$$
$$+\frac{1}{16}\Big((32-k^4-4k^2)-(5k^2-k^4+32)kx_{k.n}\Big)y^4\Big\}\Big]$$

At $y=2$, we get $D=(5k^2-k^4+32)\tau_k^2$
Substituting the values of C and D in (3.22), we attain the required result. This concludes the proof. □

3.4. Fekete-Szegö inequality for $f\in SPR(\tilde{p}_k(\alpha))$

THEOREM 3.4. *If $f\in SPR(\tilde{p}_k(\alpha))$ and $f(z)=z+\sum_{n\geq 2}^{\infty}a_nz^n$, $(\forall z\in\mathcal{E})$ then*

$$|\,a_3-\mu a_2^2\,|\leq\frac{\tau_k^2}{12}\Big(8+(4+3\,|\,\mu\,|)k^2\Big)$$

PROOF. Let $f\in SPR(\tilde{p}_k(\alpha))$, consider $a_3-\mu a_2^2$.
Using the equations (3.18), (3.19) and after that taking modulus, we have

$$|\,a_3-\mu a_2^2\,|=\frac{|\,e^{-i\alpha}\,|}{12}\Big|4p_2-3\mu p_1^2e^{-i\alpha}\Big|$$
$$\leq\frac{1}{12}\Big(4\,|\,p_2\,|+3\,|\,\mu\,||\,p_1^2\,|\,\Big)$$

Using lemma 2.3, we get

$$|\,a_3-\mu a_2^2\,|\leq\frac{\tau_k^2}{12}\Big(8+(4+3\,|\,\mu\,|)k^2\Big)$$

Corollary 3.4.1: For $\mu=1$, gives the result of R.B. Sharma et. al. [24]. That is $|\,a_3-a_2^2\,|\leq\frac{(7k^2+8)}{12}\tau_k^2$ □

3.5 Zalcman conjecture for $f\in SPR(\tilde{p}_k(\alpha))$:

THEOREM 3.5. *If $f\in SPR(\tilde{p}_k(\alpha))$ and $f(z)=z+\sum_{n\geq 2}^{\infty}a_nz^n$, $(\forall z\in\mathcal{E})$ then*

$$|\,a_2^2-a_3\,|\leq\frac{\tau_k^2}{12}(k^2+8)\tag{3.23}$$

PROOF. If $f\in SPR(\tilde{p}_k(\alpha))$ then by using the equations (3.18), (3.19) after that from lemma 2.3, we get the result given by the equation (3.23).

□

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