GANITA, Vol.74(1), 2024, 15-22

ON QUASI HYPERPERFECT AND ALMOST HYPERPERFECT NUMBERS

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Abstract

It is well-known that a natural number n is called $k-$ hyperperfect if $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}$,where $\sigma(n)$ denotes the sum of all positive divisors of n and k is a positive integer. For quasi $k-$hyperperfect number n, $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k} + 1$. We call a number n is almost $k-$ hyperperfect number, if $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k} - 1$. In this paper, we have introduced the notion of almost $k-$ hyperperfect number and also presented some certain form of quasi hyperperfect number with numerical examples upto two distinct prime factors. We also discussed about the abundancy index $I(n) = \frac{\sigma(n)}{n}$ for these two numbers along with generalized hyperperfect numbers like near hyperperfect number and deficient hyperperfect number.

2010 *Mathematics subject classification:* primary 11A25; secondary 11N25.

Keywords and phrases: Perfect number, Hyperperfect number, Abundancy index .

1. Introduction

For any positive integer n, the divisor function $\sigma(n)$ denotes the sum of all positive divisors of n. This function is multiplicative i.e., if the integers m and n are relatively prime, then $\sigma(mn) = \sigma(m)\sigma(n)$. It is well-known that a natural number n is called $k-$hyperperfect [6], if it can be expressed as $n = 1+k[\sigma(n)-n-1]$, also this expression can be written as $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}$, where k is a positive integer. Corresponding to $k = 1$, the classical perfect numbers are $1-$ hyperperfect number, i.e., if n is a perfect number [1], then $\sigma(n) = 2n$. There are only a few examples of even perfect numbers. In recent years, the idea of classical perfect numbers has been generalized in different ways. Hyperperfect numbers are one of the generalized notions of classical perfect number. The notion of hyperperfect number was first introduced by D. Minoli and R. Bear [9] and they conjectured [8, 9] that there are $k-$hyperperfect number for every positive integer k .To construct the hyperperfect number, H. J. J. te Riele presented three rules that appear in [10] and using these rules several examples of hyperperfect numbers have been discovered with upto five different prime factors. Using computer programming, J. S. McCranie [7] computed all hyperperfect numbers less than 10^{11} and found that there are only 2190 hyperperfect numbers in this range, for 1932 different values of k . In [2] A. Bege and K. Fogarasi have given list of $k-$hyperperfect numbers for some positive integer k and they also conjectured that all $2-$hyperperfect numbers are of the form $3^{m-1}(3^m-2)$, where 3^m-2 is a prime. Near hyperperfect and deficient hyperperfect numbers are the generalized notion of hyperperfect number. In 2017, B. Das and H. K. Saikia [6] introduced the notion of near hyperperfect and deficient hyperperfect number. If n is a near $k-$hyperperfect number, then $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k} + d$ and for deficient $k-$ hyperperfect number, $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k} - d$, where $d(> 1)$ is a proper divisor of n and is termed as the redundant divisor for n. Well known near perfect numbers are near $1-$hyperperfect number, i.e. if n is a near perfect number [11], then n is a solution of the equation $\sigma(n) = 2n + d$ and deficient perfect numbers are deficient $1-$ hyperperfect number i.e., i.e. if n is a deficient perfect number [11], then n is a solution of the equation $\sigma(n) = 2n - d$. Following are the types of near hyperperfect numbers [6] with two different prime factors for different values of prime q.

(i) If $p = q^l - (q-1)$ is an odd prime, then $n = q^l p$ is a near $(q-1)-$ hyperperfect number with redundant divisor q^l, where $l > 1$.

(ii) If $p = q^l - (q-1)$ is an odd prime, then $n = q^{l-1}p^2$ is a near $(q-1)-$ hyperperfect number with redundant divisor p, where $l > 1$.

Following is a type of deficient hyperperfect number [6] with two different prime factors for different values of prime q

(i) $n = q^l p$ is a deficient $(q-1)-$ hyperperfect number with redundant divisor q^l, where $p = q^{l+1} + (q-1)q^t - (q-1)$ and q are distinct primes and $l \geq t$.

For more information and numerical examples of near hyperperfect number and deficient hyperperfect number see [6]. Recall that if $\sigma(n) = 2n + 1$, then n is said to be quasi perfect[1]; if $\sigma(n) = 2n - 1$, then n is said to be almost perfect [1]. No quasi perfect numbers have been found so far [4]. G. L. Cohen showed that if quasi perfect number exists [4], then it is not divisible by $3.5.17.p$ for $p = 19....., 101$. Almost perfect numbers are of the form 2^m, where m is a positive integer. Near perfect, deficient perfect, almost perfect and quasi perfect numbers are the generalized notion of classical perfect number. In this paper, we include some results for quasi hyperperfect number with numerical examples, also define almost hyperperfect number with certain forms. We have also determine lower and upper bounds of abundancy index $I(n)$ for near hyperperfect, deficient hyperperfect, quasi hyperperfect and almost hyperperfect numbers.

2. Quasi hyperperfect number

The notion of quasi hyperperfect numbers were first introduced by A. Bege and Z. Bartha [3]. In this section the small numerical results are presented for quasi hyperperfect numbers upto two distinct prime factors of certain form numbers. A natural number n is said to be quasi $k-$hyperperfect number [3] if $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k} + 1$. In particular for $k = 1$, $\sigma(n) = 2n + 1$ which is the definition of quasi perfect number i.e., quasi $1-$ hyperperfect numbers are quasi perfect. For $k = 2$, i.e., if n is quasi $2-$ hyperperfect number, then $\sigma(n) = \frac{3}{2}n + \frac{3}{2}$. For $k = 3$, i.e., if n is quasi $3-$ hyperperfect number, then $\sigma(n) = \frac{4}{3}n + \frac{5}{3}$. In general for $k = q-1$, where q is a prime number i.e., if

n is quasi $(q-1)-$ hyperperfect number, then $\sigma(n) = \frac{q}{q-1}n + \frac{q-2}{q-1} + 1$. We have obtained the following results for quasi $k-$hyperperfect number for different values of k.

Proposition 2.1. *For any positive integer $l > 1$, if $n = p^{l-1}$, where p is a prime, then n is not a quasi $2-$hyperperfect number.*

Proof. If $n = p^{l-1}$ is a quasi $2-$hyperperfect number, then by definition $\sigma(n) = \frac{3}{2}n + \frac{3}{2}$, so we can write $2(p^l - 1) = (3p^{l-1} + 3)(p - 1)$. This equation strictly implies that $(p-3)(p^{l-1} + 3) = -8$. But the last equation has no solution for any prime p and $l > 1$. Therefore $l > 1$, $n = p^{l-1}$ is not a quasi $2-$ hyperperfect number. □

Remark 2.2. *From the proposition 2.1., it follows that for quasi $2-$hyperperfect number $\omega(n) > 1$ i.e., number of prime factors for quasi $2-$hyperperfect number is at least 2. From the following proposition one can obtain the form of quasi $2-$hyperperfect number with two distinct prime factors.*

Proposition 2.3. *For any positive integer $l \geq 1$ and $k > 1$, if $n = 3^{k-1}p^l$ is a quasi $2-$hyperperfect number,then $p = 3^k - 4$ is an odd prime greater than 3 and $l = 1$.*

Proof. Suppose $n = 3^{k-1}p^l$ is a quasi $2-$hyperperfect number, then $\sigma(n) = \frac{3}{2}n + \frac{3}{2}$, so we can write $\sigma(3^{k-1})\sigma(p^l) = \frac{3^k p^l + 3}{2}$ i.e., $(3^k - 1)(p^l + p^{l-1} + ... + p + 1) = (3^k p^l + 3)$. This equation implies that $(3^k - p)(p^{l-1} + p^{l-2} + ... + p + 1) = 4$. But the last equation has only solution when the prime p is the form $3^k - 4$ and $l = 1$. □

Proposition 2.4. *If $n = pq$ is a quasi $2-$hyperperfect number, where p and q are distinct primes, then $p = 3$ and $q = 5$.*

Proof. Suppose $n = pq$ is a quasi $2-$hyperperfect number, then $2(p + 1)(q + 1) = 3pq + 3$ and therefore $pq - 2p - 2q + 1 = 0$. This equation strictly implies that $(p - 2)(q - 2) = 3$. But the last equation has only solution when $p = 3$ and $q = 5$. □

From this proposition, it is evident that 15 is the only quasi $2-$ hyperperfect number of the form $n = pq$.

Proposition 2.5. *For any positive integer $l \geq 1$ and $k > 1$, if $n = 5^{k-1}p^l$ is a quasi $4-$ hyperperfect number,then $p = 5^k - 8$ is an odd prime greater than 5 and $l = 1$.*

Proposition 2.6. *For any positive integer $l \geq 1$ and $k > 1$, if $n = 7^{k-1}p^l$ is a quasi $6-$ hyperperfect number,then $p = 7^k - 12$ is an odd prime greater than 7 and $l = 1$.*

In general from the following proposition one can obtain a general form of quasi $(q-1)-$hyperperfect number with two distinct prime factors, where q is an odd prime.

Proposition 2.7. *If $n = q^{k-1}p$, where $p = q^k - (2q - 2)$ is an odd prime number, then n is quasi $(q - 1)-$hyperperfect number.*

Proof. From the structure of odd prime p, it is clear that p and q are relatively prime i.e., g.c.d $(p, q) = 1$, therefore $\sigma(n) = \sigma(q^{k-1}p) = \frac{q^k-1}{q-1}(p+1) = \frac{q^kp+q^k-p-1}{q-1} = \frac{qn+2q-3}{q-1} = \frac{qn}{q-1} + \frac{q-2}{q-1} + 1$. □

Remark 2.8. *From the proposition 2.7., one can determine one class of quasi* $(q-1)-$*hyperperfect number for different prime q.*

Following are the some numerical examples of quasi $k-$ hyperperfect number with two distinct prime factors for different values of k.

quasi 2−hyperperfect number: $15 = 3(3^2-4), 207 = 3^2(3^3-4), 19359 = 3^4(3^5-4)$

quasi 4−hyperperfect number: $85 = 5(5^2-8), 77125 = 5^3(5^4-8)$

quasi 6−hyperperfect number: $259 = 7(7^2-12), 16219 = 7^2(7^3-12), 819427 = 7^3(7^4-12)$

quasi 10−hyperperfect number: $1111 = 11(11^2-20), 19460551 = 11^3(11^4-20)$

3. Almost hyperperfect number

In this section we have introduced the notion of almost $k-$hyperperfect number and studied certain form of such numbers upto two distinct prime factors for different values of k. We call a number n is almost $k-$hyperperfect number if $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k} - 1$. In particular for $k = 1$, $\sigma(n) = 2n-1$,which is the definition of almost perfect number i.e., almost 1−hyperperfect numbers are almost perfect. For $k = 2$, i.e., if n is almost 2− hyperperfect number, then $\sigma(n) = \frac{3}{2}n - \frac{1}{2}$. For $k = 3$, i.e., if n is almost 3− hyperperfect number, then $\sigma(n) = \frac{4}{3}n - \frac{1}{3}$. In general for $k = q-1$, where q is a prime number i.e., if n is almost $(q-1)-$ hyperperfect number, then $\sigma(n) = \frac{q}{q-1}n + \frac{q-2}{q-1} - 1$. We have obtained the following results related to almost $k-$ hyperperfect numbers for different values of k.

Proposition 3.1. *For any positive integer $l > 1$, if $n = p^{l-1}$ is an almost* 2− *hyper perfect number, then $p = 3$.*

Proof. For $n = p^{l-1}$, by definition of divisor function, we can write $\sigma(n) = \frac{p^l-1}{p-1}$. If $n = p^{l-1}$ is an almost 2− hyperperfect number, then $\sigma(n) = \frac{3}{2}n - \frac{1}{2}$ and which implies that $\frac{p^l-1}{p-1} = \frac{3p^{l-1}-1}{2}$. Then we can write $(p-3)(p^{l-1}-1) = 0$. But this equation only has solution when $p = 3$ and $l > 1$. Therefore $n = 3^{l-1}$ is an almost 2−hyperperfect number with one prime factor. □

Proposition 3.2. *For any positive integer $l > 1$, there is no almost* 3− *hyperperfect number of the form $n = p^{l-1}$.*

Proof. If $n = p^{l-1}$ is an almost 3− hyperperfect number,then $\sigma(n) = \frac{4}{3}n - \frac{1}{3}$ and then we can write $3(p^l-1) = (p-1)(4p^{l-1}-1)$. This implies that $(p-4)(p^{l-1}-1) = 0$. But this equation has no solution for any prime p and $l > 1$. □

PROPOSITION 3.3. *If $n = p^{l-1}$ is an almost $4-$hyperperfect number, then $p = 5$ and $l > 1$.*

PROOF. If $n = p^{l-1}$ is an almost $4-$hyperperfect number, then from the equation $\sigma(n) = \frac{5}{4}n - \frac{1}{4}$, we can obtain $(p-5)(p^{l-1}-1) = 0$. But this equation has only solution when $p = 5$ for any $l > 1$. □

In general from the following proposition one can obtain a general form of quasi $(p-1)-$hyperperfect number with one prime factor p .

PROPOSITION 3.4. *For $l > 1$, if $n = p^{l-1}$,where p is a prime number, then n is an almost $(p-1)-$ hyperperfect number.*

PROOF. If $n = p^{l-1}$, then $\sigma(n) = \frac{p^l-1}{p-1} = \frac{pn}{p-1} - \frac{1}{p-1} = \frac{pn}{p-1} + \frac{p-2}{p-1} - 1$. □

REMARK 3.5. *In particular, for $p = 2$ from the proposition 3.4., one can obtain the form of all almost perfect number i.e., $n = 2^{l-1}$ is almost $1-$ hyperperfect number i.e., almost perfect number.*

PROPOSITION 3.6. *There is no almost $2-$hyperperfect number of the form $n = 3^{l-1}p^k$, where $p > 3$ is a prime.*

PROOF. If $n = 3^{l-1}p^k$ is an almost $2-$hyper perfect number, then $\sigma(n) = \frac{3}{2}n - \frac{1}{2}$. This implies that $(3^l - 1)(p^k + p^{k-1} + p^{k-2} + ... + p + 1) = (p-1)(3^l p^k - 1)$. From this equation we can write $(3^l - p)(p^{k-1} + p^{k-2} + ... + p + 1) = 0$. But this equation has no solution for any prime p and $k > 1$. □

PROPOSITION 3.7. *There is no almost $4-$hyperperfect number of the form $n = 5^{l-1}p^k$.*

PROPOSITION 3.8. *Suppose that p and q are distinct primes. For any positive integer $l > 1$ and $k \geq 1$, there is no almost $(q-1)-$hyperperfect number of the form $n = q^{l-1}p^k$.*

PROOF. From the definition of divisor function $\sigma(n)$, we can write $\sigma(n) = \frac{q^l-1}{q-1}\frac{p^{k+1}-1}{p-1}$. If $n = q^{l-1}p^k$ is an almost $(q-1)-$hyperperfect number, then $\sigma(n) = \frac{q}{q-1}n + \frac{q-2}{q-1} - 1$.From this expression, we can write $\frac{q^l-1}{q-1}\frac{p^{k+1}-1}{p-1} = \frac{qn}{q-1} + \frac{q-2}{q-1} - 1$. After simple computation, we obtain $q^l p^k - p^{k+1} - q^l + p = 0$. This equation implies that $(p^k - 1)(q^l - p) = 0$. From the last equation, we have $p = q^l$. But this is impossible since p and q are distinct primes and $l > 1$. Thus the number of the form $n = q^{l-1}p^k$ is not almost $(q-1)-$hyperperfect number. □

REMARK 3.9. *From the proposition 3.8., it is evident that the number of the form $n = q^{l-1}p^k$ with two distinct prime factors is not an almost $(q-1)-$ hyperperfect number.*

4. Abundancy index

The abundancy index $I(n)$ for any given positive integer n is associated with the divisor function $\sigma(n)$ and is defined as $I(n) = \frac{\sigma(n)}{n}$. Note that the abundancy index is also a multiplicative function since $\sigma(n)$ is multiplicative function. For classical perfect number, abundancy index $I(n) = 2$. For deficient number, $I(n) < 2$ and abundant number $I(n) > 2$. For more results on abundancy index for other generalized perfect number see [5]. In this section, we have discussed about the lower and upper bounds of abundacy index $I(n)$ for near hyperperfect, deficient hyperperfect, quasi hyperperfect and almost hyperperfect numbers. The bounds of $I(n)$ may play vital role in searching these numbers. Following proposition gives the upper bounds and lower bounds of $I(n)$, when n is a near $k-$hyperperfect number.

Proposition 4.1. *If n is a near $k-$hyperperfect number with redundant divisor d , then*

$$\frac{n(k+1)+k(d+2)}{k(n+1)} < I(n) < \frac{n(k+1)+k(2d+3)-1}{k(n+1)}.$$

Proof. If n is a near $k-$hyperperfect number with redundant divisor d, then $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k} + d$ and in that case abundancy index $I(n) = \frac{\sigma(n)}{n} = \frac{k+1}{k} + \frac{k-1}{kn} + \frac{d}{n}$. Now $\frac{n(k+1)+k(d+2)}{k(n+1)} = \frac{(n+1)(k+1)+k(d+2)-k-1}{k(n+1)} = \frac{(n+1)(k+1)+(k-1)+kd}{k(n+1)} = \frac{k+1}{k} + \frac{k-1}{k}\frac{1}{n+1} + \frac{d}{n+1} < I(n)$. Again $\frac{n(k+1)+k(2d+3)-1}{k(n+1)} = \frac{(n+1)(k+1)+2kd+3k-1-k-1}{k(n+1)} = \frac{(n+1)(k+1)+2(k-1)+2kd}{k(n+1)} = \frac{k+1}{k} + \frac{k-1}{k}\frac{2}{n+1} + \frac{2d}{n+1} > I(n)$. □

Remark 4.2. *From the Proposition 4.1.,if $k = 2$, then for near $2-$hyper perfect number $\frac{3n+2(d+2)}{2(n+1)} < I(n) < \frac{3n+4d+5}{2(n+1)}$. For quasi $k-$hyperperfect number $d = 1$, then $\frac{n(k+1)+3k}{k(n+1)} < I(n) < \frac{n(k+1)+5k-1}{k(n+1)}$. In fact, for quasi hyperperfect number n, we have the following result for abundancy index $I(n)$.*

Proposition 4.3. *If n is a quasi $k-$hyperperfect number, then*

$$\frac{n(k+1)+3k}{k(n+1)} < I(n) < \frac{n(k+1)+5k-1}{k(n+1)}.$$

Proposition 4.4. *If n is a deficient $k-$hyperperfect number with redundant divisor d, then*

$$\frac{n(k+1)-k(d-1)}{k(n+1)} < I(n) < \frac{n(k+1)-k(d-3)-1}{k(n+1)}.$$

Proof. If n is a deficient $k-$hyperperfect number with redundant divisor d, then $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k} - d$ and in that case abundancy index $I(n) = \frac{\sigma(n)}{n} = \frac{k+1}{k} + \frac{k-1}{kn} - \frac{d}{n}$. Now $\frac{n(k+1)-k(d-1)}{k(n+1)} = \frac{(n+1)(k+1)-k(d-1)-k-1}{k(n+1)} = \frac{(n+1)(k+1)+(k-1)-k(d+1)}{k(n+1)} = \frac{k+1}{k} + \frac{k-1}{k}\frac{1}{n+1} - \frac{d+1}{n+1} < I(n)$. Again $\frac{n(k+1)-k(d-3)-1}{k(n+1)} = \frac{(n+1)(k+1)-k(d-3)-1-(k+1)}{k(n+1)} = \frac{(n+1)(k+1)+2(k-1)-kd}{k(n+1)} = \frac{k+1}{k} + \frac{k-1}{k}\frac{2}{n+1} - \frac{d}{n+1} > I(n)$. □

Remark 4.5. *From the proposition 4.4.,if* $k = 2$*, then for deficient* $2-$*hyper perfect number* $\frac{3n-2(d-1)}{2(n+1)} < I(n) < \frac{3n-2d+5}{2(n+1)}$*. For almost* $k-$*hyperperfect number* $d = -1$*, then* $\frac{n(k+1)+2k}{k(n+1)} < I(n) < \frac{n(k+1)+4k-1}{k(n+1)}$*. In fact, for almost hyperperfect numbers, we have the following result for abundancy index* $I(n)$*.*

Proposition 4.6. *If* n *is an almost* $k-$*hyperperfect number, then*

$$\frac{n(k+1)+2k}{k(n+1)} < I(n) < \frac{n(k+1)+4k-1}{k(n+1)}.$$

The study of the abundancy index may lead to the discovery of different generalized hyperperfect numbers. It can be consider as useful tool in gaining a better understanding of these numbers.

5. Conclusion

We have presented a class of number of the form with two distinct prime factors which are quasi $(q-1)-$hyperperfect for some different values of prime q, but there is also future scope to search other quasi $k-$hyperperfect number. Similarly a class of number of almost $k-$hyperperfect with only one prime factor presented here for some different values of k and also presented a certain form number with two distinct prime factors which is not almost $k-$hyperperfect number for some different values of k. There is a scope of further investigation of almost $k-$hyperperfect for different values of k . The definition of bounds of abundancy index $I(n)$ may be used to determine the numbers like near hyperperfect, deficient hyperperfect, quasi hyperperfect and almost hyperperfect number in near future.

6. Acknowlgement

The authors are very much thankful to the anonymous referees for their valuable suggestions. The first author would like to thank Research Cell, Arya Vidyapeeth College(Autonomous) for financial support to carried out the research (Grant Ref. No.AVC/R.C/2021/2106).

Author contributions:
Conceptualisation: Bhabesh Das,Helen K. Saikia; *Software:* Bhabesh Das; *Writing-Original Draft:* Bhabesh Das,Helen K.saikia
Conflicts of Interest: The authors declare no conflict of interest.

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