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# SOME REMARKS ON THE CENTRAL INDEX BASED VARIOUS GENERALIZED GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS

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**Abstract**

The central index denoted by $\nu_f(r)$, of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$, is defined as $\max\left\{m : \mu_f(r) = |a_m| r^m\right\}$. In this paper, we find out some limiting values of the growth ratios forming from entire function and composite entire function on the basis of their central index using the concepts of $(p,q,t)L$-th order and $(p,q,t)L$-th type.

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## 1. Introduction, Definitions and Notations

Let $f(z)$ be an entire function defined in the open complex plane $\mathbb{C}$. For entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$, the maximum modulus symbolized as $M_f(r)$, the maximum term denoted as $\mu_f(r)$ and the central index indicated as $\nu_f(r)$ are respectively defined as $\max_{|z|=r} |f(z)|$, $\max_{n\geq 0}(|a_n| r^n)$ and $\max\left\{m : \mu_f(r) = |a_m| r^m\right\}$. Clearly, central index $\nu_f(r)$ of an entire function $f(z)$ is the greatest exponent $m$ such that $|a_m| r^m = \mu_f(r)$. All the functions $M_f(r)$, $\mu_f(r)$ and $\nu_f(r)$ are real and increasing. Taking another entire function $g(z)$, the limiting value of the ratio $\frac{\nu_f(r)}{\nu_g(r)}$ as $r \to +\infty$, is also called the growth of $f(z)$ with respect to $g(z)$ in terms of the central index.

We know that $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $x \in (0, \infty)$ and $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. Also, $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Throughout this paper, we take $a$, $l$, $p$, $q$, $m$ and $n$ as positive integers and $t \in \mathbb{N} \cup \{-1,0\}$. Now considering this, we just recall that Shen et al. [10] defined the $(m,n)$-$\varphi$ order and $(m,n)$-$\varphi$ lower order of entire function $f(z)$ which are as follows:

DEFINITION 1.1. [10] Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function and $m \geq n$. The $(m,n)$-$\varphi$ order $\rho^{(m,n)}(f,\varphi)$ and $(m,n)$-$\varphi$ lower order $\lambda^{(m,n)}(f,\varphi)$ of entire function $f(z)$ are defined as:

$$\rho^{(m,n)}(f,\varphi) = \limsup_{r\to+\infty} \frac{\log^{[m+1]} M_f(r)}{\log^{[n]} \varphi(r)}$$
$$\text{and } \lambda^{(m,n)}(f,\varphi) = \liminf_{r\to+\infty} \frac{\log^{[m+1]} M_f(r)}{\log^{[n]} \varphi(r)}.$$

If we take $m = 1$, $n = 1$ and $\varphi(r) = r$, then we respectively denote $\rho^{(2,1)}(f,r)$ and $\lambda^{(2,1)}(f,r)$ by $\rho(f)$ and $\lambda(f)$, which are classical growth indicators such as order and lower order of entire function $f(z)$. Now, He and Xiao [6] introduced an alternative definition of order and lower order of an entire function $f(z)$ in terms of its central index in the following way:

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log \nu_f(r)}{\log r} \text{ and } \lambda(f) = \liminf_{r\to+\infty} \frac{\log \nu_f(r)}{\log r}.$$

Let $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant '$a$' i.e., $\lim_{r\to+\infty} \frac{L(ar)}{L(r)} = 1$ where $L \equiv L(r)$ is a positive continuous function increasing slowly. If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r \cdot \exp^{[t+1]} L(r)$, then Definition 1.1 turns into the definitions of $(p,q,t)L$-th order and $(p,q,t)L$-th lower order of an entire function $f(z)$ (for details, see [3]) which are as follows:

$$\rho^{(p,q,t)L}(f) = \limsup_{r\to+\infty} \frac{\log^{[p+1]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}$$
$$\text{and } \lambda^{(p,q,t)L}(f) = \liminf_{r\to+\infty} \frac{\log^{[p+1]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

Shen et al. [10] also established the equivalence of the definition of $(m,n)$-$\varphi$ order of entire function in terms of maximum modulus and central index under some conditions. For details about it, one may see [10]. For particular if we consider $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r \cdot \exp^{[t+1]} L(r)$, then in view of Lemma 3.4 of [10] and Definition 1.1, one can write the following Definition.

DEFINITION 1.2. Let $f(z)$ be an entire function and $\nu_f(r)$ be the central index of $f(z)$, then

$$\rho^{(p,q,t)L}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} \nu_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}$$

and

$$\lambda^{(p,q,t)L}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} \nu_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

In order to compare the relative growth of two entire functions having same non zero finite $(p,q,t)L$-th order, one may introduce the definitions of $(p,q,t)L$-th type (respectively $(p,q,t)L$-th lower type) of entire functions having finite positive $(p,q,t)L$-th order in the following manner:

Definition 1.3. [3] Let $f(z)$ be an entire function with non-zero finite $(p,q,t)L$-th order $\rho^{(p,q,t)L}(f)$. The $(p,q,t)L$-th type denoted by $\sigma^{(p,q,t)L}(f)$ and $(p,q,t)L$-th lower type denoted by $\overline{\sigma}^{(p,q,t)L}(f)$ are respectively defined as follows:

$$\sigma^{(p,q,t)L}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_f^L(p,q,t)}}$$

and

$$\overline{\sigma}^{(p,q,t)L}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_f^L(p,q,t)}}.$$

Considering the above, here in this present paper we compare the central index of composition of two entire functions with their corresponding left and right factors under the treatment of the theories of slowly changing functions. In fact, some works in this direction have also been explored in [1] and [2]. Actually in this paper we attempt to prove some results related to the growth rates of composite entire functions on the basis of central index using the idea of $(p,q,t)L$-th order and $(p,q,t)L$-th type of an entire function. We have used the standard notations using the theory of entire and meromorphic functions which are available in [8].

## 2. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1. *[4] Let $f(z)$ and $g(z)$ are any two entire functions with $g(0) = 0$. Also let $\beta$ satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,*

$$M_f\left(c(\beta) M_g(\beta r)\right) \le M_{f\circ g}(r) \le M_f\left(M_g(r)\right).$$

Lemma 2.2. *([6], Theorems 1.9 and 1.10, or [7], Satz 4.3 and 4.4) Let $f(z)$ be any entire function, then*

$$\log \mu_f(r) = \log|a_0| + \int_0^r \frac{\nu_f(t)}{t} dt \text{ where } a_0 \ne 0,$$

*and for $r < R$,*

$$M_f(r) < \mu_f(r)\left\{\nu_f(R) + \frac{R}{R-r}\right\}.$$

## 3. Main results

In this section, we present the main results of the paper.

Theorem 3.1. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\rho^{(m,n,t)L}(g) < \lambda^{(p,q,t)L}(f) \le \rho^{(p,q,t)L}(f) < +\infty$ where $m \le p$, $n = q$. Then*

$$\lim_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p-m]} \nu_f(2r)} = 0,$$

*where* $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r) + E])] = o\{\exp^{[m-1]}[(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{\alpha}\}$ *as $r \to \infty$, for some $\alpha < \lambda^{(p,q,t)L}(f)$ and some constant $E$.*

Proof. In view of the first part of Lemma 2.2, one may obtain that

$$\log\mu_f(2r) = \log|a_0| + \int_0^{2r} \frac{\nu_f(t)}{t} dt \ge \log|a_0| + \nu_f(r)\log 2 \ \{cf.\ [5]\}. \tag{3.1}$$

Also by Cauchy's inequality, it is well known that

$$\mu_f(r) \le M_f(r) \ \{cf.\ [9]\}. \tag{3.2}$$

So for some constant $D$, one may obtain from (3.1) and (3.2) that

$$\nu_f(r)\log 2 \le \log M_f(2r) + D \ \{cf.\ [5]\}. \tag{3.3}$$

Therefore in view of (3.3) and the second part of Lemma 2.1, we obtain for all sufficiently large values of $r$ that

$$\nu_{f\circ g}(r)\log 2 \le \log M_{f\circ g}(2r) + D \le \log M_f\left(M_g(2r)\right) + D,$$

$$\begin{aligned} i.e.,\ \log^{[p]}(\nu_{f\circ g}(r)\log 2) &\le \log^{[p]}(\log M_f\left(M_g(2r)\right) + D),\\ i.e.,\ \log^{[p]}\nu_{f\circ g}(r) &\le \log^{[p+1]} M_f\left(M_g(2r)\right) + O(1), \end{aligned}$$

$$i.e.,\ \log^{[p]}\nu_{f\circ g}(r) \le (\rho^{(p,q,t)L}(f) + \varepsilon)[\log^{[q]} M_g(2r) + \exp^{[t]} L(M_g(2r))] + O(1), \tag{3.4}$$

$$i.e.,\ \log^{[p]}\nu_{f\circ g}(r) \le (\rho^{(p,q,t)L}(f) + \varepsilon)[M_g(2r) + \exp^{[t]} L(M_g(2r))] + O(1),$$

$$i.e.,\ \log^{[p]}\nu_{f\circ g}(r) \le (\rho^{(p,q,t)L}(f) + \varepsilon) \times$$

$$[\exp^{[m-1]}[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{(\rho^{(m,n,t)L}(g)+\varepsilon)} + \exp^{[t]} L(M_g(2r))] + O(1),$$

$$\begin{aligned} &i.e.,\ \log^{[p]}\nu_{f\circ g}(r)\\ \le\ &(\rho^{(p,q,t)L}(f) + \varepsilon)\exp^{[m-1]}[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{(\rho^{(m,n,t)L}(g)+\varepsilon)}\\ &+ (\rho^{(p,q,t)L}(f) + \varepsilon)\exp^{[t]} L(M_g(2r)) + O(1). \end{aligned} \tag{3.5}$$

Further for some constant $E$, one may get from Lemma 2.2, that

$$\log M_g(r) < \nu_g(r)\log r + \log \nu_g(2r) + E \ \{cf.\ [5]\}.$$

Therefore from above we obtain that

$$\begin{aligned} \log M_g(r) &< \nu_g(2r)\log r + \nu_g(2r) + E, \\ i.e.,\ \log M_g(r) &< \nu_g(2r)(1+\log r) + E, \\ i.e.,\ M_g(2r) &< \exp[\nu_g(4r)\log(e\cdot 2r) + E]. \end{aligned} \tag{3.6}$$

Hence (3.5) and (3.6) we get for all sufficiently large values of $r$ that

$$\begin{aligned} &\log^{[p]} \nu_{f\circ g}(r) \\ &\le (\rho^{(p,q,t)L}(f)+\varepsilon)\exp^{[m-1]}[(\log^{[n-1]}(2r))\exp^{[t+1]} L(2r)]^{(\rho^{(m,n,t)L}(g)+\varepsilon)} \\ &+ (\rho^{(p,q,t)L}(f)+\varepsilon)\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])] + O(1). \end{aligned} \tag{3.7}$$

Again we obtain for all sufficiently large values of $r$ that

$$\begin{aligned} &\log^{[p-1]} \nu_f(r) \ge [(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{(\lambda^{(p,q,t)L}(f)-\varepsilon)}, \\ i.e.,\ &\log^{[p-m]} \nu_f(r) \ge \exp^{[m-1]}[(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{(\lambda^{(p,q,t)L}(f)-\varepsilon)}. \end{aligned} \tag{3.8}$$

Now from (3.7) and (3.8) we get for all sufficiently large values of $r$ that

$$\begin{aligned} &\frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p-m]} \nu_f(2r)} \le \\ &\frac{(\rho^{(p,q,t)L}(f)+\varepsilon)[\exp^{[m-1]}[(\log^{[n-1]}(2r))\exp^{[t+1]} L(2r)]^{(\rho^{(m,n,t)L}(g)+\varepsilon)}}{\exp^{[m-1]}[(\log^{[q-1]} 2r)\exp^{[t+1]} L(2r)]^{(\lambda^{(p,q,t)L}(f)-\varepsilon)}} \\ &+ \frac{(\rho^{(p,q,t)L}(f)+\varepsilon)\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])] + O(1)}{\exp^{[m-1]}[(\log^{[q-1]} 2r)\exp^{[t+1]} L(2r)]^{(\lambda^{(p,q,t)L}(f)-\varepsilon)}}. \end{aligned} \tag{3.9}$$

Since $\rho^{(m,n,t)L}(g) < \lambda^{(p,q,t)L}(f)$, we can choose $\varepsilon(>0)$ in such a way that

$$\rho^{(m,n,t)L}(g) + \varepsilon < \lambda^{(p,q,t)L}(f) - \varepsilon. \tag{3.10}$$

Now let for some $\alpha < \lambda^{(p,q,t)L}(f)$,
$\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r) + E])] = o\{\exp^{[m-1]}[(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{\alpha}\}$ as $r \to +\infty$.

As $\alpha < \lambda^{(p,q,t)L}(f)$ we can choose $\varepsilon(>0)$ in such a way that

$$\alpha < \lambda^{(p,q,t)L}(f) - \varepsilon. \tag{3.11}$$

$\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])] = o\{\exp^{[m-1]}[(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{\alpha}\}$ as $r\to\infty$, so we get on using (3.11) that

$$\begin{aligned} &\frac{\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])]}{\exp^{[m-1]}[(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{\alpha}} \to 0 \text{ as } r\to\infty, \\ i.e.,\ &\frac{\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}{\exp^{[m-1]}[(\log^{[q-1]} 2r)\exp^{[t+1]} L(2r)]^{(\lambda^{(p,q,t)L}(f)-\varepsilon)}} \to 0 \text{ as } r\to\infty. \end{aligned} \tag{3.12}$$

Now in view of (3.9), (3.10) and (3.12) we obtain that

$$\lim_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p-m]}\nu_f(2r)}=0.$$

Thus the theorem follows. □

Theorem 3.2. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\rho^{(p,q,t)L}(f)<+\infty$ and $0<\lambda^{(m,n,t)L}(g)\le\rho^{(m,n,t)L}(g)<+\infty$ where $m>q$. Then for some constant $E$,*

$$\limsup_{r\to+\infty}\frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[m]}\nu_g(2r)+\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}\le\frac{\rho^{(m,n,t)L}(g)}{\lambda^{(m,n,t)L}(g)}$$

*where $\exp^{[t]}[L(\exp[(\nu_g(2r)\log(e\cdot r)+E])]=o\{\log^{[m]}\nu_g(r)\}$ as $r\to+\infty$.*

Proof. From (3.4) and (3.6), for all sufficiently large values of $r$, we have

$$\log^{[p]}\nu_{f\circ g}(r)\le(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]}M_g(2r)$$
$$+(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot[\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]]+O(1),$$

$$i.e.,\ \log^{[p]}\nu_{f\circ g}(r)\le$$
$$(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]}M_g(2r)\Big[\frac{(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]}M_g(2r)}{(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]}M_g(2r)}$$
$$+\frac{(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot[\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]]+O(1)}{(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]}M_g(2r)}\Big],$$

$$i.e.,\ \log^{[p]}\nu_{f\circ g}(r)\le(\rho^{(p,q,t)L}(f)+\varepsilon)$$
$$\times\log^{[q]}M_g(2r)\Big[1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)}{\log^{[q]}M_g(2r)}\Big],$$

$$i.e.,\ \log^{[p+1]}\nu_{f\circ g}(r)\le\log(\rho^{(p,q,t)L}(f)+\varepsilon)+\log^{[q+1]}M_g(2r)$$
$$+\log\Big[1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)}{\log^{[q]}M_g(2r)}\Big],$$

As $\log(1+x)\sim x$, where $x=\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)}{\log^{[q]}M_g(2r)}$, we get for all sufficiently large values of $r$,

$$\log^{[p+1]}\nu_{f\circ g}(r)\le\log^{[q+1]}M_g(2r)+\log(\rho^{(p,q,t)L}(f)+\varepsilon)$$
$$+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)}{\log^{[q]}M_g(2r)},$$

$$i.e.,\ \log^{[p+1]}\nu_{f\circ g}(r)\le\log^{[q+1]}M_g(2r)$$

$$\times\Big[1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\log^{[q]}M_g(2r)\cdot\log^{[q+1]}M_g(2r)}\Big],$$

$$i.e.,\ \ \log^{[p+2]}\nu_{f\circ g}(r)\le\log^{[q+2]}M_g(2r)$$

$$+\log\Big[1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\log^{[q]}M_g(r)\cdot\log^{[q+1]}M_g(r)}\Big].$$

Again using $\log(1+x)\sim x$ for $x=\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)+\log^{[q]}M_g(r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+1}\log^{[k]}M_g(2r)}$, we get from above for all sufficiently large values of $r$,

$$\log^{[p+2]}\nu_{f\circ g}(r)\le\log^{[q+2]}M_g(2r)$$

$$+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+1}\log^{[k]}M_g(2r)}.$$

Continuing this process, we get

$$\log^{[p+m-q+1]}\nu_{f\circ g}(r)\le\log^{[q+m-q+1]}M_g(2r)$$

$$+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+m-q}\log^{[k]}M_g(2r)}.$$

$$i.e.,\log^{[p+m-q+1]}\nu_{f\circ g}(r)\le\log^{[m+1]}M_g(2r)$$

$$+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{m}\log^{[k]}M_g(2r)}.$$

$$i.e.,\ \ \log^{[p+m-q+1]}\nu_{f\circ g}(r)\le(\rho^{(m,n,t)L}(g)+\varepsilon)[\log^{[n]}2r+\exp^{[t]}L(2r)]$$

$$+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{m}\log^{[k]}M_g(2r)}. \tag{3.13}$$

Again we have for all sufficiently large values of $r$ that

$$\log^{[m]}\nu_g(r)\ge(\lambda^{(m,n,t)L}(g)-\varepsilon)[\log^{[n]}r+\exp^{[t]}L(r)]$$

$$i.e.,\ \ \log^{[n]}2r+\exp^{[t]}L(2r)\le\frac{\log^{[m]}\nu_g(2r)}{(\lambda^{(m,n,t)L}(g)-\varepsilon)}. \tag{3.14}$$

Hence from (3.13) and (3.14), it follows for all sufficiently large values of $r$ that

$$\log^{[p+m-q+1]}\nu_{f\circ g}(r)\le\Big(\frac{\rho^{(m,n,t)L}(g)+\varepsilon}{\lambda^{(m,n,t)L}(g)-\varepsilon}\Big)\cdot\log^{[m]}\nu_g(2r)$$

$$+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{m}\log^{[k]}M_g(2r)},$$

$$i.e,\ \frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[m]}\nu_g(2r)+\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}$$
$$\leq\left(\frac{\rho^{(m,n,t)L}(g)+\varepsilon}{\lambda^{(m,n,t)L}(g)-\varepsilon}\right)\cdot\frac{\log^{[m]}\nu_g(2r)}{\log^{[m]}\nu_g(2r)+\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}$$
$$+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{[\log^{[m]}\nu_g(2r)+\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]]\cdot\prod_{k=q}^{m}\log^{[k]}M_g(2r)},$$

$$i.e,\ \frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[m]}\nu_g(2r)+\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}$$
$$\leq\frac{\frac{\rho^{(m,n,t)L}(g)+\varepsilon}{\lambda^{(m,n,t)L}(g)-\varepsilon}}{1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\log^{[m]}\nu_g(2r)}}$$
$$+\frac{\frac{1}{\prod_{k=q}^{m}\log^{[k]}M_g(2r)}+\frac{O(1)}{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]\cdot\prod_{k=q}^{m}\log^{[k]}M_g(2r)}}{\frac{\log^{[m]}\nu_g(2r)}{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}+1}$$
$$+\frac{\frac{\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]\cdot\prod_{k=q+1}^{m}\log^{[k]}M_g(2r)}}{\frac{\log^{[m]}\nu_g(2r)}{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}+1}.\tag{3.15}$$

Since $\exp^{[t]}[L(\exp[(\nu_g(2r)\log(e\cdot r)+E])]=o\{\log^{[m]}\nu_g(r)\}$ as $r\to+\infty$ and $\varepsilon(>0)$ is arbitrary, we have from (3.15) that

$$\limsup_{r\to+\infty}\frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[m]}\nu_g(2r)+\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}\leq\frac{\rho^{(m,n,t)L}(g)}{\lambda^{(m,n,t)L}(g)}.$$

Thus the theorem is established. □

Now we state the following theorem without its proof as it can be carried out in the line of Theorem 3.2:

THEOREM 3.3. *Let $f(z)$ and $g(z)$ be any two entire functions such that $0<\lambda^{(p,q,t)L}(f)\leq\rho^{(p,q,t)L}(f)<+\infty$ and $\rho^{(m,n,t)L}(g)<+\infty$ where $m>n=q$. Then for some constant $E$,*

$$\limsup_{r\to+\infty}\frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[p]}\nu_f(2r)+\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}\leq\frac{\rho^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(f)}$$

*where $\exp^{[t]}[L(\exp[(\nu_g(2r)\log(e\cdot r)+E])]=o\{\log^{[p]}\nu_f(r)\}$ as $r\to+\infty$.*

Theorem 3.4. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\rho^{(p,q,t)L}(f) < \infty$ and $\lambda^{(p,q,t)L}(f \circ g) = +\infty$. Then*

$$\lim_{r \to +\infty} \frac{\log^{[p]} \nu_{f \circ g}(r)}{\log^{[p]} \nu_f(r)} = +\infty.$$

Proof. If possible, let there exists a constant $\beta$ such that for a sequence of values of $r$ tending to infinity we have

$$\log^{[p]} \nu_{f \circ g}(r) \leq \beta \cdot \log^{[p]} \nu_f(r). \tag{3.16}$$

Again from the definition of $\rho^{(p,q,t)L}(f)$, we obtain for all sufficiently large values of $r$ that

$$\log^{[p]} \nu_f(r) \leq (\rho^{(p,q,t)L}(f) + \varepsilon)[\log^{[q]} r + \exp^{[t]} L(r)]. \tag{3.17}$$

Now combining (3.16) and (3.17) we obtain for a sequence of values of $r$ tending to infinity that

$$\begin{aligned} \log^{[p]} \nu_{f \circ g}(r) &\leq \beta \cdot (\rho^{(p,q,t)L}(f) + \varepsilon)[\log^{[q]} r + \exp^{[t]} L(r)] \\ \text{i.e., } \lambda^{(p,q,t)L}(f \circ g) &\leq \beta \cdot (\rho^{(p,q,t)L}(f) + \varepsilon), \end{aligned}$$

which contradicts the condition $\lambda^{(p,q,t)L}(f \circ g) = +\infty$. So for all sufficiently large values of $r$ we get that

$$\log^{[p]} \nu_{f \circ g}(r) > \beta \cdot \log^{[p]} \nu_f(r),$$

from which the theorem follows. □

In the line of Theorem 3.4, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 3.5. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\rho^{(m,n,t)L}(g) < \infty$, $\lambda^{(p,q,t)L}(f \circ g) = +\infty$ and $n = q$. Then*

$$\lim_{r \to +\infty} \frac{\log^{[p]} \nu_{f \circ g}(r)}{\log^{[m]} \nu_g(r)} = +\infty.$$

Theorem 3.6. *Let $f(z)$ and $g(z)$ be any two entire functions such that $0 < \lambda^{(p,q,t)L}(f) \leq \rho^{(p,q,t)L}(f) < +\infty$ and $\sigma^{(m,n,t)L}(g) < +\infty$ where $m < q$. Then*

$$\limsup_{r \to +\infty} \frac{\log^{[p]} \nu_{f \circ g}(r)}{\log^{[p]} \nu_f(\exp^{[q]}[\log^{[n-1]}(2r) \cdot \exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)})} \leq \frac{\rho^{(p,q,t)L}(f) \cdot \sigma^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(f)},$$

*where $\exp^{[t]}[L(\exp[\nu_g(2r) \log(e \cdot r) + E])] = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ as $r \to +\infty$, for some $\alpha < \rho^{(m,n,t)L}(g)$ and some constant $E$.*

Proof. Since $0 < \rho^{(p,q,t)L}(f) < +\infty$, then it follows from (3.4)and (3.6) for all sufficiently large values of $r$ that

$$\begin{aligned} & \log^{[p]} \nu_{f\circ g}(r) \\ \leq\ & (\rho^{(p,q,t)L}(f)+\varepsilon)[\log^{[q]} M_g(2r)+\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]]+O(1), \end{aligned}$$

$$\begin{aligned} & i.e.,\ \log^{[p]} \nu_{f\circ g}(r) \\ \leq\ & (\rho^{(p,q,t)L}(f)+\varepsilon)[\log^{[m]} M_g(2r)+\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]]+O(1), \end{aligned}$$

$$\begin{aligned} i.e.,\ \log^{[p]} \nu_{f\circ g}(r) \ \leq\ & (\rho^{(p,q,t)L}(f)+\varepsilon) \\ & \times\Big[(\sigma^{(m,n,t)L}(g)+\varepsilon)[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)} \\ & +\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]\Big]+O(1)\,. \end{aligned} \tag{3.18}$$

Also, we obtain for all sufficiently large values of $r$ that

$$\begin{aligned} & \log^{[p]} \nu_f(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho^{(m,n,t)L}(g)}) \geq \\ & \qquad (\lambda^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho^{(m,n,t)L}(g)} \\ & \qquad +(\lambda^{(p,q,t)L}(f)-\varepsilon)\exp^{[t]}[L(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho^{(m,n,t)L}(g)})], \end{aligned}$$

$$\begin{aligned} & \log^{[p]} \nu_f(\exp^{[q]}[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)}) > \\ & \qquad (\lambda^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)}. \end{aligned}$$

Now from (3.18) and above it follows for all sufficiently large values of $r$ that

$$\begin{aligned} & \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p]} \nu_f(\exp^{[q]}[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)})} \\ \leq\ & \frac{(\rho^{(p,q,t)L}(f)+\varepsilon)[(\sigma^{(m,n,t)L}(g)+\varepsilon)[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)}}{(\lambda^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)}} \\ & +\frac{(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}{(\lambda^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)}} \\ & +\frac{O(1)}{(\lambda^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)}}. \end{aligned} \tag{3.19}$$

As $\alpha < \rho^{(m,n,t)L}(g)$ and $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])] = o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to+\infty$, we obtain that

$$\lim_{r\to+\infty}\frac{\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}{[\log^{[n-1]}(2r)\cdot\exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)}}=0. \tag{3.20}$$

Since $\varepsilon(>0)$ is arbitrary, it follows from (3.19) and (3.20) that

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p]} \nu_f(\exp^{[q]}[\log^{[n-1]}(2r) \cdot \exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)})} \leq \frac{\rho^{(p,q,t)L}(f) \cdot \sigma^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(f)}.$$

□

In the line of Theorem 3.6, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 3.7. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\lambda^{(m,n,t)L}(g) > 0$, $\rho^{(p,q,t)L}(f) < +\infty$ and $\sigma^{(m,n,t)L}(g) < +\infty$ where $m < q$. Then*

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[m]} \nu_g(\exp^{[n]}[\log^{[n-1]}(2r) \cdot \exp^{[t+1]} L(2r)]^{\rho^{(m,n,t)L}(g)})} \leq \frac{\rho^{(p,q,t)L}(f) \cdot \sigma^{(m,n,t)L}(g)}{\lambda^{(m,n,t)L}(g)},$$

*where* $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r) + E])] = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ *as* $r \to +\infty$, *for some* $\alpha < \rho^{(m,n,t)L}(g)$ *and some constant* $E$.

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