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A GENERALIZED ANALYTICAL MODEL OF UNSTEADY CHLORINE TRANSPORT IN DRINKING WATER SUPPLY PIPES

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Abstract

Simple analytical models of chlorine transport in water supply pipes are available. However, such models has been lacking to describe the distribution of chlorine uniformly throughout the pipe. This paper presents an exact solution to the equation governing two-dimensional unsteady chlorine transport in water supply pipes. Applying the Laplace transform yields the solution, and the method of residues yields the inverse Laplace transform. The Bessel function is used to derive the expression for the concentration of chlorine transport. Additionally, the analytical formula for the dimensionless cup-mixing average concentration is also derived. This model is very useful and effective for analyzing related models under different initial and boundary conditions.

Keywords and phrases: Analytical Model, Chlorine Transport, Laplace Transform.

1. Introduction

Improving the quality of drinking water has always been a challenging task. Various techniques and chemicals are used to purify drinking water. As we know that pure drinking water is very essential for good health. Therefore, this study is still a growing field because of its importance to society and engineering science. At a treatment plant, drinking water is often cleaned. The most popular disinfectant, chlorine, is a very unstable oxidizing agent that may react with inorganic, organic, as well as pipe materials in supply networks. Trihalomethanes, which are thought to be carcinogenic, can be created when chlorine interacts with organic materials found in nature. The water may undergo further disinfection in some critical locations since the disinfectant breaks down as it passes through the pipes.

Chlorine concentrations at various locations may fall below a specified minimum desirable level due to the transportation of chlorine through water supply pipe. To fulfil the need for chlorine across the whole distribution network, chlorine can be further injected or an appropriate dose may be supplied at the main junction. However, finding the desired location is also a challenging task. Therefore, maintaining the desired chlorine concentrations and making sure that the proportion of transport chlorine concentrations doesn't alter above or below predetermined levels are necessary for the successful management of assuring drinking water quality.

Biswas et al. [1] developed the fundamental concept of transport of chlorine in water supply pipes. Also many other researchers (Bhadula et al. [2], Al-Jasser [3], Abokifa et al. [4], Mahrous [5], Hallam et al. [6], and Ozdemir and Ger [7] & [8]) developed various models of chlorine transport. Based on the chlorine transport model of Rossman et al. [9] and the modified two-dimensional model for turbulent flow of Biswas et al. [1], Axworthy and Karney [10] examined the transport of chlorine in water supply long pipes. Recently, Aljohani et al. [11] analyzed the steady state two dimensional chlorine transport model of water supply in pipes.

Aljohani et al. [11] presented an exact solution to the chlorine transport model developed by Biswas et al. [1] in the absence of the diffusion term for chlorine concentration in water in the $x-$direction but retain the diffusion term in the $r-$direction. An improved model of the transport of chlorine in water supply pipes would certainly predict the distribution system of chlorine concentrations. In order to enhance the quality of drinking water, an accurate model of chlorine transport in water supply pipes is required.

In this paper, we present an analytical model of chlorine transport in water supply pipes. The exact solution of a generalized model of unsteady two-dimensional chlorine transport in water supply in pipes is studied. The exact solution of equation governing the above mentioned flow is obtained by using the method of Laplace transform and inverse Laplace transform.

The Laplace transform is a very powerful and effective method for solving various problems in engineering and science. The Laplace transform was used by Ebaid and Al Sharif [12] to solve the equations governing the thermal transport of nanofluids suspended with carbon nanotubes. By using the Laplace transform, Ebaid et al. [13] found analytical solutions for a class of singular boundary value problems. A model explaining the effects of radiation on MHD Marangoni flow over a flat plate was solved analytically by Khaled [14]. A wide class of singular boundary value problems with applications in nanofluids were addressed by Ebaid et al. [15] using the Laplace transform. The Laplace transform was used by Bakodah and Ebaid [16] to address the Ambartsumen delay problem. References (Sujit Handibag and Karande [17], Liang et al. [18], Venkata Pavani et al. [19], Ebaid et al. [20] and Spiegel [21]) include many Laplace transform uses in addition to additional modifications.

2. Mathematical formulation

The equation governing the unsteady two-dimensional chlorine transport of water supplies through pipes was established by Biswas et al. [1] as follows:

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x}(Uf(r)C) = \frac{\partial}{\partial x}\left(D_L \frac{\partial C}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(rD_r \frac{\partial C}{\partial r}\right) - \kappa C, \tag{2.1}$$

where r is the radial distance from the centre of the pipe, x is the axial distance from the inlet along the pipe, C is the chlorine concentration, $f(r)$ is the flow parameter term depending on the flow regime, D_L is the chlorine diffusion term in the axial direction, and D_r is the chlorine diffusion term in the radial direction, κ is the first order decay rate which is constant in the bulk water and U is the average flow velocity throughout the distribution system.

Biswas et al. [1] have solved the non-dimensional form of equation (2.1) by neglecting the diffusivity D_L in comparison with diffusivity D_r, and also assuming $f(r) = 1$. Biswas et al. [1] solved equation (2.1) with the following boundary conditions using the separable of variables method.

$$C(0,r) = 1, 0 \le r \le 1; \frac{\partial C(x,0)}{\partial r} = 0, 0 \le x \le 1; \frac{\partial C(x,1)}{\partial r} + A_2 C = 0, 0 \le x \le 1. \tag{2.2}$$

where A_2, the dimensionless parameter, regulates how quickly chlorine decay in the distribution system.

Similar problems have been solved by many scientists working in this field using different techniques. Recently, Aljohani et al. [11] have solved a similar problem analytically using the method of the Laplace transform.

This paper deals with the analytical model of unsteady chlorine transport in pipes. We consider the average flow velocity U across the distribution system to be fixed with respect to space coordinates and time and the flow parameter $f(r) = 1$. For this, we solve the following differential equation governing the two dimensional unsteady transport of chlorine in pipes:

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}\left(D_L \frac{\partial C}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r D_r \frac{\partial C}{\partial r}\right) - \kappa C, \tag{2.3}$$

Subject to boundary conditions

$$\begin{aligned} C(x=0,r,t) = C_0; \frac{\partial}{\partial r} C(x, r=0, t) = 0, \\ D_r \frac{\partial}{\partial r} C(x, r=r_0, t) = W_d(C) C(x, r=r_0, t); \frac{\partial}{\partial x} C(x=L, r, t) = 0. \end{aligned} \tag{2.4}$$

The non-dimensional forms of equations (2.3) and (2.4) are:

$$\frac{\partial C}{\partial t} + F_V \frac{\partial C}{\partial x} = P_L \frac{\partial^2 C}{\partial x^2} + \frac{A_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) - A_1 C, \tag{2.5}$$

Subject to non-dimensional boundary conditions

$$\begin{aligned} C(x=0,r,t) = 1; \frac{\partial}{\partial r} C(x, r=0, t) = 0, \\ \frac{\partial}{\partial r} C(x, r=1, t) + A_2 C(x, r=1, t) = 0; \frac{\partial}{\partial x} C(x=1, r, t) = 0. \end{aligned} \tag{2.6}$$

where non-dimensional parameters are

$$\begin{aligned} r \to r/r_0, C \to C/C_0, x \to x/L, t \to Ut/L, A_0 = (LD_r)/(r_0^2 U), A_1 = \kappa L/U, \\ A_2 = W_d(C) r_0/D_r, P_L = D_L/UL, F_V = V/U. \end{aligned} \tag{2.7}$$

F_V- is a dimensionless velocity profile that varies on a flow and pipe properties, but we assume constant value to provide an accurate solution.

3. Exact Solution

It is well established that water compounds interactions occurring in bulk water in pipes cause chlorine to decays in distribution systems (Al-Jasser [3] & Mahrous [5]). Therefore, the simplest solution of the equation (2.5) with decay in distribution system with time may be considered as

$$C(x, r, t) = u(x, r) \exp\left(-\lambda^2 t\right). \tag{3.1}$$

where λ^2 is the chlorine decay constant. On substitution of (3.1), equations (2.5) reduces to

$$-\lambda^2 u e^{-\lambda^2 t} + F_V \frac{\partial u}{\partial x} e^{-\lambda^2 t} = P_L \frac{\partial^2 u}{\partial x^2} e^{-\lambda^2 t} + \frac{A_0}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) e^{-\lambda^2 t} - A_1 u e^{-\lambda^2 t}, \tag{3.2}$$

On simplification, we get

$$F_V \frac{\partial u}{\partial x} - P_L \frac{\partial^2 u}{\partial x^2} = \frac{A_0}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) - \left(A_1 - \lambda^2\right) u, \tag{3.3}$$

Corresponding boundary conditions (2.6) reduces to,

$$\begin{gathered} u(x=0, r, t) = e^{-\lambda^2 t} \approx 1; \frac{\partial}{\partial r} u(x, r=0) = 0, \\ \frac{\partial}{\partial r} u(x, r=1) + A_2 u(x, r=1) = 0; \frac{\partial}{\partial x} u(x=1, r) = 0. \end{gathered} \tag{3.4}$$

Tanking Laplace transform to equation (3.3) with respect to axial coordinate x, we get

$$F_V L\left\{\frac{\partial u}{\partial x}\right\} - P_L L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = L\left\{\frac{A_0}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right)\right\} - \left(A_1 - \lambda^2\right) L\{u\}, \tag{3.5}$$

using the definition of Laplace transform of derivatives, the above equation gives

$$\begin{aligned} F_V \{sU(s, r) - u(0, r)\} - P_L \left\{s^2 U(s, r) - su(0, r) - u_x(0, r)\right\} = \frac{A_0}{r} \frac{d}{dr}\left(r \frac{dU(s, r)}{dr}\right) \\ - \left(A_1 - \lambda^2\right) U(s, r) \end{aligned} \tag{3.6}$$

or

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{\left\{P_L s^2 - F_V s + \left(\lambda^2 - A_1\right)\right\}}{A_0}\right] U(s, r) = -\frac{(F_V - sP_L) u(0, r) - P_L u_x(0, r)}{A_0}, \tag{3.7}$$

Equation (3.7) is the well-known inhomogeneous Bessel differential equation of order zero where inhomogeneous part is $-\{(F_V - sP_L) u(0, r) - P_L u_x(0, r)\}/A_0$. In general, $u(0, r)$ and $u_x(0, r)$ are functions of r, but here we consider both to be constants and hence the solution of Eq. (3.7) is given as

$$U\left(s,r\right)=C_0J_0\left(\mu r\right)+C_1Y_0\left(\mu r\right)-\frac{\{(F_V-sP_L)\,u(0,r)-P_Lu_x(0,r)\}}{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}},\tag{3.8}$$

where $\mu=\sqrt{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}/A_0}$ and J_0 and Y_0 are Bessel functions and C_0 and C_1 denotes the arbitrary constants. The value of C_1 must be zero because $Y_0\rightarrow\infty$ as $r\rightarrow 0$ requires that $u(x,r)$ and the associated Laplace transform $U(s,r)$ must be bounded at $r=0$.

Thus, we can write equation (3.8) as

$$U\left(s,r\right)=C_0J_0\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}r\right)-\frac{\{(F_V-sP_L)\,u(0,r)-P_Lu_x(0,r)\}}{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}},\tag{3.9}$$

Applying Laplace transform on boundary condition (3.4) yields,

$$\frac{d}{dr}U\left(s,r=1\right)+A_2U\left(s,r=1\right)=0,\tag{3.10}$$

on differentiation of equation (3.9) with respect to s and using the boundary conditions (3.4) and $u_x(0,r)=K(const.)$, we obtain,

$$\frac{d}{dr}U\left(s,r\right)=-C_0\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}\times J_1\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}r\right),\tag{3.11}$$

where $J'(kr)=-kJ_1(kr)$. From (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned}0=&-C_0\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}\times J_1\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}\right)\\&+A_2C_0J_0\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}\right)-A_2\frac{\{(F_V-sP_L)\,u(0,1)-P_Lu_x(0,1)\}}{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}},\end{aligned}\tag{3.12}$$

we obtain

$$C_0=A_2\frac{\{(F_V-sP_L)\,u(0,1)-P_Lu_x(0,1)\}}{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}\times\frac{1}{A_2J_0\left(\mu\right)-\mu\times J_1\left(\mu\right)}\tag{3.13}$$

Substituting (3.13) into (3.9) leads to,

$$U\left(s,r\right)=A_2F\left(s,r\right)-\frac{\{(F_V-sP_L)\,u(0,r)-P_Lu_x(0,r)\}}{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}\tag{3.14}$$

Let

$$F(s,r)=\frac{\frac{\{(F_V-sP_L)u(0,1)-P_Lu_x(0,1)\}}{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}\times J_0\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}r\right)}{A_2J_0\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}\right)-\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}J_1\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}\right)}, \tag{3.15}$$

Applying inverse Laplace transform to (3.14), yields

$$u(x,r)=A_2f(x,r)+L^{-1}\left\{-\frac{\{(F_V-sP_L)\,u(0,r)-P_Lu_x(0,r)\}}{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}\right\}. \tag{3.16}$$

where $f(x,r)$ is the inverse Laplace transform of $F(s,r)$ i.e.,

$$f(x,r)=L^{-1}\left\{\frac{\frac{\{(F_V-sP_L)u(0,1)-P_Lu_x(0,1)\}}{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}J_0\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}r\right)}{A_2J_0\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}\right)-\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}J_1\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}\right)}\right\}, \tag{3.17}$$

Now, we assume $u(0,r)=1$ and let $u_x(0,r)=K$, where K is constant, above equations (3.16) and (3.17) reduces to

$$u(x,r)=A_2f(x,r)-L^{-1}\left\{\frac{(F_V-sP_L)-P_LK}{P_Ls^2-F_Vs+(\lambda^2-A_1)}\right\}. \tag{3.18}$$

where $f(x,r)$ is the inverse Laplace transform of $F(s,r)$ so that,

$$f(x,r)=L^{-1}\left\{\frac{\frac{\{(F_V-sP_L)-P_LK\}}{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}J_0\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}r\right)}{A_2J_0\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}\right)-\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}J_1\left(\sqrt{\frac{\{P_Ls^2-F_Vs+(\lambda^2-A_1)\}}{A_0}}\right)}\right\}. \tag{3.19}$$

Simplifying the equation (3.18), we obtain

$$\begin{aligned}u(x,r)=A_2f(x,r)+\cos\left(\sqrt{\left(\frac{\lambda^2-A_1}{P_L}-\frac{F_V^2}{4P_L^2}\right)}x\right)\times\exp\left(\frac{F_V}{2PL}x\right)+\\ \left(\frac{K-F_V/2P_L}{\sqrt{\left(\frac{\lambda^2-A_1}{P_L}-\frac{F_V^2}{4P_L^2}\right)}}\right)\sin\left(\sqrt{\left(\frac{\lambda^2-A_1}{P_L}-\frac{F_V^2}{4P_L^2}\right)}x\right)\times\exp\left(\frac{F_V}{2PL}x\right).\end{aligned} \tag{3.20}$$

Finding the inverse Laplace transform of $F(s,r)$ is not a very easy task. The expression of $F(s,r)$ is really complex because the Bessel functions of zero and first ordered are present in the denominator. We have applied the method of residues to evaluate the inverse Laplace transform of $F(s,r)$. The method of residues states that if a function $F(s,r)$ has pole at$s_i(i = 1,2,3,\dots,n)$, then inverse Laplace transform of $F(s,r)$ is given by

$$f(x,r) = \sum_{i=1}^{n} Res\left\{e^{s_i x} \times F(s_i,r)\right\}. \tag{3.21}$$

The poles of the $F(s,r)$ are the zeroes of the expression in the denominator i.e. roots of the following equation

$$\left\{P_L s^2 - F_V s + \left(\lambda^2 - A_1\right)\right\} \times \left\{A_2 J_0(\mu) - \mu J_1(\mu)\right\} = 0, \tag{3.22}$$

where $\mu = \sqrt{\left\{P_L s^2 - F_V s + (\lambda^2 - A_1)\right\}/A_0}$. Equation (3.22) has simple zeroes at

$$P_L s^2 - F_V s + \left(\lambda^2 - A_1\right) = 0,$$

and at

$$\mu = \sqrt{\left\{P_L s^2 - F_V s + (\lambda^2 - A_1)\right\}/A_0} = \lambda_n,$$

or at

$$P_L s^2 - F_V s + \left(\lambda^2 - A_1 - A_0\lambda_n^2\right) = 0, (n = 1,2,3,...),$$

where $\lambda_n(n = 1,2,3,...)$ are the roots of the equation

$$\left\{A_2 J_0(\lambda_n) - \lambda_n J_1(\lambda_n)\right\} = 0.$$

Thus, the simple poles of expression $F(s,r)$ are,

$$s_1 = \frac{F_V - \sqrt{F_V^2 - 4P_L(\lambda^2 - A_1)}}{2P_L}, s_2 \frac{F_V + \sqrt{F_V^2 - 4P_L(\lambda^2 - A_1)}}{2P_L}. \tag{3.23}$$

and

$$s_3 = \frac{F_V - \sqrt{F_V^2 - 4P_L(\lambda^2 - A_1 - A_0\lambda_n^2)}}{2P_L}, s_4 = \frac{F_V + \sqrt{F_V^2 - 4P_L(\lambda^2 - A_1 - A_0\lambda_n^2)}}{2P_L}. \tag{3.24}$$

Now, using the formula (3.19), one can determine the inverse Laplace transform $f(x,r)$ of $F(s,r)$ by first computing the residues of $e^{sx}F(s,r)$ at $s = s_i$ $(i = 1,2,3,\dots)$.

Residue at $s = s_1$

The residue of $F(s,r)$ at $s = s_1$ i.e. $Res\left\{e^{sx}F(s,r), s_1\right\}$ is

$$\begin{aligned}
\lim_{s\to s_1}\left\{(s - s_1)e^{sx}F(s,r)\right\} &= \lim_{s\to s_1}\left\{\frac{(s - s_1)e^{sx}\left\{(F_V - sP_L) - P_L K\right\}J_0(\mu r)}{P_L(s - s_1)(s - s_2)\left\{A_2 J_0(\mu) - \mu J_1(\mu)\right\}}\right\} \\
&= \lim_{s\to s_1}\left\{\frac{e^{sx}\left\{(F_V - sP_L) - P_L K\right\}J_0(\mu r)}{P_L(s - s_2)\left\{A_2 J_0(\mu) - \mu J_1(\mu)\right\}}\right\} \\
&= \frac{e^{s_1 x}\left\{(F_V - s_1 P_L) - P_L K\right\}J_0(0)}{P_L(s_1 - s_2)\left\{A_2 J_0(0) - 0\right\}} = \frac{e^{s_1 x}\left\{(F_V - s_1 P_L) - P_L K\right\}}{P_L A_2(s_1 - s_2)},
\end{aligned} \tag{3.25}$$

Thus, the Residue at $s = s_1$ is

$$Res\left\{e^{sx}F(s,r),s_1\right\} = \frac{e^{s_1x}\left\{(F_V - s_1P_L) - P_LK\right\}}{P_LA_2(s_1 - s_2)}. \tag{3.26}$$

where $\mu = \sqrt{\left\{P_Ls^2 - F_Vs + (\lambda^2 - A_1)\right\}/A_0}$ and $J_0(0) = 1$.

Residue at $s = s_2$
Similarly the residue of $F(s,r)$ at $s = s_2$ is

$$Res\left\{e^{sx}F(s,r),s_2\right\} = \frac{e^{s_2x}\left\{(F_V - s_2P_L) - P_LK\right\}}{P_LA_2(s_2 - s_1)}. \tag{3.27}$$

Residue at $s = s_3$

$$\begin{aligned}
Res\left\{e^{sx}F(s,r),s_3\right\} &= \lim_{s\to s_3}(s - s_3)\,e^{sx}F(s,r)\\
&= \lim_{s\to s_3}\frac{(s - s_3)\,e^{sx}\left\{(F_V - sP_L) - P_LK\right\}J_0(\mu r)}{\left\{P_Ls^2 - F_Vs + (\lambda^2 - A_1)\right\}\left\{A_2J_0(\mu) - \mu J_1(\mu)\right\}}\\
&= \lim_{s\to s_3}\frac{(s - s_3)\left\{(F_V - sP_L) - P_LK\right\}}{\left\{A_2J_0(\mu) - \mu J_1(\mu)\right\}}\times\lim_{s\to s_3}\frac{e^{sx}J_0(\mu r)}{\left\{P_Ls^2 - F_Vs + (\lambda^2 - A_1)\right\}}\\
&= \lim_{s\to s_3}G(s,r)\times\frac{e^{s_3x}J_0(\lambda_nr)}{A_0\lambda_n^2},
\end{aligned} \tag{3.28}$$

To determine the limit of the expression $G(s,r)$ at $s = s_3$, L'Hospital's rule is used as follows

$$\begin{aligned}
\lim_{s\to s_3}G(s,r) &= \lim_{s\to s_3}\frac{(s - s_3)\left\{(F_V - sP_L) - P_LK\right\}}{\left\{A_2J_0(\mu) - \mu J_1(\mu)\right\}} = \frac{\left\{(F_V - s_3P_L) - P_LK\right\}}{\frac{-1}{2\lambda_n}\left(\frac{2s_3P_L - F_V}{A_0}\right)\left[A_2J_1(\lambda_n) + \lambda_nJ_0(\lambda_n)\right]}\\
&= -\frac{2\lambda_nA_0\left\{(F_V - s_3P_L) - P_LK\right\}}{(2s_3P_L - F_V)\left[A_2J_1(\lambda_n) + \lambda_nJ_0(\lambda_n)\right]},
\end{aligned} \tag{3.29}$$

Thus,

$$\begin{aligned}
Res\left\{e^{sx}F(s,r),s_3\right\} &= -\frac{2\lambda_nA_0\left\{(F_V - s_3P_L) - P_LK\right\}}{(2s_3P_L - F_V)\left[A_2J_1(\lambda_n) + \lambda_nJ_0(\lambda_n)\right]}\times\frac{e^{s_3x}J_0(\lambda_nr)}{A_0\lambda_n^2}\\
&= -\frac{2\left\{(F_V - s_3P_L) - P_LK\right\}e^{s_3x}J_0(\lambda_nr)}{(2s_3P_L - F_V)\left[A_2J_1(\lambda_n) + \lambda_nJ_0(\lambda_n)\right]\lambda_n}.
\end{aligned} \tag{3.30}$$

Similarly, the residue at $s = s_4$ is

$$Res\left\{e^{sx}F(s,r),s_4\right\} = -\frac{2\left\{(F_V - s_4P_L) - P_LK\right\}e^{s_4x}J_0(\lambda_nr)}{(2s_4P_L - F_V)\left[A_2J_1(\lambda_n) + \lambda_nJ_0(\lambda_n)\right]\lambda_n}. \tag{3.31}$$

Now applying the formula (3.21), we get

$$f(x,r) = Res\left\{e^{s_1x}F(s_1,r)\right\} + Res\left\{e^{s_2x}F(s_2,r)\right\} + Res\left\{e^{s_3x}F(s_3,r)\right\} + Res\left\{e^{s_4x}F(s_4,r)\right\}. \tag{3.32}$$

or

$$f(x,r) = \frac{e^{s_1 x}\{(F_V - s_1 P_L) - P_L K\} - e^{s_2 x}\{(F_V - s_2 P_L) - P_L K\}}{A_2(s_1 - s_2)} - \sum_{n=1}^{\infty}\left[\frac{2\{(F_V - s_3 P_L) - P_L K\} e^{s_3 x}}{(2s_3 P_L - F_V)} + \frac{2\{(F_V - s_4 P_L) - P_L K\} e^{s_4 x}}{(2s_4 P_L - F_V)}\right] \times \frac{J_0(\lambda_n r)}{\{A_2 J_1(\lambda_n) + \lambda_n J_0(\lambda_n)\}\lambda_n}, \tag{3.33}$$

Now, substituting the values of s_1, s_2, s_3 and s_4 from (3.23) and (3.24) in the above equation, we obtained,

$$f(x,r) = -\frac{P_L e^{\left(\frac{F_V}{2P_L}x\right)}}{2A_2\sqrt{X}}\left[\left\{F_V + \sqrt{X} - 2P_L K\right\} e^{-\frac{\sqrt{X}}{2P_L}x} - \left\{F_V - \sqrt{X} - 2P_L K\right\} e^{\frac{\sqrt{X}}{2P_L}x}\right] + \sum_{n=1}^{\infty}\left[\frac{\left\{F_V + \sqrt{M} - 2P_L K\right\}}{\sqrt{M}} e^{-\frac{\sqrt{M}}{2P_L}x} - \frac{\left\{F_V - \sqrt{M} - 2P_L K\right\}}{\sqrt{M}} e^{\frac{\sqrt{M}}{2P_L}x}\right] \times \frac{J_0(\lambda_n r)\, e^{\left(\frac{F_V}{2P_L}x\right)}}{\{A_2 J_1(\lambda_n) + \lambda_n J_0(\lambda_n)\}\lambda_n}. \tag{3.34}$$

where $X = F_V^2 - 4P_L\left(\lambda^2 - A_1\right)$, $M = F_V^2 - 4P_L\left(\lambda^2 - A_1 - A_0\lambda_n^2\right)$ and λ_n is the root of the equation:

$$A_2 J_0(\lambda_n) - \lambda_n J_1(\lambda_n) = 0. \tag{3.35}$$

Substituting (3.34), equation (3.20) yields

$$u(x,r) = -\frac{P_L e^{\left(\frac{F_V}{2P_L}x\right)}}{2\sqrt{X}}\left[\left\{F_V + \sqrt{X} - 2P_L K\right\} e^{-\frac{\sqrt{X}}{2P_L}x} - \left\{F_V - \sqrt{X} - 2P_L K\right\} e^{\frac{\sqrt{X}}{2P_L}x}\right] + A_2\sum_{n=1}^{\infty}\left[\frac{\left\{F_V + \sqrt{M} - 2P_L K\right\}}{\sqrt{M}} e^{-\frac{\sqrt{M}}{2P_L}x} - \frac{\left\{F_V - \sqrt{M} - 2P_L K\right\}}{\sqrt{M}} e^{\frac{\sqrt{M}}{2P_L}x}\right] \times \frac{J_0(\lambda_n r)\, e^{\left(\frac{F_V}{2P_L}x\right)}}{\{A_2 J_1(\lambda_n) + \lambda_n J_0(\lambda_n)\}\lambda_n} + \cos\left(\sqrt{\left(\frac{\lambda^2 - A_1}{P_L} - \frac{F_V^2}{4P_L^2}\right)}x\right) e^{\left(\frac{F_V}{2PL}x\right)} + \left(\frac{K - F_V/2P_L}{\sqrt{\left(\frac{\lambda^2 - A_1}{P_L} - \frac{F_V^2}{4P_L^2}\right)}}\right)\sin\left(\sqrt{\left(\frac{\lambda^2 - A_1}{P_L} - \frac{F_V^2}{4P_L^2}\right)}x\right) e^{\left(\frac{F_V}{2PL}x\right)}, \tag{3.36}$$

Thus, the unsteady chlorine transport in water supply through pipes may be given by,

$$C(x,r,t) = -\frac{P_L e^{\left(\frac{F_V}{2P_L}x-\lambda^2 t\right)}}{2\sqrt{X}}\left[\left\{F_V + \sqrt{X} - 2P_L K\right\}e^{-\frac{\sqrt{X}}{2P_L}x} - \left\{F_V - \sqrt{X} - 2P_L K\right\}e^{\frac{\sqrt{X}}{2P_L}x}\right]$$
$$+ A_2\sum_{n=1}^{\infty}\left[\frac{\left\{F_V + \sqrt{M} - 2P_L K\right\}}{\sqrt{M}}e^{-\frac{\sqrt{M}}{2P_L}x} - \frac{\left\{F_V - \sqrt{M} - 2P_L K\right\}}{\sqrt{M}}e^{\frac{\sqrt{M}}{2P_L}x}\right]\times$$
$$\frac{J_0(\lambda_n r)\, e^{\left(\frac{F_V}{2P_L}x-\lambda^2 t\right)}}{\{A_2 J_1(\lambda_n) + \lambda_n J_0(\lambda_n)\}\lambda_n} + \cos\left(\sqrt{\left(\frac{\lambda^2 - A_1}{P_L} - \frac{F_V^2}{4P_L^2}\right)}x\right)e^{\left(\frac{F_V}{2PL}x-\lambda^2 t\right)}$$
$$+\left(\frac{K - F_V/2P_L}{\sqrt{\left(\frac{\lambda^2-A_1}{P_L} - \frac{F_V^2}{4P_L^2}\right)}}\right)\sin\left(\sqrt{\left(\frac{\lambda^2 - A_1}{P_L} - \frac{F_V^2}{4P_L^2}\right)}x\right)e^{\left(\frac{F_V}{2PL}x-\lambda^2 t\right)}. \tag{3.37}$$

4. Results and Discussions

There are several simple and useful analytical models for chlorine transport through circular pipes available in the literature. But the most accurate and simple analytical model to study the generalized model has been lacking. In this paper, we have presented analytical solutions for the generalized chlorine transport model through pipes. Recently, Aljohani et al. [11] presented an exact solution to the equation governing two-dimensional steady chlorine transport in pipes via the Laplace transform in the absence of diffusivity of chlorine in axial direction. Aljohani et al. [11] have presented an exact solution of the equation governing chlorine transport model developed by Biswas et al. [1]. We retain the diffusion diffusivity in axial direction and then present the exact solution of the two-dimensional unsteady chlorine transport model of water supply in pipes. Although many numerical models (Ozdemir and Ger [7] & [8]) have been investigated for similar flow problems, there are also some simple analytical models but they fail to examine the real model.

Cup mixing average concentration

The cup mixing average concentration in dimensionless form determined by Biswas et al. [1].

$$u_{av} = 2\int_0^1 rC(x,r,t)\,dr, \tag{4.1}$$

Substituting (3.37) into (4.1), yields

$$u_{av} = -\frac{P_L e^{\left(\frac{F_V}{2P_L}x-\lambda^2 t\right)}}{2\sqrt{X}}\left[\left\{F_V + \sqrt{X} - 2P_L K\right\} e^{-\frac{\sqrt{X}}{2P_L}x} - \left\{F_V - \sqrt{X} - 2P_L K\right\} e^{\frac{\sqrt{X}}{2P_L}x}\right]$$
$$+ 2A_2 \sum_{n=1}^{\infty}\left[\frac{\left\{F_V + \sqrt{M} - 2P_L K\right\}}{\sqrt{M}} e^{-\frac{\sqrt{M}}{2P_L}x} - \frac{\left\{F_V - \sqrt{M} - 2P_L K\right\}}{\sqrt{M}} e^{\frac{\sqrt{M}}{2P_L}x}\right]\times$$
$$\frac{J_1(\lambda_n)\, e^{\left(\frac{F_V}{2P_L}x-\lambda^2 t\right)}}{\lambda_n^2\left\{A_2 J_1(\lambda_n) + \lambda_n J_0(\lambda_n)\right\}} + \cos\left(\sqrt{\left(\frac{\lambda^2 - A_1}{P_L} - \frac{F_V^2}{4P_L^2}\right)}x\right) e^{\left(\frac{F_V}{2PL}x-\lambda^2 t\right)} \tag{4.2}$$
$$+\left(\frac{K - F_V/2P_L}{\sqrt{\left(\frac{\lambda^2 - A_1}{P_L} - \frac{F_V^2}{4P_L^2}\right)}}\right)\sin\left(\sqrt{\left(\frac{\lambda^2 - A_1}{P_L} - \frac{F_V^2}{4P_L^2}\right)}x\right) e^{\left(\frac{F_V}{2PL}x-\lambda^2 t\right)}.$$

5. Conclusions

The two-dimensional unsteady chlorine transport model for water supply in pipes solved theoretically to determine the exact solution. By using the Laplace transformation, the exact solution is found. To get the inverse Laplace, the residual approach was used. The exact solutions are represented in terms of zero order Bessel's functions of the first and second kind. Using this approach, the exact solution is agree with previously reported solution by of variable separation method.

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