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APPROXIMATE SOLUTION OF FRACTIONAL PANTOGRAPH DIFFERENTIAL EQUATIONS USING SUMUDU DECOMPOSITION METHOD

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Abstract

This paper introduces the Sumudu decomposition method (SDM) to solve the fractional pantograph differential equations for framing signals with numerical approach. The Caputo sense fractional derivative has been used to form the statement of problems. The Sumudu decomposition method is described with the help of Adomian decomposition method and which provides the successful comparison with previous methods. The features of Adomian polynomials are applied to construct the decomposition method with Sumudu transform for convergent series solution. The brief discussion about absolute errors of the derived approximations is reflected in accuracy of the results. Finally, some numerical examples are explained to discuss the effectiveness of proposed method.

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1. Introduction

In fractional calculus, the generalization of differential equations is termed as fractional differential equations. The practical process is modeled on using fractional differential equations. The fractional differential equations have huge applications in science and engineering fields such as continuum and statistics mechanics, econometric, electrochemistry, medicine, mechanics of solids, fluid mechanics, electromagnetism and signal processing [18].
Currently the analysis of fractional differential equations (FDEs) has become an active area of research due to their applications to a variety of problems in physics and engineering with integral transforms. In recent years, a number of differential equations have been generalized using fractional order derivatives to model non-local phenomena. The integral transforms were extensively applied to solve the ordinary differential equations. A few of them are: Fourier, Hankel, Laplace, Mellin etc. G. K. Watugala [19] introduced a new integral transform named as the Sumudu transform and it is

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applied to find solution of ordinary differential equation in control engineering problems. The various properties and fundamental formulae of Sumudu transform have been explained by F. B. M. Belgacem [5].

It is difficult to find the exact solution of fractional pantograph differential equations but it is possible to calculate approximate analytical solutions for these differential equations. Therefore, different numerical methods have been investigated to study the solution of fractional differential equations such as Adomian decomposition method, eigen vector expansion method, homotopy perturbation method, finite difference method, variational iteration method, Chebyshev polynomial method, Bernstein operational method, mixed and Legendre collocation method [18].

As it is well known that the delay differential equations are repeatedly seen in control system, digital images, signal processing, lasers, chemical kinetics traffic models, population dynamic and metal cutting in such many physical phenomena. Delay differential equations are divided into distinctive types. Pantograph equation is one of the main and important type of delay differential equations. Previously several numerical techniques have been implemented for solution of pantograph differential equations of non-integer order. Among such methods, U. Saeed et al. [16] introduced Hermite wavelet method for fractional pantograph delay differential equations. K. Rabiei et al. [15] presented fractional-order Boubaker polynomials method for solution of fractional pantograph delay differential equations. P. Vichitkunakorn [18] applied Taylor wavelets for finding the solution of fractional pantograph differential equations. They developed exact formula for the Riemann-Liouville fractional integral of the Taylor Wavelets. M.S. Hashemi et al. [9] introduced least-squares approximation method for solving linear fractional pantograph delay equations. They converted given problem into a minimization problem. Further, the converted problem was solved by Langrage's multiplier method. In previous research work, the Sumudu decomposition method has been successfully applied for various fractional order differential equations [3, 4, 8, 10–14, 17]. The Sumudu decomposition method is based on coupling of Sumudu transform method and Adomian decomposition method [ADM]. The ADM has been introduced by famous mathematician and physician G. Adomian [1]. This historical survey motivates us to concentrate on the fractional pantograph differential equations.

In the present paper, we successfully employed the use of Sumudu decomposition method for finding the solutions of linear and nonlinear pantograph differential equations which are given in [16, 18]. The solution of linear and nonlinear equations is obtained in closed form which is infinite Taylor's series expansion. The obtained fruitful results are significantly more correct than existing findings.

We consider linear pantograph equation [15]

$$y'(t) = a(t)y(t) + b(t)y(qt) \tag{1.1}$$

subject to initial conditions
$y^k(0) = y_0^k$

Also, we concentrate on fractional pantograph differential equation [15]

$$D_t^{\alpha} y(t) = a(t)y(t) + \sum_{r=1}^{l} b_r(t) D^{\alpha_r} y(q_r t), t \in [0, h], n-1 < \alpha \leq n \tag{1.2}$$

subject to initial conditions
$y^k(0) = y_0^k, k = 0, 1, 2, \ldots\ldots, n-1, 0 < q_r < 1, 0 \leq \alpha_r < \alpha \leq n, r = 1, 2, \ldots..l.$
where D_t^{α} is the Caputo fractional derivative, $h \leq 1, y(t)$ is an unidentified function, $b_r(t), r = 1, 2, \ldots..l$, and $a(t)$ are known and determined in $[0, h]$.
The summary of this paper is as follows. Section 2 explains basic definitions of fractional calculus, Sumudu transform and its properties, Section 3 provides the methodology of Sumudu decomposition Method (SDM) for explaining the fractional pantograph differential equations (FPDEs), Section 4 specifies convergence of Adomian decomposition method, Section 5 includes three numerical examples. Finally, Section 6 briefly concludes the summary of the paper.

2. Preliminaries of Fractional Calculus

In this section, the prerequisites of fractional calculus, definition and properties of Sumudu transform have been explained.

Definition 2.1. [18] The Riemann-Liouville fractional integral and differential operator of order $\alpha \geq 0$ of a function $f(t)$ over $[0, +\infty)$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * f(t), if \alpha > 0, \tag{2.1}$$

$I^0 f(t) = f(t), if \alpha > 0,$
where $x^{\alpha-1} * f(t)$ is the convolution product of $x^{\alpha-1}$ and $f(t)$.
For $\alpha \geq 0$, we have
1.$I^{\alpha}(af + bg) = aI^{\alpha} f + bI^{\alpha} g$, for all functions $f \& g$ defined over $[0, \infty)$ and real numbers $a \& b$.
2.$I^{\alpha}\left(I^{\beta} f(t)\right) = I^{\alpha+\beta} f(t)$, for all functions f defined over $[0, \infty)$ and $\alpha \& \beta \geq 0$.
3.$I^{\alpha} t^j = \frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)} t^{j+\alpha}$, for every $j > -1$.
In 1967, M. Caputo [6] used elementary definition of fractional differential operator to give brief summary of rigorous proofs of the formulae which have been applied in obtaining the analytic expression of Q.

Definition 2.2. [18] The Caputo fractional derivative of$f(t)$ of order $\alpha \geq 0$ over $[0, +\infty)$ is defined as

$$D^{\alpha} f(t) = I^{n-\alpha} f^n(t), for \alpha = n. \tag{2.2}$$

For $\alpha \geq 0$, we have
1.$D^{\alpha}(af + bg) = aD^{\alpha} f + bD^{\alpha} g$, for all functions $f \& g$ defined over $[0, \infty)$ and real numbers $a \& b$.
2.$D^{\alpha} I^{\alpha} f(t) = f(t)$.

3.$I^{\alpha}D^{\alpha}f(t) = f(t) - \sum_{j=0}^{\alpha-1}\frac{f^{j}(0)}{j!}t^{j}$.
4.$D^{\alpha}t^{j} = \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)}t^{j-\alpha}$, for every $j > \alpha - 1$.

Definition 2.3. [5] The Sumudu transform is defined over the set of functions

$$A = \{f(t) : \exists M, \tau_1, \tau_2 > 0, f(t) < Me^{\frac{t}{\tau_j}}, if t \in (-1)^j \times [0, \infty)\}, \tag{2.3}$$

which is defined through definite integral by using the following formula:

$$F(u) = S[f(t)] = \frac{1}{u}\int_0^{\infty} e^{\frac{-t}{u}} f(t)dt, u \in (-\tau_1, \tau_2) \tag{2.4}$$

where S is a Sumudu transform operator.
The various properties of Sumudu transform have been included in [14]. For understanding the concept of present research article, we consider some useful properties of the Sumudu transform as given below.
1.$S\{1\} = 1$,
2.$S\{t^n\} = u^n\Gamma(n+1), n > 0$,
3.$S\{f(t) \pm g(t)\} = S\{f(t)\} \pm S\{g(t)\}$.

Definition 2.4. [12] The Sumudu transform of Caputo fractional derivative is defined as follows

$$S\{D_t^{\alpha}f(t)\} = u^{-\alpha}S\{f(t)\} - \sum_{k=0}^{m-1} u^{-\alpha+k}f^{(k)}(0), m-1 < \alpha \le m. \tag{2.5}$$

3. Analysis of the Method [SDM]

The proposed work starts with a class of fractional pantograph differential equation of the form

$$D_t^{\alpha}y(t) + L(y) + N(y) = f(t), n-1 < \alpha \le n, \tag{3.1}$$

with intial condition

$$y^k(0) = y_0^k, \tag{3.2}$$

where L denotes the linear bounded operator, N denotes the nonlinear bounded operator, $f(t)$ is a given continuous function and $D_t^{\alpha}y(t)$ denotes the term of the Caputo fractional order derivative. The formation of Sumudu decomposition method depends on Sumudu transform and Adomian polynomials. Such a technique has been discussed in [14].
First, we apply Sumudu transform on both sides of equation (3.1) to obtain

$S\{D_t^\alpha y(t)\} + S\{L(y)\} + S\{N(y)\} = S\{f(t)\}$

By definition (2.4) and initial condition (3.2), we obtain

$\frac{S\{y(t)\}}{u^\alpha} - \frac{C}{u^{\alpha-k}} + S\{L(y)\} + S\{N(y)\} = S\{f(t)\}$, where $C = \sum_{k=0}^{n-1} f^{(k)}(0)$

$$S\{y(t)\} = u^k C + u^\alpha S\{f(t)\} - u^\alpha S\{L(y)\} - u^\alpha S\{N(y)\} \tag{3.3}$$

The concept of Sumudu decomposition method defines the solution $y(t)$ by the series

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{3.4}$$

and the decomposition of linear and nonlinear terms is divided into two cases.

Case I: Decomposition of linear operator $L(y) = \sum_{n=0}^{\infty} A_n$

whereA_n i.e. Adomian Polynomials of $y_0, y_1, y_2, \ldots\ldots., y_n$ that are given by using the relation

$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[L\left(\sum_{n=0}^{\infty} \lambda^n y_n\right)\right]_{\lambda=0}$ $, n = 0, 1, 2, \ldots$

The first few Adomian Polynomials are defined by

$$A_0 = L(y_0) \tag{3.5}$$

$$A_1 = y_1 L^{'}(y_0) \tag{3.6}$$

$$A_2 = y_2 L^{'}(y_0) + \frac{1}{2!} y_1^2 L^{''}(y_0) \tag{3.7}$$

$$A_3 = y_3 L^{'}(y_0) + y_1 y_2 L^{''}(y_0) + \frac{1}{3!} y_1^3 L^{'''}(y_0) \tag{3.8}$$

and so on.

Case II: Decomposition of nonlinear operator $N(y) = \sum_{n=0}^{\infty} A_n$

whereA_n i.e. Adomian Polynomials of $y_0, y_1, y_2, \ldots\ldots., y_n$ that are given by using the relation

$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty} \lambda^n y_n\right)\right]_{\lambda=0}$ $, n = 0, 1, 2, \ldots$

The first few Adomian Polynomials are defined by

$$A_0 = N(y_0) \tag{3.9}$$

$$A_1 = y_1 N^{'}(y_0) \tag{3.10}$$

$$A_2 = y_2 N^{'}(y_0) + \frac{1}{2!} y_1^2 N^{''}(y_0) \tag{3.11}$$

$$A_3 = y_3 N^{'}(y_0) + y_1 y_2 N^{''}(y_0) + \frac{1}{3!} y_1^3 N^{'''}(y_0) \tag{3.12}$$

and so on.
We put equation (3.4), Case I and Case II in (3.3), we get

$$S\left\{\sum_{n=0}^{\infty} y_n(t)\right\} = u^k C + u^\alpha S\{f(t)\} - u^\alpha S\left\{\sum_{n=0}^{\infty} A_n\right\} \tag{3.13}$$

Comparing both side of equation (3.13)

$$S\{y_0\} = u^k C + u^\alpha S\{f(t)\}, \tag{3.14}$$

$$S\{y_1\} = -u^\alpha S\{A_0\}, \tag{3.15}$$

$$S\{y_2\} = -u^\alpha S\{A_1\}, \tag{3.16}$$

In general, the recursive relation is given by

$$S\{y_n\} = -u^\alpha S\{A_{n-1}\}, n \geq 1, \tag{3.17}$$

Further, we apply inverse Sumudu transform from equation (3.14) to equation (3.17) and get

$$y_0 = F(t), \tag{3.18}$$

$$y_n = -S^{-1}[u^\alpha S\{A_{n-1}\}], n \geq 1. \tag{3.19}$$

Where $F(t)$ is a function that is obtained from the source term and prescribed initial conditions and y_n indicates the general approximate solution which provides all necessary iterations.

4. Convergence of Adomian decomposition method

In this section, we demonstrate that the proposed approximation in equation (3.19) is valid. For this purpose, K. Abbaoui and Y. Cherruault [2] suggested new ideas for proving the convergence of decomposition method. G. Adomian [1] investigated a new technique for solving exactly nonlinear functional equations of various kinds (algebraic, differential, partial differential, integral...). In this section, we prove convergence of the series solution with the help of a new formula giving the Adomian's polynomials. This formula produces the series solution as a function of the first term of the series. We have a simple formula for calculation of A_n.

$$A_n = \sum_{\alpha_1+\alpha_2+\cdots+\alpha_n} N^{(\alpha_1)}(u_0)\frac{u_1^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!}\cdots\cdots\frac{u_{n-1}^{\alpha_{n-1}-\alpha_n}}{(\alpha_{n-1}-\alpha_n)!}\frac{u_n^{\alpha_n}}{(\alpha_n)!}, n \neq 0.$$

Theorem 4.1. *With the following hypothesis,*
(a) N is C^∞ in a neghbourhood of u_0 and $\|N^{(n)}(u_0)\| \leq M'$, for any n (the derivatives of N at u_0 are bounded in norm) where N is a nonlinear operator from a Hilbert space H into H;
(b) $\|u_i\| \leq M < 1, i = 1, 2, \ldots,$ where $\|\cdot\|$ is the norm in the Hilbert space H; the series

$\sum_{n=0}^{\infty} A_n$ *is absolutely convergent and furthermore,*

$\|A_n\| \leq \left(exp\left(\pi\sqrt{\frac{2}{3}n}\right)\right)M^{'}M^n, n \geq M, n \geq M^{'}$

where $M > 0$ *and* $M^{'} > 0$ *both are finite numbers.*

THEOREM 4.2. *If* N *is* C^{∞} *and satisfies* $\|N^{(n)}(u_0)\| \leq M < 1$, *for any* $n \in N$, *then the decompositional series* $\sum_{n=0}^{\infty} u_n$ *is absolutely convergent and we have* $\|u_{n+1}\| = \|A_n\| \leq M^{n+1}n^{\sqrt{n}}\left(exp\left(\pi\sqrt{\frac{2}{3}n}\right)\right)$.

Previously Y. Cherruault [7] discussed that the Adomian technique is equivalent to determining the sequence: $S_n = y_1 + y_2 + \cdots + y_n, S_{n+1} = N(y_0 + S_n), S_0 = 0$.

THEOREM 4.3. *N being a contradiction* ($\delta < 1$), *if we assume that* $\|N_n - N\| = \epsilon_{n(n\to\infty)} \to 0$, *(satisfied in our case), then the sequence* S_n *is given by* $S_{n+1} = N_n(y_0 + S_n), S_0 = 0$ *converges towards the* S *solution of* $N(y_0 + S) = S$.

THEOREM 4.4. *(1) For every* $f \in V^{'}$, *there exists* $y \in V$ *such that:* $y - N(y) = f$ *where* V *is a Hilbert space and* $V^{'}$ *its dual.*
(2) The sequence y_n *defined by* $y_{n+1} = y_n - \varrho[N(y_0 + y_n)]$, $\varrho > 0$ *is strongly convergent in* V *and its limit* y *is the solution of* $y = N(y_0 + y)$ *for* $\varrho > 0$ *well chosen. A first consequence is that* $u = y_0 + y$ *with* $y_0 = f$ *is a solution of* $u = f + N(u)$.

These convergence theorems are powerful and easy to handle Linear (L) and Non-linear (N) terms. The series solution is convergent and successive terms y_i are easily computed. In Adomian research work, we can see the various difficult problems have been successively solved.

5. Numerical Problems

This section is dedicated to some numerical problems to show the relevance and meticulousness of our method. We have used Mathematica software for presenting all the results graphically.

Example 5.1 We consider the linear fractional pantograph differential equation[18]

$$D_t^{\alpha}y(t) = \frac{3}{4}y(t) + y\left(\frac{t}{2}\right) - t^2 + 2, 0 \leq t \leq 1, 1 < \alpha \leq 2, \tag{5.1}$$

subject to initial condition

$$y(0) = 0, y^{'}(0) = 0, \tag{5.2}$$

The exact solution of equation is given by $y(t) = t^2$ when $\alpha = 2$.
We apply Sumudu transform to both side of equation (5.1)
$S\{D_t^{\alpha}y(t)\} = S\left\{\frac{3}{4}y(t) + y\left(\frac{t}{2}\right) - t^2 + 2\right\}$

By using definition (2.4) and initial condition (5.2), we have
$\frac{Y(u)}{u^{\alpha}} - \frac{y(0)}{u^{\alpha-0}} - \frac{y'(0)}{u^{\alpha-1}} = S\left\{\frac{3}{4}y(t) + y\left(\frac{t}{2}\right) - t^2 + 2\right\}$

$$Y(u) = u^{\alpha} S\left\{\frac{3}{4}y(t) + y\left(\frac{t}{2}\right) - t^2 + 2\right\} \tag{5.3}$$

Further, we apply inverse Sumudu transform to (5.3) to get
$y(t) = S^{-1}\left[u^{\alpha}S\{-t^2+2\}\right] + S^{-1}\left[u^{\alpha}S\left\{\frac{3}{4}y(t) + y\left(\frac{t}{2}\right)\right\}\right]$
$y_0(t) = S^{-1}\{2u^{\alpha} - u^{\alpha+2}\Gamma(2+1)\} = \frac{2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}$
$y_0\left(\frac{t}{2}\right) = \frac{2t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} - \frac{2t^{\alpha+2}}{2^{\alpha+2}\Gamma(\alpha+3)}$

$$y_{n+1}(t) = S^{-1}[u^{\alpha}S\{A_n\}] \tag{5.4}$$

The linear term of FPDE is $L(y) = \frac{3}{4}y(t) + y\left(\frac{t}{2}\right)$.
From equation (3.5) to (3.7), we have

$$A_0 = \frac{3}{4}y_0(t) + y_0\left(\frac{t}{2}\right) \tag{5.5}$$

$$A_1 = \frac{3}{4}y_1(t) + y_1\left(\frac{t}{2}\right) \tag{5.6}$$

$$A_2 = \frac{3}{4}y_2(t) + y_2\left(\frac{t}{2}\right) \tag{5.7}$$

and so on.
We put n=0 in equation (5.4)

$$y_1(t) = S^{-1}[u^{\alpha}S\{A_0\}] \tag{5.8}$$

Substituting equation (5.5) in (5.8), we get
$y_1(t) = S^{-1}\left[u^{\alpha}S\left\{\frac{3}{4}y_0(t) + y_0\left(\frac{t}{2}\right)\right\}\right]$
$= \frac{3t^{2\alpha}}{2\Gamma(2\alpha+1)} - \frac{3t^{2\alpha+2}}{2\Gamma(2\alpha+3)} + \frac{2t^{2\alpha}}{2^{\alpha}\Gamma(2\alpha+1)} - \frac{2t^{2\alpha+2}}{2^{\alpha+2}\Gamma(2\alpha+3)}$
$y_1\left(\frac{t}{2}\right) = \frac{3t^{2\alpha}}{2^{2\alpha+1}2\Gamma(2\alpha+1)} - \frac{3t^{2\alpha+2}}{2^{2\alpha+3}\Gamma(2\alpha+3)} + \frac{2t^{2\alpha}}{2^{3\alpha}\Gamma(2\alpha+1)} - \frac{2t^{2\alpha+2}}{2^{3\alpha+4}\Gamma(2\alpha+3)}$
We put n=1 in equation (5.4)

$$y_2(t) = S^{-1}[u^{\alpha}S\{A_1\}] \tag{5.9}$$

Substituting equation (5.6) in (5.9), we get
$y_2(t) = S^{-1}\left[u^{\alpha}S\left\{\frac{3}{4}y_1(t) + y_1\left(\frac{t}{2}\right)\right\}\right]$
$= \frac{9t^{3\alpha}}{8\Gamma(3\alpha+1)} - \frac{9t^{3\alpha+2}}{8\Gamma(3\alpha+3)} + \frac{6t^{3\alpha}}{42^{\alpha}\Gamma(3\alpha+1)} - \frac{6t^{3\alpha+2}}{42^{\alpha+2}\Gamma(3\alpha+3)} + \frac{3t^{3\alpha}}{2^{2\alpha+1}\Gamma(3\alpha+1)} - \frac{3t^{3\alpha+2}}{2^{2\alpha+3}\Gamma(3\alpha+3)}$
$+ \frac{2t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} - \frac{2t^{3\alpha+2}}{2^{3\alpha+4}\Gamma(3\alpha+3)}$

$y_2\left(\frac{t}{2}\right) = \frac{9t^{3\alpha}}{82^{3\alpha}\Gamma(3\alpha+1)} - \frac{9t^{3\alpha+2}}{82^{3\alpha+2}\Gamma(3\alpha+3)} + \frac{6t^{3\alpha}}{42^{4\alpha}\Gamma(3\alpha+1)} - \frac{6t^{3\alpha+2}}{42^{4\alpha+4}\Gamma(3\alpha+3)} + \frac{3t^{3\alpha}}{2^{5\alpha+1}\Gamma(3\alpha+1)}$
$- \frac{3t^{3\alpha+2}}{2^{5\alpha+5}\Gamma(3\alpha+3)} + \frac{2t^{3\alpha}}{2^{6\alpha}\Gamma(3\alpha+1)} - \frac{2t^{3\alpha+2}}{2^{6\alpha+6}\Gamma(3\alpha+3)}$

We put n=2 in equation (5.4)

$$y_3(t) = S^{-1}[u^\alpha S\{A_2\}] \tag{5.10}$$

Substituting equation (5.7) in (5.10), we get

$$\begin{aligned} y_3(t) &= S^{-1}\left[u^\alpha S\left\{\tfrac{3}{4}y_2(t) + y_2\left(\tfrac{t}{2}\right)\right\}\right] \\ &= \frac{27t^{4\alpha}}{32\Gamma(4\alpha+1)} - \frac{27t^{4\alpha+2}}{32\Gamma(4\alpha+3)} + \frac{18t^{4\alpha}}{162^{\alpha}\Gamma(4\alpha+1)} - \frac{18t^{4\alpha+2}}{162^{\alpha+2}\Gamma(4\alpha+3)} + \frac{9t^{4\alpha}}{42^{2\alpha+1}\Gamma(4\alpha+1)} \\ &- \frac{9t^{4\alpha+2}}{42^{2\alpha+3}\Gamma(4\alpha+3)} + \frac{6t^{4\alpha}}{42^{3\alpha}\Gamma(4\alpha+1)} - \frac{6t^{4\alpha+2}}{42^{3\alpha+4}\Gamma(4\alpha+3)} + \frac{9t^{4\alpha}}{82^{3\alpha}\Gamma(4\alpha+1)} - \frac{9t^{4\alpha+2}}{82^{3\alpha+2}\Gamma(4\alpha+3)} \\ &+ \frac{6t^{4\alpha}}{42^{4\alpha}\Gamma(4\alpha+1)} - \frac{6t^{4\alpha+2}}{42^{4\alpha+4}\Gamma(4\alpha+3)} + \frac{3t^{4\alpha}}{2^{5\alpha+1}\Gamma(4\alpha+1)} - \frac{3t^{4\alpha+2}}{2^{5\alpha+5}\Gamma(4\alpha+3)} + \frac{2t^{4\alpha}}{2^{6\alpha}\Gamma(4\alpha+1)} \\ &- \frac{2t^{4\alpha+2}}{2^{6\alpha+6}\Gamma(4\alpha+3)} \end{aligned}$$

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The series solution is given by

$$\begin{aligned} y(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) + \ldots \\ &= \frac{2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{3}{2}\left(\frac{t^{2\alpha}}{2\Gamma(\alpha+1)} - \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)}\right) + \frac{2t^{2\alpha}}{2^{\alpha}\Gamma(2\alpha+1)} - \frac{2t^{2\alpha+2}}{2^{\alpha+2}\Gamma(2\alpha+3)} + \frac{9t^{3\alpha}}{8\Gamma(3\alpha+1)} \\ &- \frac{9t^{3\alpha+2}}{8\Gamma(3\alpha+3)} + \frac{6t^{3\alpha}}{42^{\alpha}\Gamma(3\alpha+1)} - \frac{6t^{3\alpha+2}}{42^{\alpha+2}\Gamma(3\alpha+3)} + \frac{3t^{3\alpha}}{2^{2\alpha+1}\Gamma(3\alpha+1)} - \frac{3t^{3\alpha+2}}{2^{2\alpha+3}\Gamma(3\alpha+3)} + \frac{2t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} \\ &- \frac{2t^{3\alpha+2}}{2^{3\alpha+4}\Gamma(3\alpha+3)} + \frac{27t^{4\alpha}}{32\Gamma(4\alpha+1)} - \frac{27t^{4\alpha+2}}{32\Gamma(4\alpha+3)} + \frac{18t^{4\alpha}}{162^{\alpha}\Gamma(4\alpha+1)} - \frac{18t^{4\alpha+2}}{162^{\alpha+2}\Gamma(4\alpha+3)} \\ &+ \frac{9t^{4\alpha}}{42^{2\alpha+1}\Gamma(4\alpha+1)} - \frac{9t^{4\alpha+2}}{42^{2\alpha+3}\Gamma(4\alpha+3)} + \frac{6t^{4\alpha}}{42^{3\alpha}\Gamma(4\alpha+1)} - \frac{6t^{4\alpha+2}}{42^{3\alpha+4}\Gamma(4\alpha+3)} + \frac{9t^{4\alpha}}{82^{3\alpha}\Gamma(4\alpha+1)} \\ &- \frac{9t^{4\alpha+2}}{82^{3\alpha+2}\Gamma(4\alpha+3)} + \frac{6t^{4\alpha}}{42^{4\alpha}\Gamma(4\alpha+1)} - \frac{6t^{4\alpha+2}}{42^{4\alpha+4}\Gamma(4\alpha+3)} + \frac{3t^{4\alpha}}{2^{5\alpha+1}\Gamma(4\alpha+1)} - \frac{3t^{4\alpha+2}}{2^{5\alpha+5}\Gamma(4\alpha+3)} \\ &+ \frac{2t^{4\alpha}}{2^{6\alpha}\Gamma(4\alpha+1)} - \frac{2t^{4\alpha+2}}{2^{6\alpha+6}\Gamma(4\alpha+3)} + \ldots \end{aligned}$$

In particular case $\alpha = 2$, we get approximate series solution
$y(t) = t^2$
The exact solution when $\alpha = 2$ is given by $y(t) = t^2$.
According to convergence theorem of Adomian decomposition method, the obtained infinite series is rapidly convergent [2, 7]. Table 1 shows the comparison of approximate solution of SDM with the existing literature of Fractional-order Boubaker polynomials method [15], we observe that the absolute errors between numerical and exact solutions could be reduced by choosing suitable α. The best computations of the ignorable absolute errors show the accuracy of the proposed method.

In 2D figure, we successfully compared approximate and exact solutions of linear fractional pantograph differential equation (5.1) to (5.2). The comparison shows that the obtained numerical solution is identical with the exact solution.

Example 5.2 We consider the linear fractional pantograph differential equation[16]

$$D_t^\alpha y(t) = \frac{1}{2}e^{\frac{t}{2}}y\left(\frac{t}{2}\right) + \frac{1}{2}y(t), 0 \le t \le 1, 0 < \alpha \le 1, \tag{5.11}$$

subject to initial condition

$$y(0) = 1, \tag{5.12}$$

The exact solution of equation is given by $y(t) = e^t$ when $\alpha = 1$.
We apply Sumudu transform to both side of equation (5.11)

TABLE 1. Comparison of absolute errors of SDM with Fractional-order Boubaker polynomials method for $\alpha = 2$ in the interval [0, 1].

t	SDM	Absolute Errors		Exact solution
		SDM	Fractional-order Boubaker polynomials method[15]	
0.0	0.0	0.0	7.20×10^{-18}	0.0
0.2	0.04	0.0	1.60×10^{-17}	0.04
0.4	0.16	0.0	4.27×10^{-17}	0.16
0.6	0.36	0.0	8.71×10^{-17}	0.36
0.8	0.64	0.0	1.49×10^{-16}	0.64
1	1	0.0	2.29×10^{-16}	1

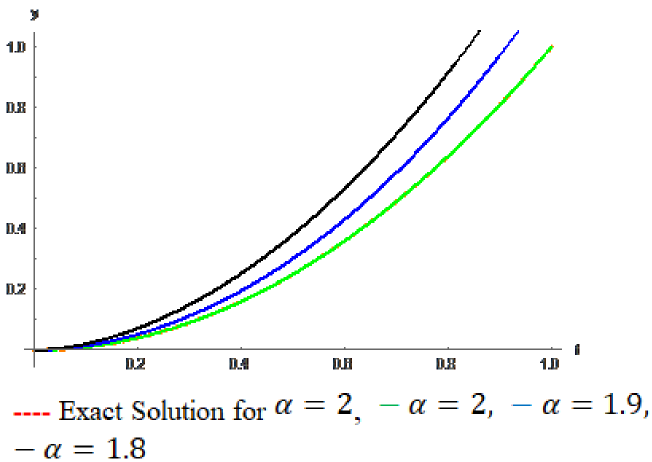

FIGURE 1. The graph of exact and numerical solutions for various vales of $\alpha = 2, 1.9, 1.8$.

$S\{D_t^\alpha y(t)\} = S\left\{\frac{1}{2}e^{\frac{t}{2}}y\left(\frac{t}{2}\right) + \frac{1}{2}y(t)\right\}$

By using definition (2.4) and initial condition (5.12), we get

$\frac{Y(u)-y(0)}{u^\alpha} = S\left\{\frac{1}{2}e^{\frac{t}{2}}y\left(\frac{t}{2}\right) + \frac{1}{2}y(t)\right\}$

$$Y(u) = 1 + u^\alpha S\left\{\frac{1}{2}e^{\frac{t}{2}}y\left(\frac{t}{2}\right) + \frac{1}{2}y(t)\right\} \tag{5.13}$$

Further, we apply inverse Sumudu transform to (5.13) to get

$y(t) = S^{-1}\left[1 + u^\alpha S\left\{\frac{1}{2}e^{\frac{t}{2}}y\left(\frac{t}{2}\right) + \frac{1}{2}y(t)\right\}\right]$

$y_0(t) = S^{-1}\{1\} = 1$

$y_0\left(\frac{t}{2}\right) = 1$

$$y_{n+1}(t) = S^{-1}[u^\alpha S\{A_n\}] \tag{5.14}$$

The linear term of FPDE is $L(y) = \frac{1}{2}e^{\frac{t}{2}}y\left(\frac{t}{2}\right) + \frac{1}{2}y(t)$.
From equation (3.5) to (3.7), we have

$$A_0 = \frac{1}{2}e^{\frac{t}{2}}y_0\left(\frac{t}{2}\right) + \frac{1}{2}y_0(t), \tag{5.15}$$

$$A_1 = \frac{1}{2}e^{\frac{t}{2}}y_1\left(\frac{t}{2}\right) + \frac{1}{2}y_1(t), \tag{5.16}$$

$$A_2 = \frac{1}{2}e^{\frac{t}{2}}y_2\left(\frac{t}{2}\right) + \frac{1}{2}y_2(t), \tag{5.17}$$

and so on.
We put n=0 in equation (5.14)

$$y_1(t) = S^{-1}[u^\alpha S\{A_0\}] \tag{5.18}$$

Substituting equation (5.15) in (5.18), we get

$$y_1(t) = S^{-1}\left[u^\alpha S\left\{\frac{1}{2}e^{\frac{t}{2}}y_0\left(\frac{t}{2}\right) + \frac{1}{2}y_0(t)\right\}\right]$$
$$= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{4\Gamma(\alpha+2)} + \frac{t^{\alpha+2}}{8\Gamma(\alpha+3)}$$
$$y_1\left(\frac{t}{2}\right) = \frac{t^\alpha}{2^\alpha\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{4\,2^{\alpha+1}\Gamma(\alpha+2)} + \frac{t^{\alpha+2}}{8\,2^{\alpha+2}\Gamma(\alpha+3)}$$

We put n=1 in equation (5.14)

$$y_2(t) = S^{-1}[u^\alpha S\{A_1\}] \tag{5.19}$$

Substituting equation (5.16) in (5.19), we get

$$y_2(t) = S^{-1}\left[u^\alpha S\left\{\frac{1}{2}e^{\frac{t}{2}}y_1\left(\frac{t}{2}\right) + \frac{1}{2}y_1(t)\right\}\right]$$

$$= \frac{1}{2}\Big[\frac{t^{2\alpha}}{2^\alpha\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{4\,2^{\alpha+1}\Gamma(2\alpha+2)} + \frac{t^{2\alpha+2}}{8\,2^{\alpha+2}\Gamma(2\alpha+3)} + \frac{t^{2\alpha+1}\Gamma(\alpha+2)}{2^{\alpha+1}\Gamma(\alpha+1)\Gamma(2\alpha+2)}$$
$$+ \frac{t^{2\alpha+2}\Gamma(\alpha+3)}{4\,2^{\alpha+2}\Gamma(\alpha+2)\Gamma(2\alpha+3)} + \frac{t^{2\alpha+3}\Gamma(\alpha+4)}{8\,2^{\alpha+3}\Gamma(\alpha+3)\Gamma(2\alpha+4)} + \frac{t^{2\alpha+2}\Gamma(\alpha+3)}{8\,2^{\alpha}\Gamma(\alpha+1)\Gamma(2\alpha+3)}$$
$$+ \frac{t^{2\alpha+3}\Gamma(\alpha+4)}{32\,2^{\alpha+1}\Gamma(\alpha+2)\Gamma(2\alpha+4)} + \frac{t^{2\alpha+4}\Gamma(\alpha+5)}{64\,2^{\alpha+2}\Gamma(\alpha+3)\Gamma(2\alpha+5)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{4\Gamma(2\alpha+2)} + \frac{t^{2\alpha+2}}{8\Gamma(2\alpha+3)}\Big]$$
$$y_2\left(\frac{t}{2}\right) = \frac{1}{2}\Big[\frac{t^{2\alpha}}{2^{3\alpha}\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{4\,2^{3\alpha+2}\Gamma(2\alpha+2)} + \frac{t^{2\alpha+2}}{8\,2^{3\alpha+4}\Gamma(2\alpha+3)} + \frac{t^{2\alpha+1}\Gamma(\alpha+2)}{2^{3\alpha+2}\Gamma(\alpha+1)\Gamma(2\alpha+2)}$$
$$+ \frac{t^{2\alpha+2}\Gamma(\alpha+3)}{4\,2^{3\alpha+4}\Gamma(\alpha+2)\Gamma(2\alpha+3)} + \frac{t^{2\alpha+3}\Gamma(\alpha+4)}{8\,2^{3\alpha+6}\Gamma(\alpha+3)\Gamma(2\alpha+4)} + \frac{t^{2\alpha+2}\Gamma(\alpha+3)}{8\,2^{3\alpha+2}\Gamma(\alpha+1)\Gamma(2\alpha+3)}$$
$$+ \frac{t^{2\alpha+3}\Gamma(\alpha+4)}{32\,2^{3\alpha+4}\Gamma(\alpha+2)\Gamma(2\alpha+4)} + \frac{t^{2\alpha+4}\Gamma(\alpha+5)}{64\,2^{3\alpha+6}\Gamma(\alpha+3)\Gamma(2\alpha+5)} + \frac{t^{2\alpha}}{2^{2\alpha}\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{4\,2^{2\alpha+1}\Gamma(2\alpha+2)}$$
$$+ \frac{t^{2\alpha+2}}{8\,2^{2\alpha+2}\Gamma(2\alpha+3)}\Big]$$

We put n=2 in equation (5.14)

$$y_3(t) = S^{-1}[u^\alpha S\{A_2\}] \tag{5.20}$$

Substituting equation (5.17) in (5.20), we get

$$y_3(t) = S^{-1}\left[u^\alpha S\left\{\frac{1}{2}e^{\frac{t}{2}}y_2\left(\frac{t}{2}\right) + \frac{1}{2}y_2(t)\right\}\right]$$

$$= \frac{1}{4}\Big[\frac{t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{4\,2^{3\alpha+2}\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}}{8\,2^{3\alpha+4}\Gamma(3\alpha+3)} + \frac{t^{3\alpha+1}\Gamma(\alpha+2)}{2^{3\alpha+2}\Gamma(\alpha+1)\Gamma(3\alpha+2)}$$
$$+ \frac{t^{3\alpha+2}\Gamma(\alpha+3)}{4\,2^{3\alpha+4}\Gamma(\alpha+2)\Gamma(3\alpha+3)} + \frac{t^{3\alpha+3}\Gamma(\alpha+4)}{8\,2^{3\alpha+6}\Gamma(\alpha+3)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+2}\Gamma(\alpha+3)}{8\,2^{3\alpha+2}\Gamma(\alpha+1)\Gamma(3\alpha+3)}$$

$$
\begin{aligned}
&+ \frac{t^{3\alpha+3}\Gamma(\alpha+4)}{32\,2^{3\alpha+4}\Gamma(\alpha+2)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+4}\Gamma(\alpha+5)}{64\,2^{3\alpha+6}\Gamma(\alpha+3)\Gamma(3\alpha+5)} + \frac{t^{3\alpha}}{2^{2\alpha}\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{4\,2^{2\alpha+1}\Gamma(3\alpha+2)}\\
&+ \frac{t^{3\alpha+2}}{8\,2^{2\alpha+2}\Gamma(3\alpha+3)} + \frac{t^{3\alpha+1}\Gamma(2\alpha+2)}{2^{3\alpha+1}\Gamma(2\alpha+1)\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}\Gamma(2\alpha+3)}{4\,2^{3\alpha+3}\Gamma(2\alpha+2)\Gamma(3\alpha+3)}\\
&+ \frac{t^{3\alpha+3}\Gamma(2\alpha+4)}{8\,2^{3\alpha+5}\Gamma(2\alpha+3)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+2}\Gamma(\alpha+2)\Gamma(2\alpha+3)}{2^{3\alpha+3}\Gamma(\alpha+1)\Gamma(2\alpha+2)\Gamma(3\alpha+3)} + \frac{t^{3\alpha+3}\Gamma(\alpha+3)\Gamma(2\alpha+4)}{4\,2^{3\alpha+5}\Gamma(\alpha+2)\Gamma(2\alpha+3)\Gamma(3\alpha+4)}\\
&+ \frac{t^{3\alpha+4}\Gamma(\alpha+4)\Gamma(2\alpha+5)}{8\,2^{3\alpha+7}\Gamma(\alpha+3)\Gamma(2\alpha+4)\Gamma(3\alpha+5)} + \frac{t^{3\alpha+3}\Gamma(\alpha+3)\Gamma(2\alpha+4)}{8\,2^{3\alpha+3}\Gamma(\alpha+1)\Gamma(2\alpha+3)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+4}\Gamma(\alpha+4)\Gamma(2\alpha+5)}{32\,2^{3\alpha+5}\Gamma(\alpha+2)\Gamma(2\alpha+4)\Gamma(3\alpha+5)}\\
&+ \frac{t^{3\alpha+5}\Gamma(\alpha+5)\Gamma(2\alpha+6)}{64\,2^{3\alpha+7}\Gamma(\alpha+3)\Gamma(2\alpha+5)\Gamma(3\alpha+6)} + \frac{t^{3\alpha+1}\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(2\alpha+1)\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}\Gamma(2\alpha+3)}{4\,2^{2\alpha+2}\Gamma(2\alpha+2)\Gamma(3\alpha+3)}\\
&+ \frac{t^{3\alpha+3}\Gamma(2\alpha+4)}{8\,2^{2\alpha+3}\Gamma(2\alpha+3)\Gamma(3\alpha+4)} + \frac{t^{3\alpha}}{2^{\alpha}\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{4\,2^{\alpha+1}\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}}{8\,2^{\alpha+2}\Gamma(3\alpha+3)}\\
&+ \frac{t^{3\alpha+1}\Gamma(\alpha+2)}{2^{\alpha+1}\Gamma(\alpha+1)\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}\Gamma(\alpha+3)}{4\,2^{\alpha+2}\Gamma(\alpha+2)\Gamma(3\alpha+3)} + \frac{t^{3\alpha+3}\Gamma(\alpha+4)}{8\,2^{\alpha+3}\Gamma(\alpha+3)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+2}\Gamma(\alpha+3)}{8\,2^{\alpha}\Gamma(\alpha+1)\Gamma(3\alpha+3)}\\
&+ \frac{t^{3\alpha+3}\Gamma(\alpha+4)}{32\,2^{\alpha+1}\Gamma(\alpha+2)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+4}\Gamma(\alpha+5)}{64\,2^{2\alpha+2}\Gamma(\alpha+3)\Gamma(3\alpha+5)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{4\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}}{8\Gamma(3\alpha+3)}\Big]
\end{aligned}
$$

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The series solution is given by

$$
\begin{aligned}
y(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) + \ldots\\
&= 1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{4\Gamma(\alpha+2)} + \frac{t^{\alpha+2}}{8\Gamma(\alpha+3)} + \frac{1}{2}\Big[\frac{t^{2\alpha}}{2^{\alpha}\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{4\,2^{\alpha+1}\Gamma(2\alpha+2)} + \frac{t^{2\alpha+2}}{8\,2^{\alpha+2}\Gamma(2\alpha+3)}\\
&+ \frac{t^{2\alpha+1}\Gamma(\alpha+2)}{2^{\alpha+1}\Gamma(\alpha+1)\Gamma(2\alpha+2)} + \frac{t^{2\alpha+2}\Gamma(\alpha+3)}{4\,2^{\alpha+2}\Gamma(\alpha+2)\Gamma(2\alpha+3)} + \frac{t^{2\alpha+3}\Gamma(\alpha+4)}{8\,2^{\alpha+3}\Gamma(\alpha+3)\Gamma(2\alpha+4)} + \frac{t^{2\alpha+2}\Gamma(\alpha+3)}{8\,2^{\alpha}\Gamma(\alpha+1)\Gamma(2\alpha+3)}\\
&+ \frac{t^{2\alpha+3}\Gamma(\alpha+4)}{32\,2^{\alpha+1}\Gamma(\alpha+2)\Gamma(2\alpha+4)} + \frac{t^{2\alpha+4}\Gamma(\alpha+5)}{64\,2^{\alpha+2}\Gamma(\alpha+3)\Gamma(2\alpha+5)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{4\Gamma(2\alpha+2)} + \frac{t^{2\alpha+2}}{8\Gamma(2\alpha+3)}\Big]\\
&+ \frac{1}{4}\Big[\frac{t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{4\,2^{3\alpha+2}\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}}{8\,2^{3\alpha+4}\Gamma(3\alpha+3)} + \frac{t^{3\alpha+1}\Gamma(\alpha+2)}{2^{3\alpha+2}\Gamma(\alpha+1)\Gamma(3\alpha+2)}\\
&+ \frac{t^{3\alpha+2}\Gamma(\alpha+3)}{4\,2^{3\alpha+4}\Gamma(\alpha+2)\Gamma(3\alpha+3)} + \frac{t^{3\alpha+3}\Gamma(\alpha+4)}{8\,2^{3\alpha+6}\Gamma(\alpha+3)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+2}\Gamma(\alpha+3)}{8\,2^{3\alpha+2}\Gamma(\alpha+1)\Gamma(3\alpha+3)}\\
&+ \frac{t^{3\alpha+3}\Gamma(\alpha+4)}{32\,2^{3\alpha+4}\Gamma(\alpha+2)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+4}\Gamma(\alpha+5)}{64\,2^{3\alpha+6}\Gamma(\alpha+3)\Gamma(3\alpha+5)} + \frac{t^{3\alpha}}{2^{2\alpha}\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{4\,2^{2\alpha+1}\Gamma(3\alpha+2)}\\
&+ \frac{t^{3\alpha+2}}{8\,2^{2\alpha+2}\Gamma(3\alpha+3)} + \frac{t^{3\alpha+1}\Gamma(2\alpha+2)}{2^{3\alpha+1}\Gamma(2\alpha+1)\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}\Gamma(2\alpha+3)}{4\,2^{3\alpha+3}\Gamma(2\alpha+2)\Gamma(3\alpha+3)}\\
&+ \frac{t^{3\alpha+3}\Gamma(2\alpha+4)}{8\,2^{3\alpha+5}\Gamma(2\alpha+3)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+2}\Gamma(\alpha+2)\Gamma(2\alpha+3)}{2^{3\alpha+3}\Gamma(\alpha+1)\Gamma(2\alpha+2)\Gamma(3\alpha+3)} + \frac{t^{3\alpha+3}\Gamma(\alpha+3)\Gamma(2\alpha+4)}{4\,2^{3\alpha+5}\Gamma(\alpha+2)\Gamma(2\alpha+3)\Gamma(3\alpha+4)}\\
&+ \frac{t^{3\alpha+4}\Gamma(\alpha+4)\Gamma(2\alpha+5)}{8\,2^{3\alpha+7}\Gamma(\alpha+3)\Gamma(2\alpha+4)\Gamma(3\alpha+5)} + \frac{t^{3\alpha+3}\Gamma(\alpha+3)\Gamma(2\alpha+4)}{8\,2^{3\alpha+3}\Gamma(\alpha+1)\Gamma(2\alpha+3)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+4}\Gamma(\alpha+4)\Gamma(2\alpha+5)}{32\,2^{3\alpha+5}\Gamma(\alpha+2)\Gamma(2\alpha+4)\Gamma(3\alpha+5)}\\
&+ \frac{t^{3\alpha+5}\Gamma(\alpha+5)\Gamma(2\alpha+6)}{64\,2^{3\alpha+7}\Gamma(\alpha+3)\Gamma(2\alpha+5)\Gamma(3\alpha+6)} + \frac{t^{3\alpha+1}\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(2\alpha+1)\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}\Gamma(2\alpha+3)}{4\,2^{2\alpha+2}\Gamma(2\alpha+2)\Gamma(3\alpha+3)}\\
&+ \frac{t^{3\alpha+3}\Gamma(2\alpha+4)}{8\,2^{2\alpha+3}\Gamma(2\alpha+3)\Gamma(3\alpha+4)} + \frac{t^{3\alpha}}{2^{\alpha}\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{4\,2^{\alpha+1}\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}}{8\,2^{\alpha+2}\Gamma(3\alpha+3)}\\
&+ \frac{t^{3\alpha+1}\Gamma(\alpha+2)}{2^{\alpha+1}\Gamma(\alpha+1)\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}\Gamma(\alpha+3)}{4\,2^{\alpha+2}\Gamma(\alpha+2)\Gamma(3\alpha+3)} + \frac{t^{3\alpha+3}\Gamma(\alpha+4)}{8\,2^{\alpha+3}\Gamma(\alpha+3)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+2}\Gamma(\alpha+3)}{8\,2^{\alpha}\Gamma(\alpha+1)\Gamma(3\alpha+3)}\\
&+ \frac{t^{3\alpha+3}\Gamma(\alpha+4)}{32\,2^{\alpha+1}\Gamma(\alpha+2)\Gamma(3\alpha+4)} + \frac{t^{3\alpha+4}\Gamma(\alpha+5)}{64\,2^{2\alpha+2}\Gamma(\alpha+3)\Gamma(3\alpha+5)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{4\Gamma(3\alpha+2)} + \frac{t^{3\alpha+2}}{8\Gamma(3\alpha+3)}\Big] + \ldots
\end{aligned}
$$

In particular case $\alpha = 1$, we get approximate series solution

$y(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

The exact solution when $\alpha = 1$ is given by $y(t) = e^t$.

The convergence analysis of Adomian decomposition method shows the rapid convergence of the obtained infinite series [2, 7]. Table 2 represents the absolute errors by the proposed method. The absolute errors of SDM are compared with the existing Hermite wavelet method. The absolute errors between numerical and exact solutions have been reduced by choosing $\alpha = 1$. It is clear that the proposed method provides more accurate numerical solutions as compared to the existing numerical results.

TABLE 2. Comparison of absolute errors of SDM with Fractional-order Hermite wavelet method for $\alpha = 1$ in the interval [0, 1].

t	SDM	Hermite wavelet method [16]	Absolute Errors		Exact Solution
			SDM	Hermite wavelet method [16] for M=17	
0.2	1.22133	1.22140	6.94×10^{-5}	3.41×10^{-25}	1.22140
0.4	1.49067	1.49182	1.16×10^{-3}	2.79×10^{-24}	1.49182
0.6	1.81600	1.82212	6.12×10^{-3}	3.94×10^{-24}	1.82212
0.8	2.20533	2.22554	2.02×10^{-2}	2.35×10^{-25}	2.22554
1	2.66667	2.71828	5.16×10^{-2}	1.58×10^{-22}	2.71828

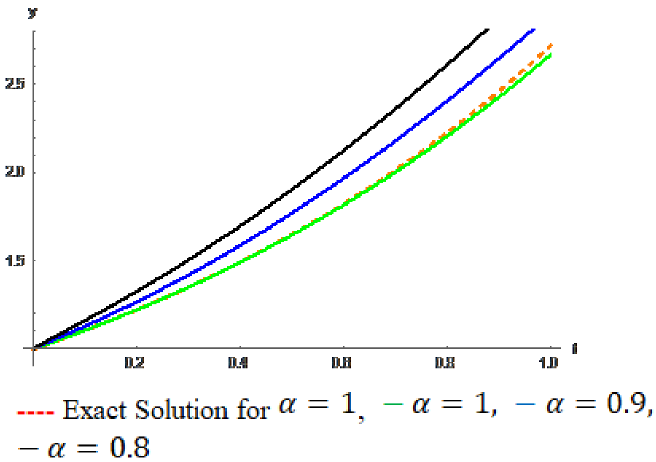

FIGURE 2. The graph of exact and numerical solutions for various vales of $\alpha = 1, 0.9, 0.8$.

In 2D figure, we successfully compared approximate and exact solutions of linear fractional pantograph differential equation (5.11) to (5.12). The comparison shows that the obtained numerical solution is identical with the exact solution.

Example 5.3 We consider the following nonlinear fractional pantograph differential equation [18]

$$D_t^{\alpha} y(t) = 1 - 2y^2\left(\frac{t}{2}\right), 0 \leq t \leq 1, 1 < \alpha \leq 2, \tag{5.21}$$

subject to initial condition

$$y(0) = 1, y^{'}(0) = 0, \tag{5.22}$$

The exact solution of equation is given by $y(t) = cost$ when $\alpha = 2$.
We take Sumudu transform to both side of equation (5.21)
$S\{D_t^{\alpha} y(t)\} = S\left\{1 - 2y^2\left(\frac{t}{2}\right)\right\}$

By using definition (2.4) and initial condition (5.22), we have
$\frac{Y(u)}{u^{\alpha}} - \frac{y(0)}{u^{\alpha-0}} - \frac{y'(0)}{u^{\alpha-1}} = 1 - 2S\left\{y^2\left(\frac{t}{2}\right)\right\}$

$$Y(u) = 1 + u^{\alpha} - 2u^{\alpha}S\left\{y^2\left(\frac{t}{2}\right)\right\} \tag{5.23}$$

Further, we apply inverse Sumudu transform to (5.23) to get
$y(t) = S^{-1}\left[1 + u^{\alpha} - 2u^{\alpha}S\left\{y^2\left(\frac{t}{2}\right)\right\}\right]$
$y_0(t) = S^{-1}\{1 + u^{\alpha}\} = 1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)}$
$y_0\left(\frac{t}{2}\right) = 1 + \frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}$

$$y_{n+1}(t) = -2S^{-1}[u^{\alpha}S\{A_n\}] \tag{5.24}$$

The non-linear term of FPDE is $N(y) = y^2\left(\frac{t}{2}\right)$.
From equation (3.5) to (3.7), we have

$$A_0 = y_0^2\left(\frac{t}{2}\right) \tag{5.25}$$

$$A_1 = 2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right) \tag{5.26}$$

$$A_2 = 2y_0\left(\frac{t}{2}\right)y_2\left(\frac{t}{2}\right) + y_1^2\left(\frac{t}{2}\right) \tag{5.27}$$

and so on.
We put n=0 in equation (5.24)

$$y_1(t) = -2S^{-1}[u^{\alpha}S\{A_0\}] \tag{5.28}$$

Substituting equation (5.25) in (5.28), we get
$y_1(t) = -2S^{-1}\left[u^{\alpha}S\left\{y_0^2\left(\frac{t}{2}\right)\right\}\right]$
$\quad = -2[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha}}{2^{\alpha}\Gamma(2\alpha+1)} + \frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha}(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}]$
$y_1\left(\frac{t}{2}\right) = -2[\frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} + \frac{2t^{2\alpha}}{2^{3\alpha}\Gamma(2\alpha+1)} + \frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{5\alpha}(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}]$
We put n=1 in equation (5.24)

$$y_2(t) = -2S^{-1}[u^{\alpha}S\{A_1\}] \tag{5.29}$$

Substituting equation (5.26) in (5.29), we get
$y_2(t) = -2S^{-1}\left[u^{\alpha}S\left\{2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)\right\}\right]$
$\quad = 8[\frac{t^{2\alpha}}{2^{\alpha}\Gamma(2\alpha+1)} + \frac{2t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} + \frac{t^{4\alpha}\Gamma(2\alpha+1)}{2^{5\alpha}(\Gamma(\alpha+1))^2\Gamma(4\alpha+1)} + \frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha}(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}$
$\quad + \frac{2t^{4\alpha}\Gamma(3\alpha+1)}{2^{4\alpha}\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)} + \frac{t^{5\alpha}\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{2^{6\alpha}(\Gamma(\alpha+1))^3\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}]$

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The series solution is given by

$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \dots$

$= 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} - 2[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha}}{2^\alpha\Gamma(2\alpha+1)} + \frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha}(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}] + 8[\frac{t^{2\alpha}}{2^\alpha\Gamma(2\alpha+1)}$

$+ \frac{2t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} + \frac{t^{4\alpha}\Gamma(2\alpha+1)}{2^{5\alpha}(\Gamma(\alpha+1))^2\Gamma(4\alpha+1)} + \frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha}(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} + \frac{2t^{4\alpha}\Gamma(3\alpha+1)}{2^{4\alpha}\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}$

$+ \frac{t^{5\alpha}\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{2^{6\alpha}(\Gamma(\alpha+1))^3\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}] + \dots$

In particular case $\alpha = 2$, we get approximate series solution

$y(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots$

The exact solution when $\alpha = 2$ is given by $y(t) = cost$.

The convergence analysis of Adomian decomposition method shows the rapid convergence of the obtained infinite series [2, 7]. We compared the absolute errors of present method with the existing Fractional-order Boubaker polynomials method [15] in Table 3. The absolute errors are reduced by using the proposed method. It means that the combination of Sumudu transform and Adomian decomposition investigate more accurate solutions as compared to the existing numerical results.

TABLE 3. Comparison of absolute errors of SDM with Fractional-order Boubaker polynomials method for $\alpha = 2$ in the interval [0, 1].

t	SDM	Absolute Errors		Exact solution
		SDM	Fractional-order Boubaker polynomials method[15]	
0.2	0.98007	8.88×10^{-8}	2.37×10^{-4}	0.98007
0.4	0.92107	5.67×10^{-6}	1.02×10^{-4}	0.92106
0.6	0.82540	6.44×10^{-5}	1.18×10^{-4}	0.82534
0.8	0.69707	3.60×10^{-4}	1.68×10^{-4}	0.69671
1	0.54167	1.36×10^{-3}	6.94×10^{-4}	0.54030

In 2D figure, we successfully compared approximate and exact solutions of nonlinear fractional pantograph differential equation (5.21) to (5.22). The comparison shows that the obtained numerical solution is identical with the exact solution.

6. Conclusions

The present numerical technique-Sumudu decomposition method has established for the set of an approximate analytical solutions of fractional pantograph differential equations. This method is easy to apply for linear and non-linear types of fractional differential equations. The obtained series solutions were efficiently compared with the exact solution in numerical and graphical ways. The comparison accomplish that the exact and approximate solutions are identical. The absolute errors of approximate and exact solutions of SDM have been successfully compared with fractional-order Boubaker polynomials method and Hermite wavelet method. We conclude that the absolute errors of SDM are less in numbers as compared to the fractional-order Boubaker polynomials method and Hermite wavelet method. From our derived results, it shows that the method is highly efficient to solve the test problems numerically. In

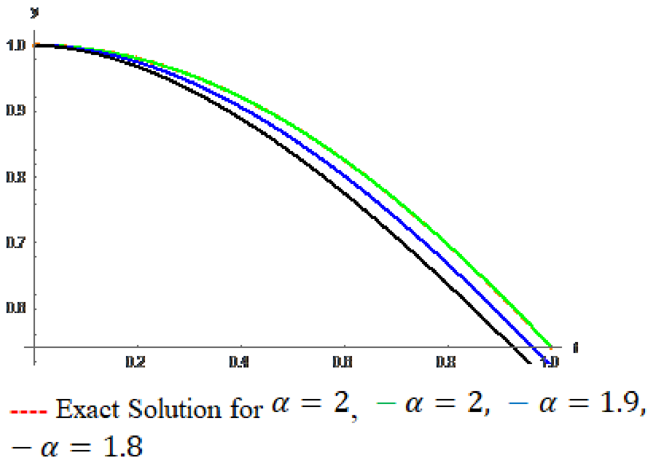

FIGURE 3. The graph of exact and numerical solutions for various vales of $\alpha = 2, 1.9, 1.8$.

addition, we have achieved acceptable approximate results for the observed problems over the interval [0, 1] only with a few numbers of iterations.

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