GANITA, Vol.74(1), 2024, 73-81

SOME COEFFICIENT INEQUALITIES FOR Q-STARLIKE AND Q-CONVEX FUNCTIONS ASSOCIATED WITH NORMALIZED BESSEL-STRUVE KERNEL FUNCTION

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Abstract

The main objective of this paper is to determine the sufficient condition for the classes of q-starlike and q-convex function associated with the Normalized Bessel - Struve Kernel Function. The Bessel Struve Kernel function defined in the unit disc is used in this study.

2010 *Mathematics subject classification:* primary 30C45; secondary 30C50, 30C80.
Keywords and phrases: Univalent function, q-derivative operator, q-starlike function, q-convex function, Bessel function, Bessel-Struve function..

1. Introduction

This paper is focused on certain coefficient inequalities for the classes of q-starlike and q- convex function of Bessel Struve kernel function.

1.1. Univalent function and q-calculus: A function $\mathcal{F}$ is known to be a Univalent function in domain $\mathfrak{D}$ if it takes different values at different points. Let z_1 and z_2 are two distinct points then $\mathcal{F}(z_1) \neq \mathcal{F}(z_2)$.

A necessary condition for a function $\mathcal{F}$ to be analytic and univalent in $\mathfrak{D}$ where $\mathfrak{D}$ is a open unit disc $\mathfrak{D} = \{z \mid , |z| < 1\}$ [16] if

$$\mathcal{F}'(z) \neq 0 \ , \qquad \forall \ \ z \in \mathfrak{D}$$

Thus, if $\mathcal{F}$ is analytic and univalent in $\mathfrak{D}$ then without loss of generality we can assume that

$$\mathcal{F}(z) = z + \sum_{n=2}^{8} a_n z^n \tag{1.1}$$

and the normalization condition is[16]

$$\mathcal{F}(0) = \mathcal{F}'(0) - 1 = 0.$$

Recently, q-calculus is studied by many researchers due to its vast applications in various areas of Mathematics, Physics and many fields of Engineering. q- analysis was firstly studied by Jackson [1-2]. Many research scholars have written a number of papers on q-starlike and Janowski function [4,5,9,11]. Recently Murli Manohar Gaur introduced sufficient condition for starlike and q-convex functions associated with normalized Gauss Hypergeometric Function [6] and Bessel function [7]. Rehman et al. [12] also discussed about some subclasses of q – starlike functions including numerous coefficient inequalities and a sufficient condition. Later, Srivastava et al. [11, 13-15] wrote a number of papers on q-starlike and q-convex function.

As we know that q-calculus is q^{th} order differential and Integral: A q-differentiation of a function $\mathcal{F}$ is defined to be [3]

$$d_q\mathcal{F} = \mathcal{F}(qz) - \mathcal{F}(z).$$

Now assume that T is the class of all analytic function $\mathcal{F}$ in the open unit disc $\mathfrak{D}$ such that

$$\mathfrak{D} = \{z \mid, |z| < 1\}$$

and

$$\mathcal{F}(z) = z + \sum_{n=2}^{8} a_n z^n, \qquad (z \in \mathfrak{D}, \mathcal{F}(0) = \mathcal{F}'(0) - 1 = 0)$$

Let any class of univalent functions is S and its subclasses are defined as T and S* then it is subclass of univalent functions as well as starlike function, Suppose $\mathcal{F} \in S^*$and satisfy the conditions [7]:

$$\mathcal{R}\left\{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right\} > 0 \qquad \forall \quad z \in \mathfrak{D}$$

and if

$$\mathcal{R}\left\{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right\} > \alpha \quad \forall \quad z,$$

then

$$\mathcal{F} \in S^*(\alpha)$$

and if $\mathcal{F}$ belong to the classes S_q^* iff

$$\left|\frac{z}{\mathcal{F}(z)} D_q\mathcal{F}(z) - \frac{1}{1-q}\right| < \frac{1}{1-q}, \ z \in \mathfrak{D}$$

Again let C^* is the class of convex function and subclass of S such that any function $z \in C^*$ then it satisfy

$$\mathcal{R}\left\{\frac{(z\mathcal{F}'(z))')}{\mathcal{F}(z)}\right\} = \mathcal{R}\left\{1 + \frac{(z\mathcal{F}''(z))}{\mathcal{F}'(z)}\right\} > 0 \qquad \forall \quad z \in \mathfrak{D}$$

Definition 1.1 [13] For $q \in (0, 1)$ the number $[\gamma]_q$ defined by

$$[\gamma]_q = \begin{cases} \frac{1-q^\gamma}{1-q}, & \gamma \in \mathbb{C} \\ \sum_{n=0}^{\gamma-1} q^\gamma, & if \quad \gamma \in \mathbb{N} = \{1, 2, 3, \ldots\} \end{cases} \tag{1.2}$$

Definition 1.2 [13] The q-derivative of a function $\mathcal{F}$ will be

$$D_q\mathcal{F}(z) = \begin{cases} \frac{\mathcal{F}(z)-\mathcal{F}(qz)}{(1-q)}, & if \;\; z \in \mathbb{C} /\{0\} \\ \mathcal{F}'(0), & if \;\; z \in 0 \end{cases} \tag{1.3}$$

If $q \to 1$ then above definition become

$$\lim_{q\to 1^-} D_q\mathcal{F}(z) = \lim_{q\to 1^-} \frac{\mathcal{F}(z) - \mathcal{F}(qz)}{(1-q)} = \mathcal{F}'(z) \tag{1.4}$$

Srivastava and Bansal [14] defined q-derivative as

$$D_q\mathcal{F}(z) = 1 + \sum_{n=2}^{8} [n]_q a_n z^{n-1} \quad \forall \; z \in \mathbb{U}, \quad z \in \mathbb{T} \tag{1.5}$$

Definition 1.3 Let f, an analytic function if $f(0) = 1$ fits to the class $\mathcal{H}[A, B]$ with the condition $-1 = B < A = 1$ if and only if

$$f(z) < \frac{1 + Az}{1 + Bz}$$

According to Janowski [10], each function $h \in \mathcal{H}$ if and only if $f \in \mathcal{H}[A, B]$, such that

$$f(z) < \frac{(A + 1)\, h(z) - (A - 1)}{(B + 1)\, h(z) - (B - 1)}, \quad z \in \mathfrak{D} \tag{1.6}$$

Definition 1.4 [10] For starlike function $S^*[A, B]$ and function $\mathcal{F} \in M$, be in this class with $-1 = B < A = 1$ if and only if

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} < \frac{1 + Az}{1 + Bz} \tag{1.7}$$

Let $\mathcal{F}$ belongs to the classes $S^*_q[A, B]$ iff

$$\mathcal{F}(z) < \frac{1 + z}{1 - qz}$$

and

$$\frac{z}{\mathcal{F}(z)} D_q\mathcal{F}(z) = \frac{(A + 1)\mathcal{F}(z) - (A - 1)}{(B + 1)\mathcal{F}(z) - (B - 1)}$$

then

$$\frac{z}{\mathcal{F}(z)} D_q\mathcal{F}(z) < \Phi(z)$$

where

$$\Phi(z) = \frac{(A+1)z + 2 + (A-1)qz}{(B+1)z + 2 + (B-1)qz} \tag{1.8}$$

[A and B satisfying the usual inequality]
*Similarly if function $\mathcal{F} \in M$ be in the class of $C^*[A, B]$ with $-1 = B < A = 1$ if and only if

$$1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} < \frac{1+Az}{1+Bz}$$

Remarks:

1. If $q \to 1$ then from (1.4) and (1.5)

$$D_q\mathcal{F}(z) \to \mathcal{F}'(z)$$

2. If $q \to 1$ then from (1.7) and (1.8)

$$S_q^*[A, B] \to S^*[A, B]$$

Bessel Struve Kernel Function-
Bessel function is the solution of the differential equation

$$z^2\mathcal{F}''(z) + bz\mathcal{F}'(z)\left(cz^2 - v^2 + (1-b)v\right)\mathcal{F}(z) = 0$$

when right hand side of above function is not zero then equation become

$$z^2\mathcal{F}''(z) + bz\mathcal{F}'(z)\left(cz^2 - v^2 + (1-b)v\right)\mathcal{F}(z) = \frac{4\left(\frac{z}{2}\right)^{v+1}}{\sqrt{\pi}\ \Gamma(v+1)}$$

Then it becomes Bessel - Struve function and its solution in power series can be written as

$$B_{v,b,c}(z) = \sum_{n=0}^{8} \frac{c^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\ \Gamma(n+1)\ \Gamma\left(\frac{n}{2}+k\right)} z^n$$

where $k = v + (b+1)/2$ and $v > -1/2$.
By using Gamma property and replacing $\frac{n}{2} \to n$

$$B_{k,c}(z) = \sum_{n=0}^{8} \frac{\left(\frac{c}{4}\right)^n}{\Gamma(n+1)\Gamma(n+k)} z^{2n}$$

Now generalisation of Bessel-Struve function

$$\psi(z) = \Gamma k B_{k,c}(z) = \sum_{n=0}^{8} \frac{\left(\frac{c}{4}\right)^n}{(k)_n n!} z^n$$

Normalisation of Bessel-Struve function

$$z\psi(z) = z\sum_{n=0}^{8} \frac{\left(\frac{c}{4}\right)^n}{(k)_n n!} z^n$$

$$\Phi(z) = z\psi(z) = z + \sum_{n=2}^{8} \frac{\left(\frac{c}{4}\right)^{n-1}}{(k)_{n-1}(n-1)!} z^n = z + \sum_{n=2}^{8} a_n z^n$$

In this paper we discuss and define some sufficient conditions of q - starlikeness and q-convexity for Bessel - Struve function by using sufficient conditions[13]. Some similar work is also done by Gour et al. [6,7], Seoudy in [12] and Goyal et al. found the sufficient conditions of starlikeness for the multivalent function[8].

Lemma 1.1 [13] Suppose if $F \in S_q^*[A, B]$ then it is achieving below condition

$$\sum_{n=2}^{8} \left(2q[n-1]_q + \left|(B+1)[n]_q - (A+1)\right|\right)|a_n| < |B-A| \tag{1.9}$$

Lemma 1.2 [13] Suppose if $F \in C_q^*[A, B]$ then it is achieving below condition

$$\sum_{n=2}^{8} [n]_q\left(2q[n-1]_q + \left|(B+1)[n]_q - (A+1)\right|\right)|a_n| < |B-A| \tag{1.10}$$

2. Main Results

Theorem 2.1 Let $P(k, q)$ be defined as follows

$$P(k,q) = \left(\frac{2q + (B+1)}{1-q} + (A+1)\right) {}_0F_1\left(-;;k;;\ \frac{c}{4}\right) - \frac{(B+3)q}{1-q} {}_0F_1\left(-;;k;;\ \frac{cq}{4}\right) + (A+B+2) \tag{2.1}$$

If the inequality $P(k, q) < |B-A|$ holds, then function $z\psi(z) \in S_q^*[A, B]$.

Proof: Here

$$z\psi(z) = z + \sum_{n=2}^{8} \frac{\left(\frac{c}{4}\right)^{n-1}}{(k)_{n-1} n-1!} z^n = z + \sum_{n=2}^{8} a_n z^n, \quad z \in \mathfrak{D}$$

From Lemma 1.1, if any function $F \in S_q^*[A, B]$ then F satisfy the above condition. So for $\Phi(z) = z\psi(z)$, it is sufficient to show that function satisfy the Lemma where

$$a_n = \frac{\left(\frac{c}{4}\right)^{n-1}}{(k)_{n-1} n-1!} \qquad \& \qquad [n]_q = \frac{1-q^n}{1-q}$$

Now, using triangle's inequality we get

$$\sum_{n=2}^{8} \left(2q[n-1]_q + \left|(B+1)[n]_q - (A+1)\right|\right)|a_n| = \sum_{n=2}^{8} \left(2q\frac{1-q^{n-1}}{1-q}\right)|a_n| +$$

$$+ \sum_{n=2}^{8} (B+1)\frac{1-q^n}{1-q}|a_n| + \sum_{n=2}^{8} (A+1)|a_n|$$

$$= \sum_{n=2}^{8}\left(\frac{2q+(B+1)}{1-q} + (A+1)\right)|a_n| - \sum_{n=2}^{8}\frac{(B+3)q^n}{1-q}|a_n|$$

$$= \left(\frac{2q+(B+1)}{1-q} + (A+1)\right)\sum_{n=1}^{8}\frac{\left(\frac{c}{4}\right)^n}{(k)_n n!} - \frac{(B+3)q}{1-q}\sum_{n=1}^{8}\frac{\left(\frac{c}{4}\right)^n}{(k)_n n!}q^n$$

As $|a_n| = a_n$

Using the definition of Gauss hypergeometric function above condition convert in

$$= \left(\frac{2q+(B+1)}{1-q} + (A+1)\right)\left(-;;k;;\frac{c}{4}\right) - \frac{(B+3)q}{1-q}{}_0F_1\left(-;;k;;\frac{cq}{4}\right) + (A+B+2)$$

$$= P(k,q)$$

and conclude that

$$\Phi(z) = z\psi(z) \in S_q^*[A,B]$$

which conclude the required result of Theorem (2.1)

Corollary 2.2 Let A = z , B = 1 then above condition become

$$P^*[k,q] = \left(\frac{2q+1}{1-q} + z + 1\right)\left(-;;k;;\frac{c}{4}\right) - \frac{4q}{1-q}{}_0F_1\left(-;;k;;\frac{cq}{4}\right) + (z+3) \quad (2.2)$$

If the inequality $P^*[k,q] < 1-z$, holds, then the function $z\psi(z) \in S_q^*[z]$.

If $z = 0$ then from (2.2)

$$P_1{}^*[k,q] = \left(\frac{2q+1}{1-q} + 1\right)\left(-;;k;;\frac{c}{4}\right) - \frac{4q}{1-q}{}_0F_1\left(-;;k;;\frac{cq}{4}\right) + 3 \quad (2.3)$$

If the inequality $P^*[k,q] < 1$, holds, then the function $z\psi(z) \in S_q^*[0]$.

Theorem 2.3 Let $Q[k,q]$, be defined as follows

$$Q[k,q] = \frac{1}{(1-q)^2}[(q+B+2+A(1-q)\left[{}_0F_1\left(-;;k;;\frac{c}{4}\right)\right] - \left(Aq(1-q)+2Bq+q^2+5q\right)\times$$

$$\times\left[{}_0F_1\left(-;;k;;\frac{cq}{4}\right)\right] + (B+3)q^2\left[{}_0F_1\left(-;;k;;\frac{cq^2}{4}\right)\right] + (A+B+2)$$

If the inequality $Q(k,q) < |B-A|$ holds, and then function $z\psi(z) \in C_q^*[A,B]$.

Proof: Here

$$z\psi(z) = z + \sum_{n=2}^{8}\frac{\left(\frac{c}{4}\right)^{n-1}}{(k)_{n-1}n-1!}z^n = z + \sum_{n=2}^{8}a_n z^n, \qquad z \in \mathfrak{D}$$

From Lemma 1.1, If any function $F \in C_q^*[A,B]$ and satisfies (1.9). So for $\Phi(z) = z\,\psi(z)$ it is sufficient that (1.10) holds, for

$$a_n = \frac{\left(\frac{c}{4}\right)^{n-1}}{(k)_{n-1}n-1!} \quad \& \quad [n]_q = \frac{1-q^n}{1-q}$$

Now, on using triangle's inequality

$$\sum_{n=2}^{8}[n]_q\left(2q[n-1]_q+\left|(B+1)\,[n]_q-(A+1)\right|\right)|a_n|\le\sum_{n=2}^{8}2q[n]_q\frac{1-q^{n-1}}{1-q}\,|a_n|$$

$$+\sum_{n=2}^{8}(B+1)\,[n]_q\frac{1-q^n}{1-q}\,|a_n|+\sum_{n=2}^{8}(A+1)[n]_q\,|a_n|$$

$$=\sum_{n=2}^{8}2q\frac{1-q^n}{1-q}\frac{1-q^{n-1}}{1-q}\,|a_n|+\sum_{n=2}^{8}(B+1)\,\frac{1-q^n}{1-q}\frac{1-q^n}{1-q}\,|a_n|+\sum_{n=2}^{8}(A+1)\frac{1-q^n}{1-q}\,|a_n|$$

$$=\sum_{n=2}^{8}\frac{2q+(B+1)+(A+1)(1-q)}{(1-q)^2}\,|a_n|+\sum_{n=2}^{8}\frac{2+(B+1)}{(1-q)^2}q^{2n}\,|a_n|$$

$$-\sum_{n=2}^{8}\frac{(A+1)\,(1-q)+2\,(B+1)+\ 2q+2}{(1-q)^2}q^n\,|a_n|$$

By using Gauss Hyper geometric function

$$=\frac{q+B+2+A(1-q)}{((1-q)^2}\left[{}_0F_1\left(-;;k\,;;\ \frac{c}{4}\right)-1\right]-\frac{Aq\,(1-q)+\ 2Bq+\ q^2+\ 5q}{(1-q)^2}\times$$

$$\times\left[{}_0F_1\left(-;;k\,;;\ \frac{cq}{4}\right)-1\right]+\frac{(B+1)\,q^2}{(1-q)^2}\left[{}_0F_1\left(-;;k\,;;\ \frac{cq^2}{4}\right)-1\right]$$

$$=\frac{1}{(1-q)^2}\,(q+B+2+A\,(1-q))\left[{}_0F_1\left(-;;k\,;;\ \frac{c}{4}\right)\right]-\left(Aq\,(1-q)+2Bq+q^2+5q\right)\times$$

$$\times\left[{}_0F_1\left(-;;k\,;;\ \frac{cq}{4}\right)\right]+(B+3)\,q^2\left[{}_0F_1\left(-;;k\,;;\ \frac{cq^2}{4}\right)\right]+(A+B+2)=Q(k,q)<|B-A|$$

Therefore, on taking (1.10) into consideration, function $z\psi\,(z)\in C_q^*\,[A,B]$.

Corollary 2.4

Let A = z, B = 1 the above theorem becomes

$$Q^*\,(k,q)=\ \frac{1}{(1-q)^2}[(q+3+z\,(1-q)\left[{}_0F_1\left(-;;k\,;;\ \frac{c}{4}\right)\right]-\left(zq\,(1-q)+2q+q^2+5q\right)\times$$

$$\times\left[{}_0F_1\left(-;;k\,;;\ \frac{cq}{4}\right)\right]+\ 4q^2\left[{}_0F_1\left(-;;k\,;;\ \frac{cq^2}{4}\right)\right]+(z+3)\qquad(2.4)$$

If the inequality $Q^*\,[k,q]<1-z$ holds, then the function $z\psi\,(z)\in C_q^*\,[z]$.

If $z = 0$ then from (2.2)

$$Q_1{}^*[k,q] = \frac{1}{(1-q)^2}[(q+3+z)\left[{}_0F_1\left(-;;k;;\frac{c}{4}\right)\right] - \left(2q+q^2+5q\right)\times$$
$$\times\left[{}_0F_1\left(-;;k;;\frac{cq}{4}\right)\right] + 4q^2\left[{}_0F_1\left(-;;k;;\frac{cq^2}{4}\right)\right] + 3 \qquad (2.5)$$

For the inequality $Q_1{}^*[k,q] < 1$, the function $z\psi(z) \in C_q^*[0]$.

3. Conclusion

We obtain the necessary conditions for a function connected to a normalised Bessel-Struve function to be q- starlikeness and q-convexity as [7]. More conditions for the q-starlikeness and q-convexity of functions can be found using the derived results. This q analysis has applications in many areas of physics and mathematics, including ordinary fractional calculus, optimum control issues, q difference and q integral equations, and q transform analysis.

Conflicts of Interest: The authors declare no conflict of interest.

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