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FUZZY NEUTROSOPHIC H-IDEALS(CLOSED) IN BCK/BCI-ALGEBRAS

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Abstract

In this article, we introduce the concepts of fuzzy neutrosophic H-ideal and fuzzy neutrosophic closed H-ideal in the context of BCK/BCI-algebras. We explore the properties and characteristics of these ideals and provide an illustrative example of a fuzzy neutrosophic H-ideal. Furthermore, we demonstrate that the pre-image of a fuzzy neutrosophic H-ideal under an onto homomorphism remains a fuzzy neutrosophic H-ideal.

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1. Introduction

L.A.Zadeh [17], a professor of computer science at the University of California, introduced the concept of fuzzy set (FS) in 1965. Fuzzy sets analyzed the degree of membership of members of set and Xi [16] applied this concept to the ideals of BCK/BCI algebras. In 1986 Atanassove [3] generalized a fuzzy set to an intuitionistic fuzzy set (IFS) by including another function called a non-membership function and Jun and Kim [8] introduced intuitionistic fuzzy ideals of BCK-algebras. Neutrosophic set theory, an extension of classical set theory, introduces the concept of ambiguity or indeterminacy, which makes it particularly valuable for solving real-world problems characterized by imprecision, uncertainty, or incomplete information.

The origins of Neutrosophic set theory can be traced to the pioneering work of mathematician and philosopher Florentin Smarandache ([12], [13]). Since then, it has attracted considerable attention and has found practical applications in decision making, expert systems, pattern recognition, image processing, and fuzzy logic. Furthermore, researchers are actively exploring new approaches in the field of Neutrosophic set theory to further improve its capabilities. In recent years, researchers have also been exploring new approaches in the field of Neutrosophic set theory. Please refer to ([10], [4], [5], [1], [14], [15]). In [2], Arockiarani et al. introduced the concept of fuzzy neutrosophic set. In [11], B.Satyanarayana et.al given the idea of intuitionistic fuzzy H-ideals of $BCK-$algebras. This article introduces the concept of fuzzy neutrosophic

H-ideals of $BCK-$algebra, fuzzy neutrosophic closed H-ideals of $BCI-$algebra and its properties.

2. Preliminaries

Definition 2.1. [6] Let $\mathcal{K}$ be a non-empty set with a binary operation "$*$"and a constant "0"is called a BCI-algebra if it satisfies the following axioms for all $\zeta, \eta, \theta \in \mathcal{K}$
(i) $((\zeta * \eta) * (\zeta * \theta)) * (\theta * \eta) = 0$
(ii) $(\zeta * (\zeta * \eta)) * \eta = 0$
(iii) $\zeta * \zeta = 0$
(iv) $\zeta * \eta = 0, \eta * \zeta = 0 \Rightarrow \zeta = \eta$.
If a BCI-algebra $\mathcal{K}$ satisfies the following identity
(v) $0 * \zeta = 0$ for all $\zeta \in \mathcal{K}$ then $\mathcal{K}$ is called a BCK-algebra.

Definition 2.2. [6] We can define a binary operation "$\leq$"on $\mathcal{K}$ by assuming $\zeta \leq \eta$ if and only if $\zeta * \eta = 0$.

Definition 2.3. [7] A non-empty subset $\mathcal{I}$ of a BCK- algebra $\mathcal{K}$ is called an ideal, if it satisfies (I-1) $0 \in \mathcal{I}$. (I-2) $\zeta * \eta \in \mathcal{I}$ and $\eta \in \mathcal{I} \Rightarrow \zeta \in \mathcal{I}$ for all $\zeta, \eta \in \mathcal{K}$.

Definition 2.4. [7] An ideal $\mathcal{I}$ of a BCI- algebra $\mathcal{K}$ is called an closed ideal, if it satisfies $0 * \zeta \in \mathcal{I}$, for all $\zeta \in \mathcal{I}$.

Definition 2.5. [9] A non-empty subset $\mathcal{I}$ of a BCK- algebra $\mathcal{K}$ is called an H-ideal, if it satisfies (HI-1) $0 \in \mathcal{I}$. (HI-2) $\zeta * (\eta * \theta) \in \mathcal{I}$ and $\eta \in \mathcal{I} \Rightarrow \zeta * \theta \in \mathcal{I}$ for all $\zeta, \eta \in \mathcal{K}$.

Definition 2.6. [17] Let $\mathcal{K}$ be a non-empty set. A fuzzy set in a set $\mathcal{K}$ is a mapping $\mathcal{N}_T : \mathcal{K} \to [0, 1]$.The complement of a fuzzy set $\mathcal{N}_T$ denoted by $(\mathcal{N}_T)^c$ is also a fuzzy set defined as $(\mathcal{N}_T)^c(\zeta) = 1 - \mathcal{N}_T(\zeta)$, for all $\zeta \in \mathcal{K}$. Also $({\mathcal{N}_T}^c)^c = \mathcal{N}_T$.

Definition 2.7. [3] An intuitionistic fuzzy set $\mathcal{N}$ in a non-empty set $\mathcal{K}$ is an object having the form $\mathcal{N} = \{(\zeta, \mathcal{N}_T(\zeta), \mathcal{N}_F(\zeta)) \mid \zeta \in \mathcal{K}\}$ where the functions $\mathcal{N}_T : \mathcal{K} \to [0, 1]$ and $\mathcal{N}_F : \mathcal{K} \to [0, 1]$ denote the grade of membership and non-membership of each element$\zeta \in \mathcal{K}$ to the set $\mathcal{N}$ respectively and $0 \leq \mathcal{N}_T(\zeta) + \mathcal{N}_F(\zeta) \leq 1$ for all $\zeta \in \mathcal{K}$.

Definition 2.8. [11] An intuitionistic fuzzy set $\mathcal{N} = (\mathcal{K}, \mathcal{N}_T, \mathcal{N}_F)$ in a BCK-algebra $\mathcal{K}$ is called a an intuitionistic fuzzy H-ideal (IFH-I) of BCK-algebra $\mathcal{K}$, if
(IFHI-1) $\mathcal{N}_T(0) \geq \mathcal{N}_T(\zeta)$ and $\mathcal{N}_F(0) \leq \mathcal{N}_F(\zeta)$
(IFHI-2) $\mathcal{N}_T(\zeta * \theta) \geq min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$
(IFHI-3) $\mathcal{N}_F(\zeta * \theta) \leq max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\}$
for all $\zeta, \eta, \theta \in \mathcal{K}$.

Definition 2.9. [11] An intuitionistic fuzzy set $\mathcal{N} = (\mathcal{K}, \mathcal{N}_T, \mathcal{N}_F)$ in a BCK-algebra $\mathcal{K}$ is called an intuitionistic fuzzy closed H-ideal of BCK-algebra $\mathcal{K}$, if it satisfies (IFHI-2), (IFHI-3) and (IFHI-4) $\mathcal{N}_T(0 * \zeta) \geq \mathcal{N}_T(\zeta)$ and $\mathcal{N}_F(0 * \zeta) \leq \mathcal{N}_F(\zeta)$ for all $\zeta \in \mathcal{K}$.

DEFINITION 2.10. [2] A fuzzy neutrosophic set (FNS) in a non-empty set $\mathcal{K}$ is a structure of the form

$$\mathcal{N} = \{\langle \zeta, \mathcal{N}_T(\zeta), \mathcal{N}_I(\zeta), \mathcal{N}_F(\zeta)\rangle \mid \zeta \in \mathcal{K}\}$$

Where $\mathcal{N}_T : \mathcal{K} \to [0,1]$, $\mathcal{N}_I : \mathcal{K} \to [0,1]$, and $\mathcal{N}_F : \mathcal{K} \to [0,1]$ represents grade of belongingness, grade of indeterminacy, and grade of non- belongingness of each element $\zeta \in \mathcal{K}$ to the set $\mathcal{N}$ respectively and $0 \leq \mathcal{N}_T(\zeta) + \mathcal{N}_I(\zeta) + \mathcal{N}_F(\zeta) \leq 3$. We shall use the symbol $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ for the FNS $\mathcal{N} = \{\langle \zeta, \mathcal{N}_T(\zeta), \mathcal{N}_I(\zeta), \mathcal{N}_F(\zeta)\rangle \mid \zeta \in \mathcal{K}\}$

3. Fuzzy Neutrosophic H-ideal

DEFINITION 3.1. A Fuzzy neutrosophic set $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ in a BCK-algebra $\mathcal{K}$ is called a fuzzy neutrosophic H-ideal of a BCK-algebra $\mathcal{K}$, if

(FNHI-1) $\mathcal{N}_T(0) \geq \mathcal{N}_T(\zeta)$, $\mathcal{N}_I(0) \geq \mathcal{N}_I(\zeta)$, and $\mathcal{N}_F(0) \leq \mathcal{N}_F(\zeta)$

(FNHI-2) $\mathcal{N}_T(\zeta * \theta) \geq min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$

(FNHI-3) $\mathcal{N}_I(\zeta * \theta) \geq min\{\mathcal{N}_I(\zeta * (\eta * \theta)), \mathcal{N}_I(\eta)\}$

(FNHI-4) $\mathcal{N}_F(\zeta * \theta) \leq max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\}$, for all $\zeta, \eta, \theta \in \mathcal{K}$.

EXAMPLE 3.2. *Let $\mathcal{K} = \{0, 1, 2, 3, 4\}$ be a BCK-algebra as defined in the Cayley Table 1.*

TABLE 1. BCK-algebra

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a fuzzy neutrosophic set defined in the Table 1. Then $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic H-ideal of $\mathcal{K}$.

TABLE 2. Fuzzy Neutrosophic H-Ideal

$\mathcal{K}$	$\mathcal{N}_T$	$\mathcal{N}_I$	$\mathcal{N}_F$
0	0.97	0.85	0.15
1	0.73	0.67	0.47
2	0.32	0.25	0.75
3	0.32	0.25	0.83
4	0.15	0.13	0.83

Definition 3.3. A fuzzy neutrosophic set $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ in a BCI-algebra $\mathcal{K}$ is called a fuzzy neutrosophic closed H-ideal of $\mathcal{K}$ if it satisfies (FNHI-2), (FNHI-3), (FNHI-4), and the following

(FNCHI) $\mathcal{N}_T(0 * \zeta) \geq \mathcal{N}_T(\zeta)$, $\mathcal{N}_I(0 * \zeta) \geq \mathcal{N}_I(\zeta)$, and $\mathcal{N}_F(0 * \zeta) \leq \mathcal{N}_F(\zeta)$ for all $\zeta \in \mathcal{K}$.

Definition 3.4. let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a fuzzy neutrosophic set in a BCK-algebra $\mathcal{K}$ the set $U(\mathcal{N}; \langle a, b\rangle) = \{\zeta \in \mathcal{K} \mid \mathcal{N}_T(\zeta) \geq a, \mathcal{N}_I(\zeta) \geq b\}$ is called upper (a, b)-level cut of $\mathcal{N}$ and the set $L(\mathcal{N}; c) = \{\zeta \in \mathcal{K} \mid \mathcal{N}_F(\zeta) \leq c\}$ is called lower $c-$level cut of $\mathcal{N}$.

Lemma 3.5. *If $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic H-ideal of BCK-algebra $\mathcal{K}$, then we have $\zeta \leq a \Rightarrow \mathcal{N}_T(\zeta) \geq \mathcal{N}_T(a)$, $\mathcal{N}_I(\zeta) \geq \mathcal{N}_I(a)$, and $\mathcal{N}_F(\zeta) \leq \mathcal{N}_F(a)$ for all $\zeta, a \in \mathcal{K}$.*

Proof. Let $\zeta, a \in \mathcal{K}$ such that $\zeta \leq a \Rightarrow \zeta * a = 0$. Write $\theta = 0$ and $\eta = a$ in (FNHI-2), (FNHI-3) and (FNHI-4). We get

$$\mathcal{N}_T(\zeta * 0) \geq min\{\mathcal{N}_T(\zeta * (a * 0)), \mathcal{N}_T(a)\}$$
$$\Rightarrow \mathcal{N}_T(\zeta) \geq min\{\mathcal{N}_T(\zeta * a), \mathcal{N}_T(a)\} = min\{\mathcal{N}_T(0), \mathcal{N}_T(a)\} = \mathcal{N}_T(a),$$
$$\mathcal{N}_I(\zeta * 0) \geq min\{\mathcal{N}_I(\zeta * (a * 0)), \mathcal{N}_I(a)\}$$
$$\Rightarrow \mathcal{N}_I(\zeta) \geq min\{\mathcal{N}_I(\zeta * a), \mathcal{N}_I(a)\} = min\{\mathcal{N}_I(0), \mathcal{N}_I(a)\} = \mathcal{N}_I(a),$$
$$\mathcal{N}_F(\zeta * 0) \leq max\{\mathcal{N}_F(\zeta * (a * 0)), \mathcal{N}_F(a)\}$$
$$\Rightarrow \mathcal{N}_F(\zeta) \leq max\{\mathcal{N}_F(\zeta * a), \mathcal{N}_F(a)\} = max\{\mathcal{N}_F(0), \mathcal{N}_F(a)\} = \mathcal{N}_F(a)$$

Therefore, $\mathcal{N}_T(\zeta) \geq \mathcal{N}_T(a)$, $\mathcal{N}_I(\zeta) \geq \mathcal{N}_I(a)$, and $\mathcal{N}_F(\zeta) \leq \mathcal{N}_F(a)$ for all $\zeta, a \in \mathcal{K}$. □

Theorem 3.6. *Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic H-ideal of a BCK-algebra $\mathcal{K}$, then so is $\odot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$.*

Proof. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic H-ideal of a BCK-algebra $\mathcal{K}$, then we have

$\mathcal{N}_T(0) \geq \mathcal{N}_T(\zeta)$, $\mathcal{N}_I(0) \geq \mathcal{N}_I(\zeta)$, and $\mathcal{N}_F(0) \leq \mathcal{N}_F(\zeta)$
$\mathcal{N}_T(\zeta * \theta) \geq min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$
$\mathcal{N}_I(\zeta * \theta) \geq min\{\mathcal{N}_I(\zeta * (\eta * \theta)), \mathcal{N}_I(\eta)\}$
$\mathcal{N}_F(\zeta * \theta) \leq max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\}$, for all $\zeta, \eta, \theta \in \mathcal{K}$.

It is sufficient to show that

$${\mathcal{N}_T}^c(0) \leq {\mathcal{N}_T}^c(\zeta) \text{ and } {\mathcal{N}_T}^c(\zeta * \theta) \leq max\{{\mathcal{N}_T}^c(\zeta * (\eta * \theta)), {\mathcal{N}_T}^c(\eta)\}.$$

Let $\mathcal{N}_T(0) \geq \mathcal{N}_T(\zeta) \Rightarrow 1 - \mathcal{N}_T(0) \leq 1 - \mathcal{N}_T(\zeta) \Rightarrow {\mathcal{N}_T}^c(0) \leq {\mathcal{N}_T}^c(\zeta)$, for any $\zeta \in \mathcal{K}$.

Take for any $\zeta, \eta, \theta \in \mathcal{K}$,

$$\mathcal{N}_T(\zeta * \theta) \geq min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$$
$$\Rightarrow 1 - \mathcal{N}_T(\zeta * \theta) \leq 1 - min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$$
$$\Rightarrow {\mathcal{N}_T}^c(\zeta * \theta) \leq max\{1 - \mathcal{N}_T(\zeta * (\eta * \theta)), 1 - \mathcal{N}_T(\eta)\}$$
$$\Rightarrow {\mathcal{N}_T}^c(\zeta * \theta) \leq max\{{\mathcal{N}_T}^c(\zeta * (\eta * \theta)), {\mathcal{N}_T}^c(\eta)\}$$

Hence, $\odot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$ is a fuzzy neutrosophic H-ideal of BCK-algebra $\mathcal{K}$. □

THEOREM 3.7. *Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic H-ideal of a BCK-algebra $\mathcal{K}$, then so is $\oplus\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$.*

PROOF. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic H-ideal of a BCK-algebra $\mathcal{K}$, then we have
$\mathcal{N}_T(0) \geq \mathcal{N}_T(\zeta)$, $\mathcal{N}_I(0) \geq \mathcal{N}_I(\zeta)$, and $\mathcal{N}_F(0) \leq \mathcal{N}_F(\zeta)$
$\mathcal{N}_T(\zeta * \theta) \geq min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$
$\mathcal{N}_I(\zeta * \theta) \geq min\{\mathcal{N}_I(\zeta * (\eta * \theta)), \mathcal{N}_I(\eta)\}$
$\mathcal{N}_F(\zeta * \theta) \leq max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\}$, for all $\zeta, \eta, \theta \in \mathcal{K}$.
It is sufficient to show that

$${\mathcal{N}_F}^c(0) \geq {\mathcal{N}_F}^c(\zeta) \text{ and } {\mathcal{N}_F}^c(\zeta * \theta) \geq min\{{\mathcal{N}_F}^c(\zeta * (\eta * \theta)), {\mathcal{N}_F}^c(\eta)\}.$$

Now $\mathcal{N}_F(0) \leq \mathcal{N}_F(\zeta) \Rightarrow 1 - \mathcal{N}_F(0) \geq 1 - \mathcal{N}_F(\zeta) \Rightarrow {\mathcal{N}_F}^c(0) \geq {\mathcal{N}_F}^c(\zeta)$ for any $\zeta \in \mathcal{K}$.
Consider for any $\zeta, \eta, \theta \in \mathcal{K}$,

$$\begin{aligned} &\mathcal{N}_F(\zeta * \theta) \leq max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\} \\ &\Rightarrow 1 - \mathcal{N}_F(\zeta * \theta) \geq 1 - max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\} \\ &\Rightarrow {\mathcal{N}_F}^c(\zeta * \theta) \geq min\{1 - \mathcal{N}_F(\zeta * (\eta * \theta)), 1 - \mathcal{N}_F(\eta)\} \\ &\Rightarrow {\mathcal{N}_F}^c(\zeta * \theta) \geq min\{{\mathcal{N}_F}^c(\zeta * (\eta * \theta)), {\mathcal{N}_F}^c(\eta)\} \end{aligned}$$

Hence, $\oplus\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic H-ideal of BCK-algebra $\mathcal{K}$. □

THEOREM 3.8. *A fuzzy neutrosophic set $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ of a BCK-algebra $\mathcal{K}$ is a fuzzy neutrosophic H-ideal of $\mathcal{K}$ if and only if $\odot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$ and $\oplus\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$ are fuzzy neutrosophic H-ideals of a BCK-algebra $\mathcal{K}$.*

THEOREM 3.9. *Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic closed H-ideal of a BCI-algebra $\mathcal{K}$, then so is $\odot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$.*

PROOF. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic closed H-ideal of a BCI-algebra $\mathcal{K}$, then we have
$\mathcal{N}_T(0 * \zeta) \geq \mathcal{N}_T(\zeta)$, $\mathcal{N}_I(0 * \zeta) \geq \mathcal{N}_I(\zeta)$, and $\mathcal{N}_F(0 * \zeta) \leq \mathcal{N}_F(\zeta)$
$\mathcal{N}_T(\zeta * \theta) \geq min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$
$\mathcal{N}_I(\zeta * \theta) \geq min\{\mathcal{N}_I(\zeta * (\eta * \theta)), \mathcal{N}_I(\eta)\}$
$\mathcal{N}_F(\zeta * \theta) \leq max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\}$, for all $\zeta, \eta, \theta \in \mathcal{K}$.
It is sufficient to show that

$${\mathcal{N}_T}^c(0 * \zeta) \leq {\mathcal{N}_T}^c(\zeta) \text{ and } {\mathcal{N}_T}^c(\zeta * \theta) \leq max\{{\mathcal{N}_T}^c(\zeta * (\eta * \theta)), {\mathcal{N}_T}^c(\eta)\}.$$

For any $\zeta \in \mathcal{K}$, we have
$\mathcal{N}_T(0 * \zeta) \geq \mathcal{N}_T(\zeta) \Rightarrow 1 - \mathcal{N}_T(0 * \zeta) \leq 1 - \mathcal{N}_T(\zeta) \Rightarrow {\mathcal{N}_T}^c(0 * \zeta) \leq {\mathcal{N}_T}^c(\zeta)$

Take for any $\zeta, \eta, \theta \in \mathcal{K}$,

$$\mathcal{N}_T(\zeta * \theta) \geq min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$$
$$\Rightarrow 1 - \mathcal{N}_T(\zeta * \theta) \leq 1 - min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$$
$$\Rightarrow {\mathcal{N}_T}^c(\zeta * \theta) \leq max\{1 - \mathcal{N}_T(\zeta * (\eta * \theta)), 1 - \mathcal{N}_T(\eta)\}$$
$$\Rightarrow {\mathcal{N}_T}^c(\zeta * \theta) \leq max\{{\mathcal{N}_T}^c(\zeta * (\eta * \theta)), {\mathcal{N}_T}^c(\eta)\}$$

Hence, $\odot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$ is a fuzzy neutrosophic closed H-ideal of a BCI-algebra $\mathcal{K}$. □

Theorem 3.10. *Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic closed H-ideal of BCI-algebra $\mathcal{K}$, then so is $\oplus\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$.*

Proof. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic closed H-ideal of a BCI-algebra $\mathcal{K}$, then we have

$\mathcal{N}_T(0 * \zeta) \geq \mathcal{N}_T(\zeta)$, $\mathcal{N}_I(0 * \zeta) \geq \mathcal{N}_I(\zeta)$ and $\mathcal{N}_F(0 * \zeta) \leq \mathcal{N}_F(\zeta)$
$\mathcal{N}_T(\zeta * \theta) \geq min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$
$\mathcal{N}_I(\zeta * \theta) \geq min\{\mathcal{N}_I(\zeta * (\eta * \theta)), \mathcal{N}_I(\eta)\}$
$\mathcal{N}_F(\zeta * \theta) \leq max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\}$, for all $\zeta, \eta, \theta \in \mathcal{K}$.

It is sufficient to show that

$${\mathcal{N}_F}^c(0 * \zeta) \geq {\mathcal{N}_F}^c(\zeta) \text{ and } {\mathcal{N}_F}^c(\zeta * \theta) \geq min\{{\mathcal{N}_F}^c(\zeta * (\eta * \theta)), {\mathcal{N}_F}^c(\eta)\}.$$

For any $\zeta \in \mathcal{K}$, we have

$$\mathcal{N}_F(0 * \zeta) \leq \mathcal{N}_F(\zeta) \Rightarrow 1 - \mathcal{N}_F(0 * \zeta) \geq 1 - \mathcal{N}_F(\zeta) \Rightarrow {\mathcal{N}_F}^c(0 * \zeta) \geq {\mathcal{N}_F}^c(\zeta).$$

Consider for any $\zeta, \eta, \theta \in \mathcal{K}$,

$$\mathcal{N}_F(\zeta * \theta) \leq max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\}$$
$$\Rightarrow 1 - \mathcal{N}_F(\zeta * \theta) \geq 1 - max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\}$$
$$\Rightarrow {\mathcal{N}_F}^c(\zeta * \theta) \geq min\{1 - \mathcal{N}_F(\zeta * (\eta * \theta)), 1 - \mathcal{N}_F(\eta)\}$$
$$\Rightarrow {\mathcal{N}_F}^c(\zeta * \theta) \geq min\{{\mathcal{N}_F}^c(\zeta * (\eta * \theta)), {\mathcal{N}_F}^c(\eta)\}$$

Hence, $\oplus\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic closed H-ideal of BCI-algebra $\mathcal{K}$. □

Theorem 3.11. *A fuzzy neutrosophic set $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ of a BCI-algebra $\mathcal{K}$ is a fuzzy neutrosophic closed H-ideal of $\mathcal{K}$ if and only if $\odot\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, {\mathcal{N}_T}^c)$ and $\oplus\mathcal{N} = ({\mathcal{N}_F}^c, \mathcal{N}_I, \mathcal{N}_F)$ are fuzzy neutrosophic closed H-ideals of a BCI-algebra $\mathcal{K}$.*

Theorem 3.12. *A fuzzy neutrosophic set $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ of a BCK-algebra $\mathcal{K}$ is a fuzzy neutrosophic H-ideal of $\mathcal{K}$ if and only if the non empty upper $(a, b)-$ level cut $U(\mathcal{N}; \langle a, b\rangle)$ and the non-empty lower $c-$level cut $L(\mathcal{N}; c)$ are H-ideals of $\mathcal{K}$ for any $a, b, c \in [0, 1]$.*

Proof. Suppose $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic H-ideal of a BCK-algebra $\mathcal{K}$, and $U(\mathcal{N}; \langle a, b\rangle)$, and $L(\mathcal{N}; c)$ are non-empty subsets of $\mathcal{K}$. For any $a, b, c \in [0, 1]$, we have

$$U(\mathcal{N};\langle a,b\rangle) = \{\zeta \in \mathcal{K} \mid \mathcal{N}_T(\zeta) \geq a, \mathcal{N}_I(\zeta) \geq b\},$$
$$L(\mathcal{N};c) = \{\zeta \in \mathcal{K} \mid \mathcal{N}_F(\zeta) \leq c\}.$$

For any $\zeta \in U(\mathcal{N};\langle a,b\rangle)$ and $\zeta \in L(\mathcal{N};c)$ we have

$$\mathcal{N}_T(\zeta) \geq a, \mathcal{N}_I(\zeta) \geq b, \text{ and } \mathcal{N}_F(\zeta) \leq c$$
$$\Rightarrow \mathcal{N}_T(0) \geq a, \mathcal{N}_I(0) \geq b, \text{ and } \mathcal{N}_F(0) \leq c$$
$$\Rightarrow 0 \in U(\mathcal{N};\langle a,b\rangle) \text{ and } 0 \in L(\mathcal{N};c).$$

Let $\zeta*(\eta*\theta), \eta \in U(\mathcal{N};\langle a,b\rangle)$, and $\zeta*(\eta*\theta), \eta \in L(\mathcal{N};c)$ implies $\mathcal{N}_T(\zeta*(\eta*\theta)) \geq a$, $\mathcal{N}_I(\zeta*(\eta*\theta)) \geq b$, $\mathcal{N}_F(\zeta*(\eta*\theta)) \leq a$, $\mathcal{N}_T(\eta) \geq a$, $\mathcal{N}_I(\eta) \geq b$, and $\mathcal{N}_I(\eta) \leq c$. Then, for all $\zeta,\eta,\theta \in \mathcal{K}$, we have

$$\mathcal{N}_T(\zeta*\theta) \geq min\{\mathcal{N}_T(\zeta*(\eta*\theta)), \mathcal{N}_T(\eta)\} \geq min\{a,a\} = a, \Rightarrow \mathcal{N}_T(\zeta*\theta) \geq a,$$
$$\mathcal{N}_I(\zeta*\theta) \geq min\{\mathcal{N}_I(\zeta*(\eta*\theta)), \mathcal{N}_I(\eta)\} \geq min\{b,b\} = b \Rightarrow \mathcal{N}_I(\zeta*\theta) \geq b,$$
$$\mathcal{N}_F(\zeta*\theta) \leq max\{\mathcal{N}_F(\zeta*(\eta*\theta)), \mathcal{N}_F(\eta)\} \leq max\{c,c\} = c \Rightarrow \mathcal{N}_F(\zeta*\theta) \leq c.$$

Therefore, $\zeta*\theta \in U(\mathcal{N};\langle a,b\rangle)$ and $\zeta*\theta \in L(\mathcal{N};c)$. Hence, $U(\mathcal{N};\langle a,b\rangle)$ and $L(\mathcal{N};c)$ are H-ideals of $\mathcal{K}$.

Conversely, suppose that $U(\mathcal{N};\langle a,b\rangle)$ and $L(\mathcal{N};c)$ are H-ideals of $\mathcal{K}$ for any $a,b,c \in [0,1]$. If possible, assume $\zeta \in \mathcal{K}$ such that

$$\mathcal{N}_T(0) < \mathcal{N}_T(\zeta),\ \mathcal{N}_I(0) < \mathcal{N}_I(\zeta), \text{ and } \mathcal{N}_F(0) > \mathcal{N}_F(\zeta).$$

$$\text{Put } a = \frac{1}{2}(\mathcal{N}_T(0) + \mathcal{N}_T(\zeta)); b = \frac{1}{2}(\mathcal{N}_I(0) + \mathcal{N}_I(\zeta)); c = \frac{1}{2}(\mathcal{N}_F(0) + \mathcal{N}_F(\zeta))$$

$$\text{Implies } a < \mathcal{N}_T(\zeta), 0 \leq \mathcal{N}_T(0) < a < 1$$
$$b < \mathcal{N}_I(\zeta), 0 \leq \mathcal{N}_I(0) < b < 1$$
$$c > \mathcal{N}_F(\zeta), 1 \geq \mathcal{N}_F(0) > c > 0$$
$$\Rightarrow \zeta \in U(\mathcal{N};\langle a,b\rangle) \text{ and } \zeta \in L(\mathcal{N};c)$$

Since $U(\mathcal{N};\langle a,b\rangle)$ and $L(\mathcal{N};c)$ are H-ideals of $\mathcal{K}$, we have $0 \in U(\mathcal{N};\langle a,b\rangle)$, and $0 \in L(\mathcal{N};c) \Rightarrow \mathcal{N}_T(0) \geq a$, $\mathcal{N}_I(0) \geq b$, and $\mathcal{N}_F(0) \leq c$ which is a contradiction. Therefore, $\mathcal{N}_T(0) \geq \mathcal{N}_T(\zeta)$, $\mathcal{N}_I(0) \geq \mathcal{N}_I(\zeta)$ and $\mathcal{N}_F(0) \leq \mathcal{N}_F(\zeta)$ for all $\zeta \in \mathcal{K}$. If possible assume $\zeta,\eta,\theta \in \mathcal{K}$ such that

$$\mathcal{N}_T(\zeta*\theta) < min\{\mathcal{N}_T(\zeta*(\eta*\theta)), \mathcal{N}_T(\eta)\} \text{ and}$$
$$\mathcal{N}_I(\zeta*\theta) < min\{\mathcal{N}_I(\zeta*(\eta*\theta)), \mathcal{N}_I(\eta)\}.$$

$$\text{Put } a = \frac{1}{2}[\mathcal{N}_T(\zeta*\theta) + min\{\mathcal{N}_T(\zeta*(\eta*\theta)), \mathcal{N}_T(\eta)\}]$$
$$b = \frac{1}{2}[\mathcal{N}_I(\zeta*\theta) + min\{\mathcal{N}_I(\zeta*(\eta*\theta)), \mathcal{N}_I(\eta)\}]$$
$$\Rightarrow a > \mathcal{N}_T(\zeta*\theta), a < min\{\mathcal{N}_T(\zeta*(\eta*\theta)), \mathcal{N}_T(\eta)\} \text{ and}$$
$$b > \mathcal{N}_I(\zeta*\theta), b < min\{\mathcal{N}_I(\zeta*(\eta*\theta)), \mathcal{N}_I(\eta)\}$$
$$\Rightarrow a > \mathcal{N}_T(\zeta*\theta), a < \mathcal{N}_T(\zeta*(\eta*\theta)), a < \mathcal{N}_T(\eta) \text{ and}$$

$$b > \mathcal{N}_I(\zeta * \theta),\ b < \mathcal{N}_I(\zeta * (\eta * \theta)), b < \mathcal{N}_I(\eta)$$
$$\Rightarrow \zeta * \theta \notin U(\mathcal{N}; \langle a, b\rangle), \zeta * (\eta * \theta) \in U(\mathcal{N}; \langle a, b\rangle), \eta \in U(\mathcal{N}; \langle a, b\rangle)$$

This contradicts the H-ideal $U(\mathcal{N}; \langle a, b\rangle)$.
Therefore,

$$\mathcal{N}_T(\zeta * \theta) \geq min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\},$$
$$\mathcal{N}_I(\zeta * \theta) \geq min\{\mathcal{N}_I(\zeta * (\eta * \theta)), \mathcal{N}_I(\eta)\} \text{ for any } \zeta, \eta, \theta \in \mathcal{K}.$$

Similarly, we can prove $\mathcal{N}_F(\zeta*\theta) \leq max\{\mathcal{N}_F(\zeta*(\eta*\theta)), \mathcal{N}_F(\eta)\}$ for any $\zeta, \eta, \theta \in \mathcal{K}$. Hence, $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic H-ideal of a BCK-algebra $\mathcal{K}$. □

Theorem 3.13. *A fuzzy neutrosophic set $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ of a BCI-algebra $\mathcal{K}$ is a fuzzy neutrosophic closed H-ideal of $\mathcal{K}$ if and only if the non empty upper $(a, b)-$level cut $U(\mathcal{N}; \langle a, b\rangle)$ and the non-empty lower $c-$level cut $L(\mathcal{N}; c)$ are closed H-ideals of the BCI-algebra $\mathcal{K}$ for any $a, b, c \in [0, 1]$.*

Proof. Suppose $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic closed H-ideal of a BCI-algebra $\mathcal{K}$. We have

$$\mathcal{N}_T(0 * \zeta) \geq \mathcal{N}_T(\zeta), \mathcal{N}_I(0 * \zeta) \geq \mathcal{N}_I(\zeta), \text{ and } \mathcal{N}_F(0 * \zeta) \leq \mathcal{N}_F(\zeta)$$

for any $\zeta \in \mathcal{K}$. For $\zeta \in U(\mathcal{N}; \langle a, b\rangle)$, we have

$$\Rightarrow \zeta \in \mathcal{K} \text{ and } \mathcal{N}_T(\zeta) \geq a, \mathcal{N}_I(\zeta) \geq b$$
$$\Rightarrow \mathcal{N}_T(0 * \zeta) \geq a, \mathcal{N}_I(0 * \zeta) \geq b$$
$$\Rightarrow 0 * \zeta \in U(\mathcal{N}; \langle a, b\rangle).$$

Also, for $\zeta \in L(\mathcal{N}; c)$, we have

$$\Rightarrow \zeta \in \mathcal{K} \text{ and } \mathcal{N}_F(\zeta) \leq c$$
$$\Rightarrow \mathcal{N}_F(0 * \zeta) \leq c$$
$$\Rightarrow 0 * \zeta \in L(\mathcal{N}; c).$$

Therefore, $U(\mathcal{N}; \langle a, b\rangle)$ and $L(\mathcal{N}; c)$ are closed H-ideals of $\mathcal{K}$.

Conversely, suppose that $U(\mathcal{N}; \langle a, b\rangle)$ and $L(\mathcal{N}; c)$ are closed H-ideals of $\mathcal{K}$ for any $a, b, c \in [0, 1]$. If possible, assume $\zeta \in \mathcal{K}$ such that

$$\mathcal{N}_T(0 * \zeta) < \mathcal{N}_T(\zeta), \mathcal{N}_I(0 * \zeta) < \mathcal{N}_I(\zeta) \text{ and } \mathcal{N}_F(0 * \zeta) > \mathcal{N}_F(\zeta).$$

Take,

$$a = \tfrac{1}{2}(\mathcal{N}_T(0 * \zeta) + \mathcal{N}_T(\zeta)) \quad \Rightarrow \mathcal{N}_T(0 * \zeta) < a < \mathcal{N}_T(\zeta)$$
$$b = \tfrac{1}{2}(\mathcal{N}_I(0 * \zeta) + \mathcal{N}_I(\zeta)) \quad \Rightarrow \mathcal{N}_I(0 * \zeta) < b < \mathcal{N}_I(\zeta)$$
$$c = \tfrac{1}{2}(\mathcal{N}_F(0 * \zeta) + \mathcal{N}_F(\zeta)) \quad \Rightarrow \mathcal{N}_F(\zeta) < c < \mathcal{N}_F(0 * \zeta)$$

$\Rightarrow \zeta \in U(\mathcal{N}; \langle a, b\rangle$ and $\zeta \in L(\mathcal{N}; c)$, but $0 * \zeta \notin U(\mathcal{N}; \langle a, b\rangle)$, $0 * \zeta \notin L(\mathcal{N}; c)$. This contradicts the H-ideals $U(\mathcal{N}; \langle a, b\rangle$ and $L(\mathcal{N}; c)$. Hence, $\mathcal{N}_T(0 * \zeta) \geq \mathcal{N}_T(\zeta)$, $\mathcal{N}_I(0 * \zeta) \geq \mathcal{N}_I(\zeta)$, and $\mathcal{N}_F(0 * \zeta) \leq \mathcal{N}_F(\zeta)$ for any $\zeta \in \mathcal{K}$. The proof of following inequalities is similar to the proof of converse part in Theorem 3.12.

$$\mathcal{N}_T(\zeta * \theta) \geq min\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$$
$$\mathcal{N}_I(\zeta * \theta) \geq min\{\mathcal{N}_I(\zeta * (\eta * \theta)), \mathcal{N}_I(\eta)\}$$
$$\mathcal{N}_F(\zeta * \theta) \leq max\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\} \text{ for any } \zeta, \eta, \theta \in \mathcal{K}.$$

Hence, $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a fuzzy neutrosophic closed H-ideal of a BCK-algebra $\mathcal{K}$. □

Corollary 3.14. *If $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a fuzzy neutrosophic H-ideal of a BCK-algebra $\mathcal{K}$, then the sets $\mathcal{J}_1 = \{\zeta \in \mathcal{K} \mid \mathcal{N}_T(\zeta) = \mathcal{N}_T(0)\}$, $\mathcal{J}_2 = \{\zeta \in \mathcal{K} \mid \mathcal{N}_I(\zeta) = \mathcal{N}_I(0)\}$, and $\mathcal{J}_3 = \{\zeta \in \mathcal{K} \mid \mathcal{N}_F(\zeta) = \mathcal{N}_F(0)\}$ are H-ideals of $\mathcal{K}$.*

Proof. Since $0 \in \mathcal{K}$, $\mathcal{N}_T(0) = \mathcal{N}_T(0)$, $\mathcal{N}_I(0) = \mathcal{N}_I(0)$, and $\mathcal{N}_F(0) = \mathcal{N}_F(0)$ implies $0 \in \mathcal{J}_1, \mathcal{J}_2$, and $\mathcal{J}_3$, so $\mathcal{J}_1 \neq \phi$, $\mathcal{J}_2 \neq \phi$, and $\mathcal{J}_3 \neq \phi$. Let $\zeta * (\eta * \theta) \in \mathcal{J}_1$ and $\eta \in \mathcal{J}_1 \Rightarrow \mathcal{N}_T(\zeta * (\eta * \theta)) = \mathcal{N}_T(0), \mathcal{N}_T(\eta) = \mathcal{N}_T(0)$. Since $\mathcal{N}_T(\zeta * \theta) \geq min\,\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\} \Rightarrow \mathcal{N}_T(\zeta * \theta) \geq \mathcal{N}_T(0)$ but $\mathcal{N}_T(0) \geq \mathcal{N}_T(\zeta * \theta)$.

It follows that $\zeta * \theta \in \mathcal{J}_1$ for any $\zeta, \eta, \theta \in \mathcal{K}$. Hence, $\mathcal{J}_1$ is a H-ideal of $\mathcal{K}$. Similarly, we can prove $\mathcal{J}_2$ and $\mathcal{J}_3$ are H-ideal of $\mathcal{K}$. □

Definition 3.15. Let f be a mapping on a set $\mathcal{K}$ and $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ a fuzzy neutrosophic set in $\mathcal{K}$ then the fuzzy sets u, v and w on$f(\zeta)$, defined by $u(\eta) = \sup_{\zeta \in f^{-1}(\eta)} \mathcal{N}_T(\zeta), v(\eta) = \sup_{\zeta \in f^{-1}(\eta)} \mathcal{N}_I(\zeta)$ and $w(\eta) = \inf_{\zeta \in f^{-1}(\eta)} \mathcal{N}_F(\zeta)$ called image of $\mathcal{N}$ under f. If u, v *and* w are fuzzy sets in $f(\zeta)$ then the fuzzy set $\mathcal{N}_T = uof$, $\mathcal{N}_I = vof$ and $\mathcal{N}_F = wof$ are called the pre-image of u, v and w under f.

Theorem 3.16. *Let $f : \mathcal{K} \to \mathcal{K}'$ be an onto homomorphism of BCK-algebras. If $\mathcal{N}' = (\mathcal{K}', u, v, w)$ is a fuzzy neutrosophic H-ideal of BCK-algebra $\mathcal{K}'$, then the pre-image of $\mathcal{N}'$ under f is a fuzzy neutrosophic H-ideal of BCK-algebra $\mathcal{K}$.*

Proof. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$, where $\mathcal{N}_T = uof$, $\mathcal{N}_I = vof$ and $\mathcal{N}_F = wof$ is the pre-image of $\mathcal{N}' = (\mathcal{K}', u, v, w)$. Since $\mathcal{N}' = (\mathcal{K}', u, v, w)$ is a fuzzy neutrosophic H-ideal of BCK-algebra $\mathcal{K}'$ we have $u(0') \geq u(f(\zeta)) = \mathcal{N}_T(\zeta), v(0') \geq v(f(\zeta)) = \mathcal{N}_I(\zeta), w(0') \leq w(f(\zeta)) = \mathcal{N}_F(\zeta)$. On the other hand $u(0') = u(f(0)) = \mathcal{N}_T(0), v(0') = v(f(0)) = \mathcal{N}_I(0), w(0') = w(f(0)) = \mathcal{N}_F(0)$. Therefore $\mathcal{N}_T(0) \geq \mathcal{N}_T(\zeta)$, $\mathcal{N}_I(0) \geq \mathcal{N}_I(\zeta)$ and $\mathcal{N}_F(0) \leq \mathcal{N}_F(\zeta)$ for all $\zeta \in \mathcal{K}$. Now we show that $\mathcal{N}_T(\zeta * \theta) \geq min\,\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$;
$\mathcal{N}_I(\zeta * \theta) \geq min\,\{\mathcal{N}_I(\zeta * (\eta * \theta)), \mathcal{N}_I(\eta)\}$ and
$\mathcal{N}_F(\zeta * \theta) \leq max\,\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\}$ for any $\zeta, \eta, \theta \in \mathcal{K}$. We have $\mathcal{N}_T(\zeta * \theta) = u(f(\zeta * \theta)) = u(f(\zeta) * f(\theta)) \geq min\left\{u(f(\zeta) * (\eta' * f(\theta))), u(\eta')\right\}$ for $\eta' \in \mathcal{K}'$. Since f is onto homomorphism, there is $\eta \in \mathcal{K}$ such that $f(\eta) = \eta'$. Thus,

$$\begin{aligned}
\mathcal{N}_T(\zeta * \theta) &\geq min\{u(f(\zeta) * (\eta' * f(\theta))), u(\eta')\} \\
&= min\,\{u(f(\zeta) * (f(\eta) * f(\theta))), u(f(\eta))\} \\
&= min\,\{u(f(\zeta * (\eta * \theta))), u(f(\eta))\} \\
&= min\,\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\} \text{ for all } \zeta, \eta, \theta \in \mathcal{K}.
\end{aligned}$$

Therefore, the result $\mathcal{N}_T(\zeta * \theta) \geq min\,\{\mathcal{N}_T(\zeta * (\eta * \theta)), \mathcal{N}_T(\eta)\}$ is true for all $\zeta, \eta, \theta \in \mathcal{K}$, because $\eta^{'}$ is an arbitrary element of $\mathcal{K}^{'}$ and f is onto mapping. Now $\mathcal{N}_I(\zeta * \theta) = w(f(\zeta * \theta)) = w(f(\zeta) * f(\theta)) \leq max\left\{w(f(\zeta) * (\eta^{'} * f(\theta))), w(\eta^{'})\right\}$ for $\eta^{'} \in \mathcal{K}^{'}$. Since f is onto homomorphism, there is $\eta \in \mathcal{K}$ such that $f(\eta) = \eta^{'}$. Thus,

$$\begin{aligned}\mathcal{N}_I(\zeta * \theta) &\leq max\{w(f(\zeta) * (\eta^{'} * f(\theta))), w(\eta^{'})\}\\ &= max\,\{w(f(\zeta) * (f(\eta) * f(\theta))), w(f(\eta))\}\\ &= max\,\{w(f(\zeta * (\eta * \theta))), w(f(\eta))\}\\ &= max\{\mathcal{N}_I(\zeta * (\eta * \theta)), \mathcal{N}_I(\eta)\} \text{ for all } \zeta, \eta, \theta \in \mathcal{K}.\end{aligned}$$

Therefore, the result $\mathcal{N}_I(\zeta * \theta) \leq max\,\{\mathcal{N}_I(\zeta * (\eta * \theta)), \mathcal{N}_I(\eta)\}$ is true for all $\zeta, \eta, \theta \in \mathcal{K}$, because $\eta^{'}$ is an arbitrary element of $\mathcal{K}^{'}$ and f is onto mapping. Similarly, we can prove $\mathcal{N}_F(\zeta * \theta) \leq max\,\{\mathcal{N}_F(\zeta * (\eta * \theta)), \mathcal{N}_F(\eta)\}$ for all$\zeta, \eta, \theta \in \mathcal{K}$. Hence the pre-image $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ of $\mathcal{N}^{'} = (\mathcal{K}^{'}, u, v, w)$ under f is a fuzzy neutrosophic H-ideal of BCK-algebra $\mathcal{K}$. □

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