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LIGHTLIKE HYPERSURFACES OF AN INDEFINITE QUASI-SASAKIAN STATISTICAL MANIFOLD

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Abstract

Introducing an indefinite quasi-Sasakian statistical manifold and exploring its lightlike hypersurfaces are the two main goals of this research. We establish certain relationships between induced geometrical objects and dual connections on a lightlike hypersurface of an indefinite quasi-Sasakian statistical manifold. Furthermore, we provide examples of indefinite quasi-Sasakian statistical manifolds and their lightlike hypersurfaces.

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1. Introduction

A statistical manifold is a new mathematical domain that uses tools from differential geometry to model information, analyze statistical inference, study information loss, and estimation [16]. Statistical manifolds have numerous uses in fields including artificial intelligence, machine learning, and neural networks [2, 7]. Information geometry is a significant and fascinating field in statistical manifolds. According to information geometry, the statistical manifolds are defined as follows.
An important and interesting area in statistical manifolds is information geometry. From information geometry [21] point of view, the statistical manifolds are defined as following.
Let $P(\chi)$ be the collection of probabilities specified on $\chi \subset R$ as shown below.

$$P(\chi) = \{p(m) : \chi \to R \mid p(m) > 0, \int_{\chi} p(m)d(m) = 1\}.$$

Suppose $x = [x^1, x^2, x^3, ..., x^n] \in O \subset R^n$, then $\widetilde{M}^0 = \{p(m, x) \in P(\chi) \mid m \in \chi,\ x \in O\}$ is statistical model (manifold). Let $l(m, x) = log\ p(m, x)$ and $\partial_i^l = \frac{\partial l}{\partial x_i}$, $\forall\ i = 1, 2...n$. Define the component of inner product g by

$$g_{ij} = \int \partial_i l(m, x)\partial_j l(m, x)p(m, x)dx.$$

$g = [g_{ij}]$ is Fisher information metric of $\widetilde{M}^0$ which is symmetric and positive semi-definite. Taking into account $\alpha \in R$ and

$$L_\alpha(p) = \begin{cases} \frac{2}{1-\alpha} p^{\frac{1-\alpha}{2}}, & \alpha \neq 1 \\ \log p, & \alpha = 1 \end{cases}$$

we insert $\Gamma^\alpha_{ijk} = \int \partial_i \partial_j L_\alpha(p(m,x)) \partial_k L_{-\alpha}(p(m,x)) dm$. The following equation utilizis the function Γ^α_{ijk} to define affine connections ∇^α:
$g(\nabla^\alpha_{\partial_i} \partial_j,\ \partial_k) = \Gamma^\alpha_{ijk}$. The connections ∇^α are known as α-connections in information geometry. If g is Riemannian metric, the Levi-Civita connection is represented by ∇^0. The α-connections ∇^α satisfy the following condition:

$$Pg(R, Z) = g(\nabla^\alpha_P R, Z) + g(\nabla^{-\alpha}_P Z, R).$$

In 1975, Effron [3] tried to express the significance of differential geometry in statistics. The concept of statistical manifolds was developed by Ammari's idea for α-connections [19, 20]. Infact, a statistical manifold of probability space is a Riemannian manifold $(\widetilde{M}^0, g)$ that, under certain conditions, admits dual connections ∇^α and $\nabla^{-\alpha}$. Further, Vos [17] also discovered the basic equations of submanifolds of statistical manifolds. For submanifolds of statistical manifolds with constant curvature, Aydin [11] developed generalised Weingarten inequalities. Prasad [18] studied Transversal hypersurface of Kenmotsu manifolds. Furuhata [4] explores hypersurfaces of statistical manifolds. Balgeshir [12] studied submanifolds of Sasakian statistical manifolds. Bahadir [14] studied lightlike geometry of an indefinite Sasakian statistical manifolds, Ahmad [13] studied lightlike submanifolds of an indefinite LP-Sasakian statistical manifold.
These studies act as our motivation while we define indefinite quasi-Sasakian statistical manifolds and investigate their lightlike geometry. The paper is organized as follows: In section 2, we define statistical manifolds from differential geometry point of view. Further, an indefinite quasi-Sasakian statistical manifold is defined and some results are provided for future implementation. In this section, indefinite quasi-Sasakian statistical manifolds are introduced, and their characterisation theorem is established. At last, an example on an indefinite quasi-Sasakian manifold is provided.
Section 3 examines lightlike hypersurfaces of indefinite quasi-Sasakian statistical manifolds. We describe the parallelness, totally geodicity and integrability of certain distributions.

2. Preliminaries

Let $(\widetilde{M}^0, \widetilde{g})$ be an $(m+2)-$dimensional semi-Riemannian manifold with index $(\widetilde{g}) = q \geqslant 1$. Assume that (M, g) is a hypersurface of $(\widetilde{M}^0, \widetilde{g})$ with $g = \widetilde{g}|_M$. According to citations from [8–10], M is referred to as a lightlike (null or degenerate) hypersuface if the induced metric g on M is degenerate. In this instance, a null vector field called $\zeta \neq 0$ exists on M such that

$$g(\zeta, P) = 0,\ \forall\ \ P \in \Gamma(TM).$$

At any point p in M, the radical or null space of $T_P M$ is a subspace $Rad\ T_p M$ defined by

$$Rad\ T_p M = \{\ \zeta \in T_p M : g_p(\zeta, P) = 0, P \in \Gamma(TM)\ \}.$$

For a lightlike hypersurface of $(\widetilde{M}^0, \overline{g})$ the nullity degree of g is 1. Given that any null vector is orthogonal to itself and that g is degenerate, $T_p M^\perp$ is also null and

$$Rad\ T_p M = T_p M \cap T_p M^\perp.$$

We have $Rad\ T_p M = T_p M^\perp$ because $\dim T_p M^\perp = 1$ and $\dim Rad\ T_p M = 1$. It is spanned by the null vector field ζ and is referred to as the radical distribution $Rad\ TM$. The screen bundle of M is also known as the complementary vector bundle $S(TM)$ of the Rad;TM in the TM. We point out that any screen bundle is non-degenerate. This implies that

$$TM = Rad\ (TM) \perp S(TM), \tag{2.1}$$

where the orthogonal direct sum is denoted by $\perp$. Screen transversal bundle, which has rank 2, is the complementary vector bundle of $S(TM)^\perp$ of $S(TM)$ in $T\widetilde{M}^0$. Due to the fact that $Rad\ TM$ is a lightlike subbundle of $S(TM)^\perp$ there exist a unique local section N of $S(TM)^\perp$ such that

$$\overline{g}(N, N) = 0, \quad \overline{g}(\zeta, N) = 1.$$

In this case, N is transversal to M and $\{\zeta, N\}$ is a local frame field of $S(TM)^\perp$, and there exists a line subbundle $ltr(TM)$ of $T\widetilde{M}^0$, known as the lightlike transversal bundle and locally spanned by N. As a result, the following decompositions exist:

$$T\widetilde{M}^0 = TM \oplus ltr(TM) = S(TM) \perp Rad\ TM \oplus ltr(TM), \tag{2.2}$$

Whereas $\oplus$ is the direct sum but not an orthogonal one [8, 9]. We have the following Gauss and Weingarten formulas, respectively, in light of the spilitting (2.2),

$$\overline{\nabla}_P R = \nabla_P R + h(P, R), \tag{2.3}$$

$$\overline{\nabla}_P N = -A_N P + \nabla^t_P N \tag{2.4}$$

for any $P, R \in \Gamma(TM)$, where $\nabla_P R,\ A_N P \in \Gamma(TM)$ and $h(P, R),\ \nabla^t_P N \in \Gamma(ltr(TM))$. If we set $B(P, R) = \overline{g}(h(P, R), \zeta)$ and $\tau(P) = \overline{g}((\nabla^t_P N), \zeta)$, then (2.3) and (2.4) becomes

$$\overline{\nabla}_P R = \nabla_P R + B(P, R)N, \tag{2.5}$$

$$\overline{\nabla}_P N = -A_N P + \tau(P)N \tag{2.6}$$

respectively.The shape operator and second fundamental form of the lightlike hypersurfaces M, respectively, are denoted by the letters A and B in this context [8]. Let J be the projection of $S(TM)$ on M. Then, we can write any $P \in \Gamma(TM)$,

$$P = JP + \eta(P)\zeta,$$

where η is a 1-form given by

$$\eta(P) = \overline{g}(P, N).$$

According to (2.5), we have

$$(\nabla_P g)(R, Z) = B(P, R)\eta(Z) + B(P, Z)\eta(R)$$

for all $P, R, Z \in \Gamma(TM)$, in which the induced Connection ∇ is a non-metric connection on M. Using (2.1), we have

$$\nabla_P W = \nabla^*_P W + h^*(P, W) = \nabla^*_P W + C(P, W)\zeta,$$

$$\nabla_P \zeta = -A^*_\zeta P - \tau(P)\zeta$$

for all $P \in \Gamma(TM)$, $W \in \Gamma(S(TM))$, where $\nabla^*_P W$ and $A^*_\zeta P$ belongs to $\Gamma(S(TM))$. In this context, C, A^*_ζ, and ∇^* are referred to as the local second fundamental form, the local shape operator, and the induced connection on $S(TM)$, respectively. Moreover, we have

$$g(A^*_\zeta P, W) = B(P, W), \quad g(A^*_\zeta P, N) = 0, \quad B(P, \zeta) = 0, \quad g(A_N P, N) = 0. \tag{2.7}$$

Furthermore, we have the following from the first and third equations of (2.7)

$$A^*_\zeta \zeta = 0.$$

We now define a few fundamental statistical concepts.

Definition 2.1. [4] Assume that $\widetilde{M}^0$ is a smooth manifold. Let $\widetilde{D}$ be an affine connection with the torsion tensor $T^{\widetilde{D}}$ and $\widetilde{g}$ a semi-Riemannian metric on $\widetilde{M}^0$. If so, the pair $(\widetilde{D}, \widetilde{g})$ is referred to as statistical structure on $\widetilde{M}^0$ if

(1) $(\widetilde{D}_P \widetilde{g})(R, Z) - (\widetilde{D}_R \widetilde{g})(P, Z) = \widetilde{g}(T^{\widetilde{D}}(P, R), Z)$, for all $P, R, Z \in \Gamma(T\widetilde{M}^0)$ and
(2) $T^{\widetilde{D}} = 0$.

Definition 2.2. A semi-Riemannian manifold is denoted as $(\widetilde{M}^0, \widetilde{g})$. With regard to the metric $\widetilde{g}$ two affine connections $\widetilde{D}$ and $\widetilde{D}^*$ on $\widetilde{M}^0$ are said to be dual, if

$$Z\widetilde{g}(P, R) = \widetilde{g}(\widetilde{D}_Z P, R) + \widetilde{g}(P, \widetilde{D}^*_Z R) \tag{2.8}$$

for all $P, R, Z \in \Gamma(T\widetilde{M}^0)$.
A statistical manifold will be represented by $(\widetilde{M}^0, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. If $\widetilde{\nabla}$ is Levi-Civita connection of $\widetilde{g}$, then

$$\widetilde{\nabla} = \frac{1}{2}(\widetilde{D} + \widetilde{D}^*). \tag{2.9}$$

In (2.8), if we set $\widetilde{D}^* = \widetilde{D}$, then Levi-Civita connection is formed.

Lemma 2.3. *[6] For statistical manifold $(\widetilde{M}^0, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$, we set $\overline{\mathbb{K}} = \widetilde{D} - \widetilde{\nabla}$. Then we have*

$$\overline{\mathbb{K}}(P,R) = \overline{\mathbb{K}}(R,P), \quad \widetilde{g}(\overline{\mathbb{K}}((P,R),Z) = \widetilde{g}(\overline{\mathbb{K}}((P,Z),R) \tag{2.10}$$

for any $P, R, Z \in \Gamma(TM)$.
Conversely, for a Riemannian metric g, if $\overline{\mathbb{K}}$ satisfies (2.10), the pair $(\widetilde{D} = \widetilde{\nabla} + \overline{\mathbb{K}}, \ \widetilde{g})$ is statistical structure on $\widetilde{M}^0$.

Suppose that (M, g) is a submanifold of $(\widetilde{M}^0, \widetilde{g})$. If (M, g, D, D^*) is statistical manifold, then (M, g, D, D^*) is known to as statistical submanifold of $(\widetilde{M}^0, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$, where D, D^* are affine dual connections on M and $\widetilde{D}, \widetilde{D}^*$ are affine dual connections on $\widetilde{M}^0$ [4, 17, 20]
Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}^0, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Therefore, Gauss and Weingarten formulas with respect to dual connections are presented by [4]

$$\widetilde{D}_P R = D_P R + B(P,R)N, \tag{2.11}$$

$$\widetilde{D}_P N = -A_N P + \tau(P)N, \tag{2.12}$$

$$\widetilde{D}^*_P R = D^*_P R + B^*(P,R)N, \tag{2.13}$$

$$\widetilde{D}^*_P N = -A^*_N P + \tau^*(P)N \tag{2.14}$$

for all $P, R \in \Gamma(TM)$, $N \in \Gamma(ltrTM)$ and $D_P R$, $D^*_P R$, $A_N P$, $A^*_N P \in \Gamma(TM)$ and

$$B(P,R) = \widetilde{g}(\widetilde{D}_P R, \zeta), \ \tau(P) = \widetilde{g}(\widetilde{D}_P N, \zeta),$$

$$B^*(P,R) = \widetilde{g}(\widetilde{D}^*_P R, \zeta), \ \tau^*(P) = \widetilde{g}(\widetilde{D}^*_P N, \zeta).$$

Here, the induced connections on M, the second fundamental forms and the Weingarten mappings with respect to $\widetilde{D}$ and $\widetilde{D}^*$ are denoted by D, D^*, B, B^*, A_N and A^*_N respectively.
Using the equation (2.8) and Gauss formulas, we obtain

$$Pg(R,Z) = g(\widetilde{D}_P R, Z) + g(R, \widetilde{D}^*_P Z) = g(D_P R, Z) + g(R, D^*_P Z) + B(P,R)\eta(Z) + B^*(P,Z)\eta(R). \tag{2.15}$$

In consequence of equation (2.15), we have the follwing result.

Theorem 2.4. *[15] Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}^0, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then*
(i) Induced connections D and $\widetilde{D}^*$do not necessarily have to be dual.
(ii) With respect to the dual connections, a lightlike hypersurface of a statistical manifold need not be a statistical manifold.
In (2.15), using Gauss and Weingarten formulas, we obtain

$$(D_P g)(R,Z) + (D^*_P g)(R,Z) = B(P,R)\eta(Z) + B(P,Z)\eta(R) + B^*(P,R)\eta(Z) + B^*(P,Z)\eta(R). \tag{2.16}$$

Proposition 2.5. *[15] Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}^0, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then the following assertions are true:*

(i) Induced connections D and D^* are symmetric connection.

(ii) The second fundamental forms B and B^* are symmetric.

Proposition 2.6. *[15] Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}^0, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then second fundamental form B and B^* are not degenerate.*

Additionally, due to $\widetilde{D}$ and $\widetilde{D}^*$ are dual connections we obtain

$$B(P, R) = g(\overline{A}^*_\zeta P, R) + B^*(P, \zeta)\eta(R), \tag{2.17}$$

$$B^*(P, R) = g(\overline{A}_\zeta P, R) + B(P, \zeta))\eta(R). \tag{2.18}$$

Using (2.17) and (2.18) we get

$$\overline{A}^*_\zeta \zeta + \overline{A}_\zeta \zeta = 0.$$

[14] A differentiable semi-Riemannian manifold $(\widetilde{M}^0, \widetilde{g})$ of dimension $n = 2m + 1$, a (1,1) tensor field $\widetilde{\phi}$, a contravariant vector field v, a 1-form η and a Riemannian metric $\widetilde{g}$ should be admitted, which satisfy

$$\widetilde{\phi}v = 0,\ \eta(\widetilde{\phi}P) = 0,\ \eta(v) = \epsilon, \tag{2.19}$$

$$\widetilde{\phi}^2(P) = -P + \eta(P)v,\ \widetilde{g}(P, v) = \epsilon\eta(P), \tag{2.20}$$

$$\widetilde{g}(\widetilde{\phi}P, \widetilde{\phi}R) = \widetilde{g}(P, R) - \epsilon\eta(P)\eta(R),\ \epsilon = \mp 1 \tag{2.21}$$

for all the vector fields P, R on $\widetilde{M}^0$. When an almost contact metric manifold performs

$$(\widetilde{\nabla}_P\widetilde{\phi})R = -\epsilon\eta(R)AP - \widetilde{g}(AP, R)v, \tag{2.22}$$

$$\widetilde{\nabla}_P v = A\widetilde{\phi}P. \tag{2.23}$$

$\widetilde{M}^0$ is known as an indefinite quasi-Sasakian manifold [1, 22]. We make the assumption that the vector field v in this study is spacelike.

Definition 2.7. Let $(\widetilde{g}, \widetilde{\phi}, v)$ be an indefinite quasi-Sasakian structure on $\widetilde{M}^0$. A quadruplet $(\widetilde{D} = \widetilde{\nabla} + \mathbb{K}, \widetilde{g}, \widetilde{\phi}, v)$ is called an indefinite quasi-Sasakian statistical structure on $\widetilde{M}^0$ if $(\widetilde{D}, \widetilde{g})$ is a statistical structure on $\widetilde{M}^0$ and the formula

$$\overline{\mathbb{K}}(P, \widetilde{\phi}R) = -\widetilde{\phi}\mathbb{K}(P, R) \tag{2.24}$$

holds for any $P, R \in \Gamma(T\widetilde{M}^0)$.Then $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$ is said to an indefinite quasi-Sasakian statistical manifold.

An indefinite quasi-Sasakian statistical manifold will be represented by $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. We remark that if $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$ is an indefinite quasi-Sasakian statistical manifold, so is $(\widetilde{M}^0, \widetilde{D}^*, \widetilde{g}, \widetilde{\phi}, v)$ [5, 6].

THEOREM 2.8. *Let* $(\widetilde{M}^0, \widetilde{D}, \widetilde{g},)$ *be a statistical manifold and* $(\widetilde{g}, \widetilde{\phi}, v)$ *an almost contact metric structure on* $\widetilde{M}^0$. $(\widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$ *is an indefinite quasi-Sasakian statistical structure if and only if the following conditions hold:*

$$\widetilde{D}_P\widetilde{\phi}R - \widetilde{\phi}\widetilde{D}^*_P R = \widetilde{g}(R, v)AP - \widetilde{g}(AP, R)v, \tag{2.25}$$

$$\widetilde{D}_P v = \widetilde{\phi}(AP) + \widetilde{g}(\widetilde{D}_P v, v)v \tag{2.26}$$

for all the vector fields P, R *on* $\widetilde{M}^0$.

Proof. Using $(\overline{\mathbb{K}} = \widetilde{D} - \widetilde{\nabla})$ we get

$$\widetilde{D}_P\widetilde{\phi}R - \widetilde{\phi}\widetilde{D}^*_P R = (\widetilde{\nabla}_P\widetilde{\phi})R + \overline{\mathbb{K}}(P, \widetilde{\phi}R) + \widetilde{\phi}\overline{\mathbb{K}}(P, R)$$

for all vector P and R on $\widetilde{M}^0$. If we consider Definition 2.7 and the equation (2.22), we have the formula (2.25). If we take $\widetilde{D}^*$ instead of $\widetilde{D}$ in (2.25), we have

$$\widetilde{D}^*_P\widetilde{\phi}R - \widetilde{\phi}\widetilde{D}_P R = \widetilde{g}(R, v)AP - \widetilde{g}(AP, R)v. \tag{2.27}$$

Substituting v for R in (2.27), we have the equation (2.26).
Conversely using (2.25), we have

$$\widetilde{\phi}(\widetilde{D}_P\widetilde{\phi}^2 R - \widetilde{\phi}\widetilde{D}^*_P\widetilde{\phi}R) = 0.$$

Assume (2.20) and (2.26) as well, we get

$$0 = -\widetilde{\phi}\widetilde{D}_P R - \widetilde{g}(R, v)AP + \widetilde{g}(AP, v)v + \widetilde{D}^*_P\widetilde{\phi}R - \widetilde{g}(\widetilde{\phi}P, \widetilde{\phi}R)v.$$

From (2.21), this equation gives us (2.27). We will now use (2.25) and (2.27) to prove (2.22) and (2.24).

$$(\widetilde{\nabla}_P\widetilde{\phi})R - \widetilde{g}(R, v)AP + \widetilde{g}(AP, R)v = \overline{\mathbb{K}}(P, \widetilde{\phi}R) + \widetilde{\phi}\overline{\mathbb{K}}(P, R),$$

and

$$(\widetilde{\nabla}_P\widetilde{\phi})R - \widetilde{g}(R, v)AP + \widetilde{g}(AP, R)v = -\overline{\mathbb{K}}(P, \widetilde{\phi}R) - \widetilde{\phi}\overline{\mathbb{K}}(P, R).$$

The last two equations satisfy (2.22) and (2.24).

Example 2.9 Let $(R^3_1, \widetilde{g}, \widetilde{M}^0)$ of the three dimensional semi-Riemannian manifold, where $\widetilde{g}$ is of the signature $(+, +, -)$ with respect to the canonical basis (ρ_1, ρ_2, ρ_3) and (u_1, u_2, u_3) be the standard coordinate system of $\widetilde{M}^0$. Then the vector fields

$$\rho_1 = u_3\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_3},$$

$$\rho_2 = -\frac{\partial}{\partial u_2} \quad and \quad \rho_3 = \frac{\partial}{\partial u_3}$$

are linearly independent at each point of $\widetilde{M}^0$.
Let $\widetilde{g}$ be a semi-Riemannian metric defined by

$$\widetilde{g}(\rho_1, \rho_3) = \widetilde{g}(\rho_2, \rho_3) = \widetilde{g}(\rho_1, \rho_2) = 0,$$

$$\widetilde{g}(\rho_1,\rho_1) = \widetilde{g}(\rho_2,\rho_2) = 1, \widetilde{g}(\rho_3,\rho_3) = -1.$$

Let $\widetilde{\phi}$ be an $(1,1)$ tensor field defined by

$$\widetilde{\phi}(\rho_1) = \rho_2,\ \widetilde{\phi}(\rho_2) = \rho_1,\ \widetilde{\phi}(\rho_3) = 0$$

and η be the 1-form defined by $\eta(Z) = \widetilde{g}(Z,\rho_3)$ for any Z is vector field on $\widetilde{M}^0$,
then by the linearity property of η and $\widetilde{\phi}$ we have $\eta(\rho_3) = -1$,

$$\widetilde{\phi}^2 Z = -Z + \eta(Z)\rho_3,\ \ \widetilde{g}(\widetilde{\phi}Z,\widetilde{\phi}W) = \widetilde{g}(Z,W) - \eta(Z)\eta(W)$$

for all vector fields Z, W on $\widetilde{M}$. Taking $\rho_3 = v$, $(\widetilde{\phi}, v, \eta, g)$ defines an almost contact structure. Let ∇ be the Levi-Civita connection with respect to the Riemannian metric $\widetilde{g}$, then we have

$$[\rho_1,\rho_2] = -\rho_3, \quad [\rho_1,\rho_3] = \rho_1, \quad [\rho_2,\rho_3] = 0.$$

In view of Koszul's formula, we have

$$\widetilde{\nabla}_{\rho_1}\rho_3 = \rho_2\ , \qquad \widetilde{\nabla}_{\rho_1}\rho_2 = -\rho_3\ , \qquad \widetilde{\nabla}_{\rho_1}\rho_1 = 0,$$

$$\widetilde{\nabla}_{\rho_2}\rho_3 = \rho_1\ , \qquad \widetilde{\nabla}_{\rho_2}\rho_1 = \rho_3\ , \qquad \widetilde{\nabla}_{\rho_2}\rho_2 = 0,$$

$$\widetilde{\nabla}_{\rho_3}\rho_1 = -\rho_2\ , \qquad \widetilde{\nabla}_{\rho_3}\rho_2 = \rho_1\ , \qquad \widetilde{\nabla}_{\rho_3}\rho_3 = 0.$$

We observe that $(\widetilde{\phi}, v, \eta, \widetilde{g})$ satisfied the formula $\widetilde{\nabla}_X v = A\widetilde{\phi}X$.
Hence, $\widetilde{M}$ is a three-dimensionalindefinite quasi-Sasakian manifolds with the structure function $A = 1$. If we choose difference tensor $\mathbb{K}(X,Y) = \widetilde{g}(Y,v)\widetilde{g}(X,v)v$, then $(\widetilde{D} = \widetilde{\nabla} + \mathbb{K}, \widetilde{g}, \widetilde{\phi}, v)$ is an indefinite quasi-Sasakian statistical manifold on $\widetilde{M}$.

Example 2.10 Let $(R^3_1, \widetilde{g}, \widetilde{M}^0)$ of the three dimensional semi-Riemannian manifold, where $\widetilde{g}$ is of the signature $(+,+,-)$ with respect to the canonical basis (r_1, r_2, r_3) and (v_1, v_2, v_3) be the standard coordinate system of $\widetilde{M}^0$. Then the vector fields

$$r_1 = e^{-v_3}\frac{\partial}{\partial v_1},$$

$$r_2 = e^{-v_3}(\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}),$$

$$r_3 = -\frac{\partial}{\partial v_3}$$

are linearly independent at each point of $\widetilde{M}^0$.
Let $\widetilde{g}$ be a semi-Riemannian metric defined by

$$\widetilde{g}(r_1, r_3) = \widetilde{g}(r_2, r_3) = \widetilde{g}(r_1, r_2) = 0,$$

$$\widetilde{g}(r_1, r_1) = \widetilde{g}(r_2, r_2) = 1,\ \ \widetilde{g}(r_3, r_3) = -1.$$

Let $\widetilde{\phi}$ be a $(1,1)-$tensor field defined by

$$\widetilde{\phi}(r_1) = -r_1,\ \widetilde{\phi}(r_2) = -r_2,\ \widetilde{\phi}(r_3) = 0$$

and η be the 1-form defined by $\eta(Z) = \widetilde{g}(Z, r_3)$ for any Z is vector field on $\widetilde{M}^0$,

then by the linearity property of η and $\widetilde{\phi}$ we have $\eta(r_3) = -1$,

$$\widetilde{\phi}^2 Z = -Z + \eta(Z) r_3,\ \ \widetilde{g}(\widetilde{\phi}Z, \widetilde{\phi}W) = \widetilde{g}(Z, W) - \eta(Z)\eta(W)$$

for all vector fields $Z,\ W$ on $\widetilde{M}$. Taking $r_3 = u$, $(\widetilde{\phi}, u, \eta, g)$ defines an almost contact structure. Let ∇ be the Levi-Civita connection with respect to the Riemannian metric $\widetilde{g}$, then we have

$$[r_1, r_2] = 0, \qquad [r_1, r_3] = -r_1, \qquad [r_2, r_3] = -r_2.$$

In view of Koszul's formula, we have

$$\begin{aligned}
\widetilde{\nabla}_{r_1} r_3 &= -r_1\ , & \widetilde{\nabla}_{r_1} r_2 &= 0\ , & \widetilde{\nabla}_{r_1} r_1 &= -r_3,\\
\widetilde{\nabla}_{r_2} r_3 &= -r_2\ , & \widetilde{\nabla}_{r_2} r_1 &= 0\ , & \widetilde{\nabla}_{r_2} r_2 &= -r_3,\\
\widetilde{\nabla}_{r_3} r_1 &= 0\ , & \widetilde{\nabla}_{r_3} r_2 &= 0\ , & \widetilde{\nabla}_{r_3} r_3 &= 0.
\end{aligned}$$

We observe that $(\widetilde{\phi}, u, \eta, \widetilde{g})$ satisfied the formula $\widetilde{\nabla}_X v = A\widetilde{\phi}X$.
Hence, $\widetilde{M}$ is a three-dimensional indefinite quasi-Sasakian manifolds with the structure function $A = -1$. If we choose difference tensor $\mathbb{K}(X, Y) = \widetilde{g}(Y, u)\widetilde{g}(X, u)u$, then $(\widetilde{D} = \widetilde{\nabla} + \mathbb{K}, \widetilde{g}, \widetilde{\phi}, u)$ is an indefinite quasi-Sasakian statistical manifold on $\widetilde{M}$.

3. Lightlike hypersurface of indefinite quasi-Sasakian statistical manifolds

Definition 3.1. Let (M, g, D, D^*) be a hypersurface of an indefinite quasi-Sasakian statistical manifold $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$.The quadruplet (M, g, D, D^*) is called lightlike hypersurface of an indefinite quasi-Sasakian statistical manifold $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$ if the induced metric g is degenerate.

Let $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$ be a $(2m+1)-$ dimensional quasi-Sasakian statistical manifold and (M, g) be a lightlike hypersurface of $\widetilde{M}^0$, where v is the structure vector field is tangent to M. For any $\zeta \in \Gamma(Rad\ TM)$ and $N \in \Gamma(ltr(TM))$, in view of (2.19)-(2.21), we have

$$\widetilde{g}(\zeta, v) = 0,\ \widetilde{g}(N, v) = 0, \tag{3.1}$$

$$\widetilde{\phi}^2\zeta = -\zeta,\ \widetilde{\phi}^2 N = -N.$$

Also, using (2.11) and (2.26) we obtain

$$B(\zeta, v) = 0,\ B(v, v) = 0,$$

$$B^*(\zeta, v) = 0,\ B^*(v, v) = 0.$$

Proposition 3.2. *Let* $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$ *be a* $(2m+1)$*-dimensional indefinite quasi-Sasakian statistical manifold and* (M, g, D, D^*) *be its lightlike hypersurface such that the structure vector field* v *is tangent to* M*. Then we have*

$$g(\widetilde{\phi}A\zeta, \zeta) = 0, \tag{3.2}$$

$$g(\widetilde{\phi}A\zeta, N) = -g(\zeta, \widetilde{\phi}AN) = g(A^*_N\zeta, v), \tag{3.3}$$

$$g(\widetilde{\phi}A\zeta, \widetilde{\phi}AN) = A^2, \tag{3.4}$$

where ζ *is a local section of* $Rad\ TM$ *and* N *is a local section of* $ltr(TM)$*.*

Proof. Using (2.11) and (2.26), we have

$$\begin{aligned} g(\widetilde{\phi}A\zeta, \zeta) &= g(\widetilde{D}_\zeta v - \widetilde{g}(\widetilde{D}_\zeta v, v)v, \zeta) \\ &= g(D_\zeta v + B(\zeta, v)N, \zeta), \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} g(\widetilde{\phi}A\zeta, N) &= g(\widetilde{D}_\zeta v - \widetilde{g}(\widetilde{D}_\zeta v, v)v, N) \\ &= -g(v, \widetilde{D}^*_\zeta N) \\ &= g(A^*_N\zeta, v). \end{aligned}$$

From (2.21) and (3.1), we have

$$\begin{aligned} g(\widetilde{\phi}A\zeta, \widetilde{\phi}AN) &= g(A\zeta, AN) - \epsilon\eta(A\zeta)\epsilon\eta(AN) \\ &= A^2 g(\zeta, N) - \epsilon A g(\zeta, v) A.g(N, v) \\ A^2.g(\zeta, N) &= A^2. \end{aligned}$$

In view of Proposition 3.2 it is possible to make the following decomposition :

$$S(TM) = \{\widetilde{\phi}Rad\ TM \oplus \widetilde{\phi}ltr(TM)\} \perp L_0 \perp < v >, \tag{3.5}$$

where L_0 is non-degenerate and $\widetilde{\phi}$-invariant distribution of rank $2m-4$ on M. If we denote the following distributions on M

$$L = Rad\ TM \perp \widetilde{\phi}Rad\ TM \perp L_0, \quad L' = \widetilde{\phi}ltr(TM), \tag{3.6}$$

then L is invariant and L' is anti-invariant distributions under $\widetilde{\phi}$. We also have

$$TM = L \oplus L' \perp < v > . \tag{3.7}$$

The two null vector fields U and W, as well as their 1-forms u and w, are now considered as follows:

$$U = -\widetilde{\phi}N, \ u(P) = \widetilde{g}(P, W), \tag{3.8}$$

$$W = -\widetilde{\phi}\zeta, \ w(P) = \widetilde{g}(P, U). \tag{3.9}$$

Then, for any $P \in \Gamma(T\widetilde{M}^0)$, we have

$$P = SP + u(P)U, \tag{3.10}$$

where S projection morphism of $T\widetilde{M}^0$ on the distribution L. Applying $\widetilde{\phi}$ in (3.10), we obtain

$$\widetilde{\phi}P = \widetilde{\phi}SP + u(P)\widetilde{\phi}U,$$
$$\widetilde{\phi}P = \phi P + u(P)N, \tag{3.11}$$

where ϕ is a tensor field of type (1,1) formed by M by $\phi P = \widetilde{\phi}SP$. Again, applying ϕ to the equation (3.11) and using (2.19)-(2.21) we have

$$\widetilde{\phi}^2 P = \widetilde{\phi}\phi P + u(P)\widetilde{\phi}N,$$
$$-P + g(P, v)v = \phi^2 P - u(P)U,$$

which means that

$$\phi^2 P = -P + g(P, v)v + u(P)U. \tag{3.12}$$

Now, applying ϕ to the equation (3.12) and then since ϕU=0, we have $\phi^3 + \phi = 0$, which gives that ϕ is an f-structure.

Definition 3.3. Let (M, g, D, D^*) be a hypersurface of an indefinite quasi-Sasakian statistical manifold $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. The quadruplet (M, g, D, D^*) is called a screen semi-invariant lightlike hypersurface of an indefinite quasi-Sasakian statistical manifold $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$, if

$$\widetilde{\phi}(ltr(TM)) \subset S(TM),$$
$$\widetilde{\phi}(Rad\ TM) \subset S(TM).$$

We note that a screen semi-invariant lightlike hypersurface is a hypersurface of an indefinite quasi-Sasakian statistical manifold.
In view of (3.8) and (3.9), we have

$$\widetilde{g}(U, W) = 1.$$

Thus, $< U > \oplus < W >$ is non-degenerate vector bundle of $S(TM)$ with rank 2. If we consider (3.5) and (3.6), we get

$$S(TM) = \{U \oplus W\} \perp L_0 \perp < v >$$

and

$$L = Rad\ TM \perp < W > \perp L_0,\ L^{'} =< U > .$$

Thus, for any $X \in \Gamma(TM)$, we can write

$$P = JP + QP + g(P, v)v, \tag{3.13}$$

where J and Q are the projection of TM into L and $L^{'}$. Thus, we can write $QP = u(P)U$. Using (2.19)-(2.21), (3.11) and (3.13), we have

$$\phi^2 P = -P + g(P, v)v + u(P)U,$$

where $\widetilde{\phi}JP = \phi P$. We can easily see that

$$g(\phi P, \phi R) = g(P, R) - g(P, v)g(R, v) - u(P)w(R) - u(R)w(P)$$

for any $P, R \in \Gamma(TM)$. Also we have the following identities :

$$g(\phi P, R) = g(P, \phi R) - u(P)\eta(R) - u(R)\eta(P),$$

$$\phi v = 0, \ \ g(\phi P, v) = 0.$$

Thus, we have that if (M, g, D, D^*) be a lightlike hypersurface of an indefinite quasi-Sasakian statistical manifold $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. Then, ϕ need not be a almost contact structure.

Lemma 3.4. *Let (M, g, D, D^*)be a lightlike hypersurface of an indefinite quasi-Sasakian statistical manifold $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. For any $P, R \in \Gamma(TM)$, we have the following identities:*

$$D_P\phi R - \phi D^*_P R = -B^*(P, R)U + u(R)A_N P + g(R, v)AP - g(AP, R)v, \tag{3.14}$$

$$D_P(u(R)) - u(D^*_P R) = -B(P, \phi R) - u(R)\tau(P). \tag{3.15}$$

Proof. Using Gauss and Weingarten formulas in (2.25), we have

$$D_P\phi R + B(P, \phi R)N - u(R)A_N P + u(R)\tau(P)N + D_P(u(R))N - u(D^*_P R)N + B^*(P, R)U$$
$$= g(R, v)AP - g(AP, R)v. \tag{3.16}$$

If we take the tangential and transversal parts of (3.16), we get (3.14) and (3.15).

Lemma 3.5. *Let (M, g, D, D^*)be a lightlike hypersurface of an indefinite quasi-Sasakian statistical manifold $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. For any $P, R \in \Gamma(TM)$, we have the following identities:*

$$D^*_P\phi R - \phi D_P R = -B(P, R)U + u(R)A^*_N P + g(R, v)AP - g(AP, R)v, \tag{3.17}$$

$$D^*_P(u(R)) - u(D_P R) = -B^*(P, \phi(R)) - u(R)\tau^*(P). \tag{3.18}$$

Proof. In (2.27), using Gauss and Weingarten formulas, we have

$$D^*_P\phi R + B^*(P, \phi R)N - u(R)A^*_N P + u(R)\tau^*(P)N + D^*_P(u(R))N - u(D_P R)N + B(P, R)U$$
$$= g(R, v)AP - g(AP, R)v.$$

If we take the tangential and transversal parts of last equation, we get (3.17) and (3.18).

Proposition 3.6. *Let (M, g, D, D^*) be a lightlike hypersurface of an indefinite quasi-Sasakian statistical manifold $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. We have the following expressions for any $X, Y \in \Gamma(TM)$:*

(i) If the vector field U is parallel with respect to D^*, then

$$A_N P = u(A_N P)U + \tau(A_N P)v, \ \ \tau(P) = 0, \tag{3.19}$$

(ii) If the vector field U is parallel with respect to D, then

$$A^*_N P = u(A^*_N P)U + \tau(A^*_N P)v, \ \ \tau^*(P) = 0. \tag{3.20}$$

Proof. On replacing R in (3.14) by U, we have

$$-\phi D^*_P U = A_N P - B^*(P,U)U.$$

Applying ϕ in last equation and using (3.12), we obtain

$$D^*_P U - g(D^*_P U, v)v - u(D^*_P U)U = \phi A_N P.$$

If U is parallel with respect to D^*, then $\phi(A_N P) = 0$. From (3.11) we have $\widetilde{\phi} A_N P = u(A_N P)N$. Applying $\widetilde{\phi}$ on last equation and using (2.20) we obtain $A_N P = u(A_N P)U + \tau(A_N P)v$. Now if we take U instead of R in the equation (3.15), we have

$$\tau(P) = 0.$$

Similarly, we can also obtain (3.20) by same procedure.

Proposition 3.7. *Let (M, g, D, D^*) be a lightlike hypersurface of an indefinite quasi-Sasakian statistical manifold $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. For any $P, R \in \Gamma(TM)$, we have the following expressions:*

(i) If the vector field W is parallel with respect to D, then

$$\overline{A_\zeta} P = u(\overline{A_\zeta} P)U + g(\overline{A_\zeta} P, v)v, \;\; \tau(P) = 0, \tag{3.21}$$

(ii) If the vector field W is parallel with respect to D^*, then

$$\overline{A_\zeta}^* P = u(\overline{A_\zeta}^* P)U + g(\overline{A_\zeta}^* P, v)v, \;\; \tau^*(P) = 0. \tag{3.22}$$

Proof. Taking ζ instead of R in (3.17), we get

$$D^*_P \phi\zeta - \phi D_P \zeta = -B(P,\zeta)U + u(\zeta)A^*_N P + g(\zeta, v)AP - g(AP, \zeta)v,$$

$$-\phi D_P \zeta = -B(P,\zeta)U.$$

If W is parallel with respect to D, using (3.9) in the equation, we get

$$-\phi[-\overline{A}_\zeta P - \tau(P)\zeta] = -B(P,\zeta)U,$$

$$-\phi\overline{A}_\zeta P - \tau(P)\phi\zeta = -B(P,\zeta)U,$$

$$-\phi\overline{A}_\zeta P - \tau(P)W = -B(P,\zeta)U.$$

Applying $\widetilde{\phi}$ and using (3.12), we get

$$-\overline{A_\zeta} P + u(\overline{A_\zeta} P)U + g(\overline{A_\zeta} P, v)v = \tau(P)\zeta.$$

Comparing screen and radical parts of last equation we obtain (3.21). Similarly, we can obtain (3.22) by using (3.9) and (3.12).

Definition 3.8. [5, 23] Let (M, g, D, D^*) be a hypersurface of an indefinite quasi-Sasakian statistical manifold $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$. Then

(i) M is called totally geodesic with respect to $\widetilde{D}$ if $B = 0$.

(ii) M is called totally geodesic with respect to $\widetilde{D}^*$ if $B^* = 0$.

THEOREM 3.9. *Let* (M, g, D, D^*) *be a lightlike hypersurface of an indefinite quasi-Sasakian statistical manifold* $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$*. Then*

(i) M is totally geodesic with respect to $\widetilde{D}$ *if and only if*

$$D_P\phi R - \phi D^*_P R = g(AP, R)v, \forall P \in \Gamma(TM),\ R \in \Gamma(L), \tag{3.23}$$

$$A_N P = -\phi D^*_P U,\ \forall\ P \in \Gamma(TM). \tag{3.24}$$

(ii) M is totally geodesic with respect to $\widetilde{D}^*$ *if and only if*

$$D^*_P\phi R - \phi D_P R = g(AP, R)v,\ \ \forall\ P \in \Gamma(TM), R \in \Gamma(L), \tag{3.25}$$

$$A^*_N P = -\phi D_P U,\ \ \forall\ P \in \Gamma(TM). \tag{3.26}$$

Proof. For any $P \in \Gamma(TM)$, we know that $u(R) = 0$. Then by the equation (3.14) and (3.17), we have

$$D_P\phi R - \phi D^*_P R = -B^*(P, R)U - g(AP, R)v,$$

$$D^*_P\phi R - \phi D_P R = -B(P, R)U - g(AP, R)v.$$

Again replacing R by U in (3.14) and (3.17), respectively, we also have

$$A_N P = -\phi D^*_P U + B^*(P, U)U,$$

$$A^*_N P = -\phi D_P U + B(P, U)U.$$

Using definition (3.7) in last four equations we get our results.

THEOREM 3.10. *Let* (M, g, D, D^*) *be a screen semi-invariant hypersurface of an indefinite quasi-Sasakian statistical manifold* $(\widetilde{M}^0, \widetilde{D}, \widetilde{g}, \widetilde{\phi}, v)$*. The following assertions are equivalent :*

(i) The distribution $L\perp < v >$ *is integrable.*

(ii) $B^*(P, \phi R) = B^*(\phi P, R),\ \forall\ \ P,\ R \in \Gamma(L\perp < v >)$,

(iii) $B(P, \phi R) = B(\phi P, R),\ \forall\ \ P,\ R \in \Gamma(L\perp < v >)$.

Proof. (i) We know that $P \in \Gamma(L\perp < v >)$ if and only if $u(P) = \widetilde{g}(P, W) = 0$. For any $P,\ R \in \Gamma(L\perp < v >)$, we also know that

$$(D_P u)R = P(u(R)) - u(D_P R)$$

and

$$(D_R u)P = R(u(P)) - u(D_R P).$$

Solving above last two equations, we get

$$u[P, R] = 0.$$

This means that the distribution $(L\perp < v >)$ is integrable.
(ii) by (i) $(L\perp < v >)$ is integrable, this means that

$$u(D_R P) - u(D_P R) = 0.$$

By (3.18) we obtain (ii). Similarly, we obtain (iii) by using (3.16).

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References

[1] A. Sarkar, A. Sil and A. K. Paul, *On three-Dimensional quasi-Sasakian manifolds and magnetic curves*, Applied Mathematics E-Notes, **19** (2019) 55-64.

[2] B. Bartlett, *A^* "generative" model for computing electromagnetic field solutions*, Department of Applied Physics, Stanford University (2018). http://cs229.stanford.edu/proj2018/report/233.pdf.

[3] B. Effron, *Defining the curvature of a statistical problem (with applications to second order efficiency)*, Ann. Statist., **3** (1975), 1189-1242.

[4] H. Furuhuta, *Hypersurfaces in statistical manifolds* Differential Geom. Appl., **27** (2009), 420-429.

[5] H. Furuhata and I. Hasegawa, *Submanifold theory in holomorphic statistical manifolds*, Geometry of Cauchy-Riemann submanifolds, Singapore: Springer, (2016), 179-215.

[6] H. Furuhata, I. Hasegawa, Y. Okuyama, K. Sato and M. H. Shahid, *Sasakian statistical manifolds*, J. Geom. Phys., **117** (2017) 179-186.

[7] J.V.D. Gucht, J. Davelaar, L. Hendriks, O. Porth, H. Olivares, Y. Mizuno, M. C. Fromm and H. Falcke, *Deep Horizon: A machine learning network that recovers accreting black hole parameters*, Astronomy-Astrophysics, **636** (2020), 1-12.

[8] K. L. Duggal and A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, (1996).

[9] K. L. Duggal and D. H. Jin, *Null curves and hypersurfaces of semi-Riemannian manifolds*, Hackensack, NJ: World Scientific Publishing Co. Pte. Ltd., (2007).

[10] K. L. Duggal and B. Sahin, *Differential Geometry of lightlike submanifolds*, Frontiers in Mathematics, Basel: Birkhauser Verlag, (2010).

[11] M. E. Aydin, A. Mihai and I. Mihai, *Some inequalities on submanifolds in statistical manifolds of constant curvature*, Filomat, **29** (2015), 465-476.

[12] M. Balgeshir, *On submanifolds of Sasakian statistical manifolds*, Bol. Soc. Paran. Mat., **40** (2022) 1-6.

[13] M. Ahmad, M. Alam, *Lightlike submanifold of an indefinite LP-Sasakian statistical manifold*, SER. MATH. INFORM., FACTA UNIVERSITATIS (NIS), **38(4)** (2023), 697-711.

[14] O. Bahadir, *On lightlike geometry of indefinite Sasakian statistical manifold*, AIMS Mathematics, **11** (2021), 12845-12862.

[15] O. Bahadir and M. M. Tripathi, *Geometry of lightlike hypersurfaces of a statistical manifold*, (2019). http://arxiv.org/abs/1901.09251

[16] O. Calin and C. Udriste, *Geometric modelling in probability and statics*, Springer (2014).

[17] P. W. Vos, *Fundamental equations for statistical submanifolds with applications to the Bartlett correction*, Ann. Inst. Statist. Math., **41** (1989), 429-450.

[18] R. Prasad and M. M. Tripathi, *Transversal hypersurface of Kenmotsu manifolds*, Indian Journal of Pure and Applied Mathematics, **34 (3)** 2003, 443-452.

[19] S. Amari, *Differential Geometry of curved exponential families-curvature and information loss*, Ann. Statist., **10** (1982), 357-385.

[20] S. Amari, *Differential Geometry Methods in statistics*, In: Lecture Notes in Statistics, New York: Springer, **28** (1985).
[21] S. Amari and H. Nagaoka, *Methods of information geometry*, Oxford, U.K.: AMS/Oxford university press, **191** (2000).
[22] S. Kanemaki, *Quasi-Sasakian manifolds*, Tohoku. Math. Journ., **29** (1977), 227-233.
[23] T. Kurose, *Conformal-projective geometry of statistical manifolds*, Interdiscip. Inform. Sci., **8** (2002), 89-100.

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