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η-RICCI SOLITONS ON PARA-KENMOTSU MANIFOLDS

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Abstract

This study presents an exploration of η-Ricci solitons on a Para-Kenmotsu manifold. This work aims to examine η-Ricci solitons on Para-Kenmotsu manifolds that meet the constraint $C \cdot D = 0$. Additionally, we provide findings on η-solitons on Para-Kenmotsu manifolds that possess the properties of conformal flat, projective flat, and concircularly flat. Furthermore, we get outcomes for η-Ricci solitons in Para-Kenmotsu manifolds that possess a Codazzi type of Ricci tensor and a cyclic parallel Ricci tensor. Finally, we provide an illustrative instance of a manifold that demonstrates the presence of proper η-Ricci solitons.

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1. Introduction

The idea of Ricci flow, as well as the introduction of the notion of Ricci soliton and its existence, were proposed by Hamilton [14]. The use of this particular idea is necessary to address Thurston's geometric conjecture, which posits that a three-dimensional manifold may be decomposed geometrically if it is closed. Hamilton's classification provides a comprehensive categorization of compact four-dimensional manifolds exhibiting positive curvature. The equation governing the Ricci flow may be expressed as follows

$$\frac{\partial}{\partial t} g_{ij}(t) = -2C_{ij}. \tag{1.1}$$

A Ricci soliton is a mathematical concept that serves as an extension of an Einstein metric. It has been defined precisely on a Riemannian manifold denoted as (M, g) [7]. A Ricci soliton is defined as a triple (g, Ω, Λ), where g represents a Riemannian metric, Ω represents a vector field (referred to as a potential vector field), and Λ represents a real scalar such that

$$£_{\Omega} g + 2D + 2\Lambda g = 0, \tag{1.2}$$

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where D represent the Ricci tensor of the manifold M, and let $\pounds_{\Omega}$ symbolize the Lie derivative operator applied to the vector field Ω. According to the [14], the Ricci soliton exhibits three distinct behaviours depending on the value of the Λ shrinking, steady, and expanding for negative, zero, and positive values of Λ, respectively. A Ricci soliton in which the vector field Ω is identically zero may be simplified to the Einstein equation. The metrics that satisfy the equation (1.2) are of significant interest and use in the field of physics. They are sometimes referred to as quasi-Einstein metrics [21], [22].The equation of Ricci soliton has also been the subject of investigation by theoretical physicists in the context of string theory.
Friedmann is responsible for the earliest contributions made in this approach, and he is the one who addresses certain elements of it [9]. Ricci solitons were first presented in the field of Riemannian geometry [15] as solutions of the Ricci flow that exhibit self-similarity. These solitons have shown to be significant in the study of singularities associated with the Ricci flow. Ricci solitons have been extensively investigated in several scholarly works by multiple writers, including references [16], [17], [18], [19] and numerous others.
The concept of η-Ricci solitons was developed by Cho and Kimura [12] as an extension of Ricci solitons, specifically for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is defined as a tuple $(g, \Omega, \Lambda, \varrho)$, where Ω represents a vector field on the manifold M, and Λ and ϱ are real scalars. The Riemannian metric g associated with the soliton satisfies a certain equation

$$\pounds_{\Omega} g + 2D + 2\Lambda g + 2\varrho\eta \otimes \eta = 0, \tag{1.3}$$

where D is the Ricci tensor corresponding to the metric tensor g.
In most cases
(i). In the scenario where $\varrho = 0$, the η-Ricci soliton will be simplified to the Ricci soliton.
(ii). If the condition $\varrho \neq 0$ is satisfied, the η-Ricci soliton is referred to as a proper η-Ricci soliton.
Several authors recently carried out a study on a η-Ricci soliton, as published in multiple publications such as [2], [5], [8], [10]. Their investigations have revealed numerous intriguing geometric characteristics.
In their work, Gray [1] presents the concept of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor. A Riemannian manifold or semi-Riemannian manifold is considered to possess a cyclic parallel Ricci tensor denoted as D of type $(0, 2)$, which is non-zero and fulfills a certain condition

$$(\nabla_{M_1} D)(M_2, M_3) + (\nabla_{M_2} D)(M_3, M_1) + (\nabla_{M_3} D)(M_1, M_2) = 0 \tag{1.4}$$

A Para-Kenmotsu manifold is characterized by having a Ricci tensor D of type $(0, 2)$ that is non-zero and adheres to the Codazzi condition

$$(\nabla_{M_3} D)(M_1, M_2) = (\nabla_{M_1} D)(M_2, M_3) \tag{1.5}$$

for any values of M_1, M_2, and M_3 that lie on M.

$$(\nabla_{M_1}D)(M_2,M_3)-(\nabla_{M_2}D)(M_1,M_3)=\frac{1}{2(n-1)}\Big[(M_1a^1)g(M_2,M_3)-(M_2a^1)g(M_1,M_3)\Big], \tag{1.6}$$

where a^1 represents the scalar curvature.
The projective curvature tensor N [13] in a manifold (M,g) is defined by the following

$$N(M_1,M_2)M_3=C(M_1,M_2)M_3-\frac{1}{n-1}\Big[g(M_2,M_3)GM_1-g(M_1,M_3)GM_2\Big] \tag{1.7}$$

The Concircular curvature tensor L [16] in an n-dimensional Riemannian manifold M is defined by

$$L(M_1,M_2)M_3=C(M_1,M_2)M_3-\frac{a^1}{n-1}\Big[g(M_2,M_3)M_1-g(M_1,M_3)M_2\Big], \tag{1.8}$$

let G be the Ricci tensor operator, which is defined by the equation $D(M_1,M_2)=g(GM_1,M_2)$, where M_1, M_2, and M_3 are elements of the $\Gamma(M)$, which represents the vector fields of the manifold M.
The aforementioned findings have inspired me to investigate η-Ricci solitons on Para-Kenmotsu manifolds to satisfy specific curvature requirements. The paper is structured as follows. In the 2 section, we provide a concise overview of a Para-Kenmotsu manifold. Section 3 focuses on the examination of Ricci solitons and η-Ricci solitons in Para-Kenmotsu manifolds. In the 4 section, we examine η-Ricci solitons on Para-Kenmotsu manifolds that fulfil the condition $C\cdot D=0$. The study of η-Ricci solitons on Para-Kenmotsu manifolds that are conformally flat, projectively flat, and con circularly flat is discussed in section 5. In addition to these, we investigate η-Ricci solitons in Para-Kenmotsu manifolds that possess Codazzi-type Ricci tensors and cyclic parallel Ricci tensors in section 6. At long last in section 7, we present a three-dimensional example of a Para-Kenmotsu manifold that possesses a η-Ricci soliton.

2. Preliminaries

Consider a n-dimensional smooth manifold denoted by (M,ψ,η,δ,g). Here, ψ represents a $(1,1)$ tensor field, δ represents a vector field, and η represents a 1-form such that

$$\psi^2M_1=-M_1+\eta(M_1)\delta, \qquad \eta(\delta)=1 \tag{2.1}$$

thus implying

$$(a)\,\psi\delta=0, \quad (b)\,\eta(\psi M_1)=0, \quad (c)\,rank(\psi)=n-1. \tag{2.2}$$

The tuple (ψ,η,δ,g) is referred to as an almost para contact metric structure, whereas M is denoted as an almost para contact manifold. Furthermore, in the case that M has a semi-Riemannian metric g such that

$$(a)\,g(M_1,\delta)=\eta(M_1), \quad (b)\,g(\psi M_1,\psi M_2)=-g(M_1,M_2)+\eta(M_1)\eta(M_2). \tag{2.3}$$

The tuple (ψ, δ, η, g) is referred to as an almost para contact metric structure, and the manifold M is denoted as an almost para contact metric manifold [11].
The definition of the basic 2-form Φ for an essentially paracontact metric structure $(M, \psi, \delta, \eta, g)$ is given by the equation $\Phi(M_1, M_2) = g(M_1, \psi M_2)$. If $\Phi = d\eta$, then the manifold may be classified as a paracontact metric manifold, and the metric g is said to be associated with it. A paracontact metric manifold is considered to be normal if the expression $[\psi, \psi](M_1, M_2) + 2d\eta(M_1, M_2)\delta = 0$ holds where $[\psi, \psi](M_1, M_2) = \psi^2[M_1, M_2] + [\psi M_1, \psi M_2] - \psi[\psi M_1, \psi M_2] - \psi[M_1, \psi M_2]$. In a Para-Kenmotsu manifold, the following formulas are established [3], [20].

$$(\nabla_{M_1}\psi)M_2 = g(\psi M_1, M_2)\delta - \eta(M_2)\psi M_1 \tag{2.4}$$

$$\nabla_{M_1}\delta = M_1 - \eta(M_1)\delta, \tag{2.5}$$

$$(\nabla_{M_1}\eta)M_2 = g(M_1, M_2) - \eta(M_1)\eta(M_2), \tag{2.6}$$

$$\eta(C(M_1, M_2)M_3) = -\eta(M_1)g(M_2, M_3) + \eta(M_2)g(M_1, M_3), \tag{2.7}$$

$$C(M_1, M_2)\delta = \eta(M_1)M_2 - \eta(M_2)M_1, \tag{2.8}$$

$$C(\delta, M_1)M_2 = \eta(M_2)M_1 - g(M_1, M_2)\delta, \tag{2.9}$$

$$C(\delta, M_1)\delta = M_1 - \eta(M_1)\delta, \tag{2.10}$$

$$D(M_1, \delta) = (1 - n)\eta(M_1), \tag{2.11}$$

$$G\delta = (1 - n)\delta, \tag{2.12}$$

$$(\pounds_\delta g)(M_1, M_2) = -2\{g(M_1, M_2) - \eta(M_1)\eta(M_2)\}, \tag{2.13}$$

let D be the Ricci tensor, C represent the Riemannian curvature tensor field, and ∇ symbolize the Levi-Civita connection associated with the metric tensor g. The expression for the curvature tensor of a three-dimensional Riemannian manifold is as follows:

$$C(M_1, M_2)M_3 = [D(M_2, M_3)M_1 - D(M_1, M_3)M_2 + g(M_2, M_3)GM_1 - g(M_1, M_3)GM_2] - \frac{a^1}{2}\Big[g(M_3, M_2)M_1 - g(M_1, M_3)M_2\Big] \tag{2.14}$$

Let D and a^1 be the Ricci tensor and scalar curvature, respectively. Additionally, let G represent the Ricci operator, which is defined as $g(GM_1, M_2) = D(M_1, M_2)$.

3. Ricci and η-Ricci solitons in the context of Para-Kenmotsu manifolds

Considering a para-contact metric manifold symbolized by $(M, \psi, \eta, \delta, g)$. Let us consider the given equation

$$\pounds_\delta g + 2D + 2\Lambda g + 2\varrho\eta \otimes \eta = 0 \tag{3.1}$$

where Lie derivative operator, denoted as $\pounds_\delta$, pertains to a vector field δ. The Ricci curvature tensor field is represented by D, while Λ and ϱ are real constants. By expressing $\pounds_\delta g$ in terms of Levi-Civita connection ∇, we may get the following expression:

$$2D(M_1, M_2) = -g(\nabla_{M_1}\delta, M_2) - g(M_1, \nabla_{M_2}\delta) - 2\Lambda g(M_1, M_2) - 2\varrho\eta(M_1)\eta(M_2), \tag{3.2}$$

for every given M_1, $M_2, \in \Gamma(M)$.
An expression $(g, \delta, \Lambda, \varrho)$ that accomplishes the relation (3.2) is referred to as a η-Ricci soliton over the manifold M. If the condition $\varrho = 0$ is satisfied, the triple (g, δ, Λ) may be identified as a Ricci soliton [14]. Furthermore, the soliton can be classified as either shrinking, steady, or expanding based on the values of Λ, specifically when $\Lambda < 0$, $\Lambda = 0$, or $\Lambda > 0$, respectively [6].
Now it is understood that in a Lorentzian para-Kenmotsu manifold $\nabla_{M_1}\delta = M_1 - \eta(M_1)\delta$, consequently (3.2) turns out to be

$$2S(M_1, M_2) = -g(M_1 - \eta(M_1)\delta, M_2) - g(M_1, M_2 - \eta(M_2)\delta) - 2\Lambda g(M_1, M_2) \\ -2\varrho\eta(M_1)\eta(M_2),$$

Upon solving this equation, we get the solution

$$D(M_1, M_2) = -(1 + \Lambda)g(M_1, M_2) + (1 - \varrho)\eta(M_1)\eta(M_2), \tag{3.3}$$

for any two elements, M_1 and M_2 are members of the set $\Gamma(M)$.
By substituting $M_2 = \delta$ into the aforementioned equation, we get

$$D(M_1, \delta) = -(\varrho + \Lambda)\eta(M_1), \tag{3.4}$$

In the given context, the Ricci operator G is characterized by its definition

$$GM_1 = -(1 + \Lambda)M_1 + (1 - \varrho)\eta(M_1)\delta. \tag{3.5}$$

In light of the fact that (2.11) is contrasted with (3.5), it is evident that we possess

$$\varrho + \Lambda = n - 1. \tag{3.6}$$

The set of data $(M, \delta, \Lambda, \varrho)$ that accommodates (3.1) is referred to as a η-Ricci soliton on the manifold M [12].
Consequently, the following theorem may be asserted.

THEOREM 3.1. *Consider a manifold M that is n-dimensional and has the structure of a Para-Kenmotsu manifold. If a manifold allows for the existence of a η-Ricci soliton $(M, \delta, \Lambda, \varrho)$, then the manifold M is a η-Einstein manifold in the form (3.3), and the scalar values Λ and ϱ are connected by the equation $\varrho + \Lambda = n - 1$.*

Specifically, by substituting $\varrho = 0$ into the equation (3.3) and (3.6), we get the expressions $D(M_1, M_2) = -(1+\Lambda)g(M_1, M_2)+\eta(M_1)\eta(M_2)$ and $\Lambda = n-1$, respectively. Consequently, we have

Corollary 3.2. *Consider a manifold M that is n-dimensional and has the structure of a Para-Kenmotsu manifold. In the case when the manifold exhibits a Ricci soliton* (g, δ, Λ)*, it may be concluded that M is a* η*-Einstein manifold. Furthermore, the expanding or shrinking of the manifold is determined by whether the vector field* δ *is spacelike or timelike, respectively.*

4. η-Ricci solitons on Para-Kenmotsu manifolds Exhibiting the condition $C \cdot D = 0$

In this section, we will examine a Lorentzian para-Kenmotsu manifold of dimension n that has a η-Ricci soliton satisfying the condition $C \cdot D = 0$. This condition indicates that

$$(C(M_1, M_2) \cdot D)(M_3, M_4) = 0 \tag{4.1}$$

According to (4.1), we may derive

$$D(C(M_1, M_2)M_3, M_4) + D(M_3, C(M_1, M_2)M_4) = 0. \tag{4.2}$$

By replacing $M_1 = \delta$ into equation (4.2), we may get the following expression

$$D(C(\delta, M_2)M_3, M_4) + D(M_3, C(\delta, M_2)M_4) = 0. \tag{4.3}$$

By substituting the expression of D from equation (3.3) and using the symmetries of C, we can determine

$$(\varrho - 1)[\eta(M_2)g(M_1, M_3) + \eta(M_3)g(M_1, M_2) + 2\eta(M_1)\eta(M_2)\eta(M_3)] = 0, \tag{4.4}$$

Through the process of substitution, $M_3 = \delta$ into equation (4.4), we get the following expression

$$(\varrho - 1)[g(M_1, M_2) + \eta(M_1)\eta(M_2)] = 0, \tag{4.5}$$

Based on the above information, it can be deduced that the value of ϱ is equal to 1. The value of Λ may be determined as $\Lambda = (2 - n)$ based on the relationship given in (3.6).
Consequently, it is possible to assert the following theorem:

Theorem 4.1. *Consider a Para-Kenmotsu manifold of dimension n, denoted by* $(g, \delta, \Lambda, \varrho)$*. Suppose this manifold admits a proper* η*-Ricci soliton satisfying* $C \cdot D = 0$*. In this case, it may be concluded that* $\varrho = 1$ *and* $\Lambda = (n - 2)$*.*

Based on the aforementioned theorem, we obtain:

Corollary 4.2. *In the context of a Para-Kenmotsu manifold M that satisfies the condition* $C \cdot D = 0$*, it may be shown that there does not exist a Ricci soliton with the potential vector field* δ*.*

5. η-Ricci solitons of Para-Kenmotsu manifolds satisfying some curvature Restrictions

This article aims to examine several curvature constraints, including the characteristics of conformally flat, projectively flat, and concircularly flat, in the context of a Para-Kenmotsu manifold exhibiting a η-Ricci soliton.
By carrying out a covariant differentiation on the equation (3.3) with respect to the variable M_3, we may be able to arrive at the following expression

$$(\nabla_{M_3} D)(M_1, M_2) = -(\varrho - 1)[(\nabla_{M_3}\eta)(M_1)\eta(M_2) + \eta(M_1)(\nabla_{M_3}\eta)(M_2)]$$
$$= -(\varrho - 1)[g(M_1, M_3)\eta(M_2) + g(M_2, M_3)\eta(M_1) - 2\eta(M_1)\eta(M_2)\eta(M_3)] \quad (5.1)$$

By substituting equation (5.1) into equation (1.6), we get

$$-(\varrho - 1)[g(M_1, M_3)\eta(M_2) - g(M_2, M_3)\eta(M_1)] = \frac{1}{2(n-1)}\Big[(M_1 a^1)g(M_2, M_3) - (M_2 a^1)g(M_1, M_3)\Big] \quad (5.2)$$

Substituting $M_1 = \delta$ into equation (5.2) and using equation (2.3), we get

$$2(n-1)(\varrho - 1)g(\psi M_2, \psi M_3) = \eta(M_3)(M_2 a^1), \quad (5.3)$$

By substituting M_3 with ψM_3 in equation (5.3), we get

$$2(n-1)(\varrho - 1)g(\psi M_2, \psi M_3) = 0, \quad (5.4)$$

Consequently,

$$\varrho = 1. \quad (5.5)$$

By using the relationship (3.6), the value of lambda may be determined as $\Lambda = (n-2)$. Therefore, we may assert the following theorem:

THEOREM 5.1. *A Para-Kenmotsu manifold deemed conformally flat has the property of admitting a proper η-Ricci soliton with $\Lambda = (n-2)$ and $\varrho = 1$.*

Once again, by using equations (3.3) and (3.5) in equation (2.14), we may derive

$$C(M_1, M_2)M_3 = -\Big(\frac{a^1}{2} + 2\Lambda + 2\Big)[g(M_2, M_3)M_1 - g(M_1, M_3)M_2] - (\varrho - 1)[g(M_2, M_3)\eta(M_1) - g(M_1, M_3)\eta(M_2)]\delta - (\varrho - 1)[\eta(M_2)M_1 - \eta(M_1)M_2]\eta(M_3) \quad (5.6)$$

with the help of (3.5) and (5.6) in (1.7), we get

$$N(M_1, M_2)M_3 = -\frac{(a^1 + 3\Lambda + 3)}{2}\Big[g(M_2, M_3)M_1 - g(M_1, M_3)M_2\Big] - (\varrho - 1)[g(M_2, M_3)\eta(M_1) - g(M_1, M_3)\eta(M_2)]\delta - \frac{1}{2}(\varrho - 1)[\eta(M_2)M_1 - \eta(M_1)M_2]\eta(M_3). \quad (5.7)$$

By substituting $M_1 = \psi M_1$ and $M_2 = \psi M_2$ into equation (5.7), we get

$$\frac{(a^1 + 3\Lambda + 3)}{2}[g(\psi M_2, M_3)\psi M_1 - g(\psi M_1, M_3)\psi M_2] = 0 \tag{5.8}$$

By computing the inner product of equation (5.8) with an arbitrary vector field M_4, we may get the following expression

$$\frac{(a^1 + 3\Lambda + 3)}{2}[g(\psi M_2, M_3)g(\psi M_1, M_4) - g(\psi M_1, M_3)g(\psi M_2, M_4)] = 0 \tag{5.9}$$

By contracting M_1 and M_3 in equation (5.9) and using the fact that $Tr(\psi) = 0$ and equation (2.3), we get

$$(a^1 + 3\Lambda + 3)[g(M_2, M_4) - \eta(M_2)\eta(M_4)] = 0 \tag{5.10}$$

By Contracting M_2 and M_4 in the aforementioned equation, we obtain

$$\Lambda = -\frac{a^1}{3} - 1 \tag{5.11}$$

By using equations (3.6) and (5.11), we derive $\varrho = n + \frac{a^1}{3}$. Since Λ is a constant, it may be shown from equation (5.11) that a^1 is also constant. Therefore, we may assert the following theorem:

Theorem 5.2. *A Para-Kenmotsu manifold deemed projectively flat has the property of admitting a proper η-Ricci soliton with* $\Lambda = (-\frac{a^1}{3} - 1)$ *and* $\varrho = (n + \frac{a^1}{3})$.

By using equations (3.3) and (5.6) into equation (1.8), we get

$$L(M_1, M_2)M_3 = -2\left(\frac{a^1}{3} + \Lambda + 1\right)\Big[g(M_2, M_3)M_1 - g(M_1, M_3)M_2\Big] - (\varrho - 1)[g(M_2, M_3)\eta(M_1) - g(M_1, M_3)\eta(M_2)]\delta - (\varrho - 1)[\eta(M_2)M_1 - \eta(M_1)M_2]\eta(M_3). \tag{5.12}$$

By inserting $M_1 = \psi M_1$ and $M_2 = \psi M_2$ into equation (5.12), we get

$$\left(\frac{a^1}{3} + \Lambda + 1\right)[g(\psi M_2, M_3)\psi M_1 - g(\psi M_1, M_3)\psi M_2] = 0 \tag{5.13}$$

By computing the inner product of equation (5.13) with an arbitrary vector field M_4, we may get the following expression

$$\left(\frac{a^1}{3} + \Lambda + 1\right)[g(\psi M_2, M_3)g(\psi M_1, M_4) - g(\psi M_1, M_3)g(\psi M_2, M_4)] = 0 \tag{5.14}$$

By contracting M_1 and M_3 in equation (5.14) and using the fact that $Tr(\psi) = 0$ and equation (2.3), we get

$$\left(\frac{a^1}{3} + \Lambda + 1\right)[g(M_2, M_4) - \eta(M_2)\eta(M_4)] = 0. \tag{5.15}$$

Contracting M_2 and M_4 in the aforementioned equation, we get

$$\Lambda = -\frac{a^1}{3} - 1 \tag{5.16}$$

By using equations (3.6) and (5.16), the value of ϱ may be determined as $\varrho = \frac{a^1}{3} + n$. Given that Λ is a constant, it may be inferred from equation (5.16) that a^1 is also constant.
Therefore, we may assert the following theorem:

Theorem 5.3. *A Para-Kenmotsu manifold deemed concircular flat has the property of admitting a proper η-Ricci soliton with* $\Lambda = (-\frac{a^1}{3} - 1)$ *and* $\varrho = (n + \frac{a^1}{3})$.

6. η-Ricci solitons on Para-Kenmotsu manifolds that possess the properties of Codazzi type of Ricci tensor and Cyclic parallel Ricci tensor

This section focuses on the investigation of η-Ricci solitons within the context of Para-Kenmotsu manifolds. Specifically, we consider Para-Kenmotsu manifolds that exhibit the characteristics of a Codazzi type of Ricci tensor and a Cyclic parallel Ricci tensor.
By computing the covariant derivative of equation (3.3) with respect to the vector field M_3, we get

$$(\nabla_{M_3} D)(M_1, M_2) = (1 - \varrho)[(\nabla_{M_3}\eta)(M_1)\eta(M_2) + \eta(M_1)(\nabla_{M_3}\eta)(M_2)] \tag{6.1}$$

By using equations (3.3) and (3.3), we derive

$$(\nabla_{M_3} D)(M_1, M_2) = (1 - \varrho)[\eta(M_1)g(M_2, M_3) + \eta(M_2)g(M_1, M_3) - 2\eta(M_1)\eta(M_2)\eta(M_3)] \tag{6.2}$$

According to the hypothesis, it may be said that the Ricci tensor, denoted by D, has the characteristic of being of Codazzi type. Subsequently

$$(\nabla_{M_3} D)(M_1, M_2) = (\nabla_{M_1} D)(M_2, M_3) \tag{6.3}$$

Utilizing equation (6.2), equation (6.3) is expressed in the following way

$$(1 - \varrho)[g(M_2, M_3)\eta(M_1) - g(M_1, M_2)\eta(M_3)] = 0 \tag{6.4}$$

By substituting $M_3 = \delta$ into equation (6.4), we get

$$(1 - \varrho)[\eta(M_1)\eta(M_2) - g(M_1, M_2)] = 0, \tag{6.5}$$

Consequently, it can be deduced that the value of ϱ is equal to 1.
By using the relationship expressed in equation (3.6), the value of Λ may be determined as $\Lambda = (n - 2)$.
Therefore, we may assert the following theorem:

Theorem 6.1. *Consider a Para-Kenmotsu manifold of dimension n, denoted by $(g, \delta, \Lambda, \varrho)$. Suppose this manifold admits a proper η-Ricci soliton. In the case when the manifold has a Ricci tensor of Codazzi type, it follows that $\varrho = 1$ and $\Lambda = (n - 2)$.*

Based on the aforementioned theorem, we obtain:

COROLLARY 6.2. *In the context of a Para-Kenmotsu manifold M that satisfies the condition, Ricci tensor of Codazzi type, it may be shown that there does not exist a Ricci soliton with the potential vector field* δ.

Consider a n-dimensional Para-Kenmotsu manifold with an η-Ricci soliton denoted by $(g, \delta, \Lambda, \varrho)$. If the manifold has a cyclic parallel Ricci tensor, then equation (1.4) is satisfied.
By applying the covariant derivative to equation (3.3) and using equation (2.6), we get the following expression.

$$(\nabla_{M_3} D)(M_1, M_2) = (1-\varrho)[\eta(M_1)g(M_2, M_3) + \eta(M_2)g(M_1, M_3) - 2\eta(M_1)\eta(M_2)\eta(M_3)], \tag{6.6}$$

In a similar vein, we possess

$$(\nabla_{M_1} D)(M_2, M_3) = (1-\varrho)[\eta(M_2)g(M_3, M_1) + \eta(M_3)g(M_2, M_1) - 2\eta(M_1)\eta(M_2)\eta(M_3)], \tag{6.7}$$

and

$$(\nabla_{M_2} D)(M_3, M_1) = (1-\varrho)[\eta(M_3)g(M_1, M_2) + \eta(M_1)g(M_3, M_2) - 2\eta(M_1)\eta(M_2)\eta(M_3)], \tag{6.8}$$

By using equations (6.6) to (6.8) within the context of equation (1.4), it is determined

$$2(1-\varrho)[\eta(M_1)g(M_2, M_3)+\eta(M_2)g(M_1, M_3)+\eta(M_3)g(M_1, M_2)-3\eta(M_1)\eta(M_2)\eta(M_3)] = 0, \tag{6.9}$$

Now, substituting $M_3 = \delta$ into equation (6.9), we derive

$$2(1-\varrho)[g(M_1, M_2) - \eta(M_1)\eta(M_2)] = 0, \tag{6.10}$$

Consequently, it can be deduced that the value of ϱ is equal to 1. By using the relationship expressed in equation (3.6), the value of Λ may be determined as $\Lambda = (n-2)$.
Therefore, it is possible to assert the following proposition:

THEOREM 6.3. *Consider a Para-Kenmotsu manifold of dimension n, denoted by* $(g, \delta, \Lambda, \varrho)$. *Suppose this manifold admits a proper* η*-Ricci soliton. In the case when the manifold has cyclic parallel Ricci tensor, it follows that* $\varrho = 1$ *and* $\Lambda = (n-2)$.

Based on the aforementioned theorem, we obtain:

COROLLARY 6.4. *In the context of a Para-Kenmotsu manifold M that satisfies the condition, cyclic parallel Ricci tensor, it may be shown that there does not exist a Ricci soliton with the potential vector field* δ.

7. An instance of a three-dimensional Para-Kenmotsu manifold that allows for a η-Ricci soliton

In this analysis, we will now direct our attention towards the three- dimensional manifold.

$$M = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3 : z \neq 0\} \tag{7.1}$$

Let γ_1, γ_2, and γ_3 be the standard coordinates in $\mathbb{R}^3$.
The vector fields mentioned

$$\sigma^1 = \frac{\partial}{\partial\gamma_1}, \quad \sigma^2 = \frac{\partial}{\partial\gamma_2}, \quad and \quad \sigma^3 = \gamma_1\frac{\partial}{\partial\gamma_1} + \gamma_2\frac{\partial}{\partial\gamma_2} + \frac{\partial}{\partial\gamma_3} = \delta, \tag{7.2}$$

are linearly independent at every point in the set M.
Let g be the Lorentzian metric that is defined by
$g(\sigma^1, \sigma^3) = g(\sigma^2, \sigma^3) = g(\sigma^1, \sigma^2) = 0,$
$g(\sigma^1, \sigma^1) = g(\sigma^2, \sigma^2) = g(\sigma^3, \sigma^3) = 1$
Let η be the 1- form that is defined by

$$\eta(\sigma^3) = g(M_3, \sigma^3) = g(M_3, \delta), \tag{7.3}$$

for a given vector field M_3 on manifold M.
Consider the $(1, 1)$- tensor field to be denoted by ψ

$$\psi(\sigma^1) = \sigma^2, \;\; \psi(\sigma^2) = \sigma^1, \;\; \psi(\sigma^3) = 0. \tag{7.4}$$

Subsequently, by using the linearity properties of both ψ and g, we may deduce

$$\eta(\sigma^3) = 1, \quad \psi^2 M_3 = -M_3 + \eta(M_3)\sigma^3,$$

$$g(\psi M_3, \psi M_4) = -g(M_3, M_4) + \eta(M_3)\eta(M_4),$$

for every vector field M_3 and M_4 defined on the manifold M.
It is evident that

$$\eta(\sigma^1) = 0, \quad \eta(\sigma^2) = 0, \quad \eta(\sigma^3) = 1. \tag{7.5}$$

Consequently, for $\sigma^3 = \delta$, the structure (ψ, δ, η, g) establishes an almost contact metric structure on M [4].

Let us assume that the Levi-Civita connection is denoted by ∇ with regard to the Lorentzian metric g. Then there is also

$$[\sigma^1, \sigma^2] = 0, \quad [\sigma^1, \sigma^3] = \sigma^1, \quad [\sigma^2, \sigma^3] = \sigma^2, \tag{7.6}$$

By using Koszul's formula, we may express the Levi-Civita connection ∇ in terms of the metric tensor g that is,

$$2g(\nabla_{M_1} M_2, M_3) = M_1 g(M_{M_2}, M_3) + M_2 g(M_3, M_1) - M_3 g(M_1, M_2) - g(M_1, [M_2, M_3]) - g(M_2, [M_1, M_3]) + g(M_3, [M_1, M_2]),$$

It is possible to easily compute

$$\nabla_{\sigma^1}\sigma^3 = \sigma^1, \quad \nabla_{\sigma^1}\sigma^2 = 0, \quad \nabla_{\sigma^1}\sigma^1 = -\sigma^3, \tag{7.7}$$

$$\nabla_{\sigma^2}\sigma^3 = \sigma^2, \quad \nabla_{\sigma^2}\sigma^3 = \sigma^3, \quad \nabla_{\sigma^2}\sigma^1 = 0, \tag{7.8}$$

$$\nabla_{\sigma^3}\sigma^3 = 0, \quad \nabla_{\sigma^3}\sigma^2 = 0, \quad \nabla_{\sigma^3}\sigma^1 = 0. \tag{7.9}$$

In light of what has been discussed so far, it is clear that the manifold under investigation satisfies ∇, which may be written as

$$\nabla_{M_3}\delta = M_3 - \eta(M_3)\delta, \quad and \quad (\nabla_{M_3}\psi)M_4 = g(\psi M_3, M_4) - \eta(M_4)\psi M_3 \tag{7.10}$$

In addition to this, the Riemannian curvature tensor, denoted by D may be expressed as

$$C(M_1, M_2)M_3 = \nabla_{M_1}\nabla_{M_2}M_3 - \nabla_{M_2}\nabla_{M_1}M_3 - \nabla_{[M_1,M_2]}M_3. \tag{7.11}$$

Following that

$$C(\sigma^1,\sigma^2)\sigma^3 = 0, \;\; C(\sigma^2,\sigma^3)\sigma^3 = -\sigma^2, \;\; C(\sigma^1,\sigma^3)\sigma^3 = -\sigma^1, \tag{7.12}$$

$$C(\sigma^1,\sigma^2)\sigma^2 = \sigma^1, \;\; C(\sigma^2,\sigma^3)\sigma^2 = -\sigma^3, \;\; C(\sigma^1,\sigma^3)\sigma^2 = 0, \tag{7.13}$$

$$C(\sigma^1,\sigma^2)\sigma^1 = \sigma^2, \;\; C(\sigma^2,\sigma^3)\sigma^1 = 0, \;\; C(\sigma^1,\sigma^3)\sigma^1 = \sigma^3, \tag{7.14}$$

The Ricci tensor, denoted by D, may then be obtained by

$$D(\sigma^1,\sigma^1) = -2 \;\; D(\sigma^2,\sigma^2) = 2, \;\; D(\sigma^3,\sigma^3) = -2. \tag{7.15}$$

$$D(\sigma^1,\sigma^2) = 0 \;\; D(\sigma^1,\sigma^3) = 0, \;\; D(\sigma^2,\sigma^3) = 0. \tag{7.16}$$

By referring to equation (3.3), it can be shown that $D(\sigma^1,\sigma^1)$ and $D(\sigma^2,\sigma^2)$ are both equal to $-(1+\Lambda)$. The equation $D(\sigma^3,\sigma^3) = -(\Lambda+\varrho)$ implies that $\Lambda = 1$ and $\varrho = 1$. The collection of data $(g,\delta,\Lambda,\varrho)$, where $\Lambda = 1$ and $\varrho = 1$, may be used to establish the existence of a η-Ricci soliton on the para Kenmotsu manifold M.

8. Discussions

We investigate some interesting results on η-Ricci solitons on Para-Kenmotsu manifolds and give example on such manifold. We also discuss Ricci and η-Ricci solitons on such manifolds. Certain curvature restrictions on such manifolds are also derived.

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