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INITIAL COEFFICIENT ESTIMATES FOR SUBCLASSES OF BI-UNIVALENT FUNCTIONS INVOLVING q-DERIVATIVE OPERATOR

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Abstract

In this paper, we introduce and investigate two new subclasses of functions of class Σ of analytic and bi-univalent functions defined in open unit disk, which are associated with q-derivative operator. Further, we obtain estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions of these subclasses and some consequences of the results are also pointed out.

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1. Introduction

Let $\mathcal{A}$ denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0, n \in \mathbb{N}), \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all univalent functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$, for details (see [1]). For $0 < q < 1$, we introduce the family of new functions defined as follows:

$$R_{\Sigma}^{\mu,q}(\beta) = \left\{ f \in \mathcal{A} : \Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda D_q f(z) \left(\frac{f(z)}{z}\right)^{\mu-1} \right) > \beta \right\}$$

where D_q stands for q-derivative of the function $f(z)$ introduced by Jackson [2], for $q \to 1^-$ it reduces to the class of analytic function introduced by Ding et. al. [3].
For the function $f(z) \in \mathcal{A}$ given by series (1.1) and $0 < q < 1$, the q-derivative of a function $f(z)$ is defined by (see [4, 5]).

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}; \qquad (z \neq 0, q \neq 0).$$

From above Equation, we can write

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [k]_q a_k z^{k-1} \tag{1.2}$$

where

$$[k]_q = \frac{1 - q^k}{1 - q}.$$

Since, each $f \in \mathcal{S}$ is univalent, they are invertible for some part of unit disk $\mathcal{U}$. In fact, the Koebe One Quarter Theorem (see [1]) says that f^{-1} exists at least on $\{z \in \mathbb{C} : |z| < 1/4\}$ for each $f \in \mathcal{S}$. Thus every $f \in \mathcal{S}$ has an inverse f^{-1} given by

$$f^{-1} f(z) = z, \quad z \in \mathcal{U}$$

and

$$f(f^{-1}(w)) = w, \ |w| < r_0 f, \ r_0 f \geq 1/4,$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent $\mathcal{U}$. Let Σ denote the class of bi-univalent functions.

In 1967, Lewin [6] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. On the other hand, Brannan and clunie [7] conjectured that $|a_2| \leq \sqrt{2}$. Natamyahu [8], showed that $\max |a_2| = \frac{4}{3}$. But the coefficient problem for each of the following Taylor-Maclaurin coefficients $|a_n|$, $(n \in \mathbb{N} \setminus \{1, 2\})$ is still an open problem.

Brannan and Taha [9], many researchers [10–17] have recently introduced and investigated several interesting subclasses of the bi-univalent function of class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

The main object of present paper is to find estimates on the coefficients $|a_2|$ and $|a_3|$ for the subclasses $\mathcal{H}_{\Sigma}^{\mu}(q, \alpha, \lambda)$ and $\mathcal{H}_{\Sigma}^{\mu}(q, \beta, \lambda)$ of the functions of the class Σ using q-differential operator.
In order to derive our main results, we shall need the following lemma [18].

Lemma 1.1. *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathcal{U}$, for which $\Re\{h(z)\} > 0$ $(z \in \mathcal{U})$, where*

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \ (z \in \mathcal{U}).$$

2. Coefficient bounds for the function class $\mathcal{H}_{\Sigma}^{\mu}(q, \alpha, \lambda)$

DEFINITION 2.1. A functions $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{\mu}(q, \alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda D_q f(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2}; \tag{2.1}$$
$$(0 < \alpha \leq 1, \lambda \geq 1, \mu \geq 0, z \in \mathcal{U})$$

and

$$\left|\arg\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda D_q g(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2}; \tag{2.2}$$
$$(0 < \alpha \leq 1, \lambda \geq 1, \mu \geq 0, w \in \mathcal{U})$$

where the function $g = f^{-1}$ is given by (1.3).

Observe that for $q \to 1^-$, the class $\mathcal{H}_{\Sigma}^{\mu}(q, \alpha, \lambda)$ reduces to $\mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda)$ introduced and investigated by [19], for $q \to 1^-$and $\mu = 1$, the class $\mathcal{H}_{\Sigma}^{\mu}(q, \alpha, \lambda)$ reduces to $\mathscr{B}_{\Sigma}(\alpha, \lambda)$ introduced and investigated by [20], for $q \to 1^-$and $\mu, \lambda = 1$, the class $\mathcal{H}_{\Sigma}^{\mu}(q, \alpha, \lambda)$ reduces to $\mathcal{H}_{\Sigma}^{\alpha}$ introduced and investigated by [21].

THEOREM 2.2. *If $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\Sigma}^{\mu}(q, \alpha, \lambda)$; $(0 < \alpha \leq 1, \lambda \geq 1, 0 < q < 1, \mu \geq 0)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha\left(\mu - \lambda + \lambda[3]_q + (\mu-1)\left(\frac{\mu}{2} - \lambda + \lambda[2]_q\right)\right) + (1-\alpha)\left(\mu - \lambda + \lambda[2]_q\right)^2}} \tag{2.3}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\mu - \lambda + \lambda[2]_q)^2} + \frac{2\alpha}{(\mu - \lambda + \lambda[3]_q)}. \tag{2.4}$$

PROOF. From Equations (2.1) and (2.2) it follows that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda D_q f(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = (l(z))^{\alpha} \tag{2.5}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda D_q g(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = (m(w))^{\alpha} \quad (z, w \in \mathcal{U}), \tag{2.6}$$

where

$$l(z) = 1 + c_1 z + c_2 z^2 + \dots$$
$$m(w) = 1 + d_1 w + d_2 w^2 + \dots$$

are in $\mathcal{P}$.
Now, equating the coefficients in (2.5) and (2.6), we get

$$(\mu - \lambda + \lambda[2]_q)a_2 = \alpha c_1 \tag{2.7}$$

$$(\mu - \lambda + \lambda[3]_q)a_3 + (\mu - 1)\left(\frac{\mu}{2} - \lambda + \lambda[2]_q\right)a_2^2 = \alpha c_2 + \frac{\alpha(\alpha - 1)}{2}c_1^2 \tag{2.8}$$

and

$$-(\mu - \lambda + \lambda[2]_q)a_2 = \alpha d_1 \tag{2.9}$$

$$\begin{aligned} &-(\mu - \lambda + \lambda[3]_q)a_3 + \left[2(\mu - \lambda + \lambda[3]_q) + (\mu - 1)\left(\frac{\mu}{2} - \lambda + \lambda[2]_q\right)\right]a_2^2 \\ &= \alpha d_2 + \frac{\alpha(\alpha - 1)}{2}d_1^2. \end{aligned} \tag{2.10}$$

From Equations (2.7) and (2.9), we get

$$c_1 = -d_1 \tag{2.11}$$

and

$$2(\mu - \lambda + \lambda[2]_q)^2 a_2^2 = \alpha^2(c_1^2 + d_1^2). \tag{2.12}$$

Adding Equations (2.8) and (2.10), we obtain

$$\left[2(\mu - \lambda + \lambda[3]_q) + 2(\mu - 1)\left(\frac{\mu}{2} - \lambda + \lambda[2]_q\right)\right]a_2^2 = \alpha(c_2 + d_2) + \frac{\alpha(\alpha - 1)}{2}(c_1^2 + d_1^2). \tag{2.13}$$

Using Equation (2.12) in Equation (2.13), we find that

$$\begin{aligned} &\left\{2\alpha\left[(\mu - \lambda + \lambda[3]_q) + (\mu - 1)\left(\frac{\mu}{2} - \lambda + \lambda[2]_q\right)\right]\right. \\ &\qquad \left. + (1 - \alpha)(\mu - \lambda + \lambda[2]_q)^2\right\}a_2^2 = \alpha^2(c_2 + d_2) \end{aligned}$$

Applying Lemma 1.1 to above equality, we obtain

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha\left[(\mu - \lambda + \lambda[3]_q) + (\mu - 1)\left(\frac{\mu}{2} - \lambda + \lambda[2]_q\right)\right] + (1 - \alpha)(\mu - \lambda + \lambda[2]_q)^2}}$$

which is required estimate on $|a_2|$.
Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8) we obtain that

$$2(\mu - \lambda + \lambda[3]_q)a_3 - 2(\mu - \lambda + \lambda[3]_q)a_2^2 = \alpha(c_2 - d_2) + \frac{\alpha(\alpha - 1)}{2}(c_1^2 - d_1^2). \tag{2.14}$$

Using Equations (2.11) and (2.12) in Equation (2.14), we get

$$a_3 = \frac{\alpha(c_2 - d_2)}{2(\mu - \lambda + \lambda[3]_q)} + \frac{\alpha^2(c_1^2 + d_1^2)}{2(\mu - \lambda + \lambda[2]_q)^2}.$$

Now applying Lemma 1.1, we get

$$|a_3| \leq \frac{2\alpha}{(\mu - \lambda + \lambda[3]_q)} + \frac{4\alpha^2}{(\mu - \lambda + \lambda[2]_q)^2}.$$

This completes the proof of Theorem 2.2. □

If we choose $q \to 1^-$ in Theorem 2.2, we get following consequence.

Corollary 2.1. *[19] The function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma^\mu(\alpha, \lambda)$; $(0 < \alpha \leq 1, \lambda \geq 1, \mu \geq 0)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha(\mu + 2\lambda - \lambda^2) + (\mu + \lambda)^2}} \quad \text{and} \quad |a_3| \leq \frac{4\alpha^2}{(\mu + \lambda)^2} + \frac{2\alpha}{\mu + 2\lambda}.$$

If we take $\mu = 1$ and $q \to 1^-$ in Theorem 2.2, we get following result.

Corollary 2.2. *[20] The function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\alpha, \lambda)$; $(0 < \alpha \leq 1, \lambda \geq 1)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha(1 + 2\lambda - \lambda^2) + (1 + \lambda)^2}} \quad \text{and} \quad |a_3| \leq \frac{4\alpha^2}{(1 + \lambda)^2} + \frac{2\alpha}{1 + 2\lambda}.$$

If we take $\mu = 1$, $\lambda = 1$ and $q \to 1^-$ in Theorem 2.2, we get following corollary.

Corollary 2.3. *[21] The function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\alpha)$; $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha + 2}} \quad \text{and} \quad |a_3| \leq \frac{\alpha(3\alpha + 2)}{3}.$$

If we take $\mu = 0$, $\lambda = 1$ and $q \to 1^-$ in Theorem 2.2, we get following corollary.

Corollary 2.4. *[22] The function $f(z)$ given by (1.1) be in the class $S_\Sigma^*[\alpha]$; $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha + 1}} \quad \text{and} \quad |a_3| \leq \alpha(4\alpha + 1).$$

3. Coefficient bounds for the function class $\mathcal{H}_\Sigma^\mu(q, \beta, \lambda)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma^\mu(q, \beta, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \Re\left\{(1 - \lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda D_q f(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\} > \beta; \tag{3.1}$$

$$(0 \leq \beta < 1, \lambda \geq 1, \mu \geq 0, z \in \mathcal{U})$$

and

$$\Re\left\{(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda D_q g(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right\}>\beta; \tag{3.2}$$
$$(0\le\beta<1,\lambda\ge 1,\mu\ge 0,w\in\mathcal{U})$$

where the function $g=f^{-1}$ is given by (1.3).

THEOREM 3.2. *The function $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\Sigma}^{\mu}(q,\beta,\lambda)$; $(0\le\beta<1,\lambda\ge 1,\mu\ge 0,0<q<1)$. Then*

$$|a_2|\le\min\left\{\frac{2(1-\beta)}{\mu-\lambda+\lambda[2]_q},\sqrt{\frac{2(1-\beta)}{\mu-\lambda+\lambda[3]_q+(\mu-1)\left(\frac{\mu}{2}-\lambda+\lambda[2]_q\right)}}\right\} \tag{3.3}$$

and

$$|a_3|\le\min\left\{\frac{4(1-\beta)^2}{(\mu-\lambda+\lambda[2]_q)^2}+\frac{2(1-\beta)}{\mu-\lambda+\lambda[3]_q},\right.$$
$$\left.\frac{2(1-\beta)}{\mu-\lambda+\lambda[3]_q}+\frac{2(1-\beta)}{\mu-\lambda+\lambda[3]_q+(\mu-1)\left(\frac{\mu}{2}-\lambda+\lambda[2]_q\right)}\right\} \tag{3.4}$$

PROOF. It follows from (3.1) and (3.2) that

$$\Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda D_q f(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\}=\beta+(1-\beta)l(z) \tag{3.5}$$

and

$$\Re\left\{(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda D_q g(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right\}=\beta+(1-\beta)m(w) \tag{3.6}$$
$$(z,w\in\mathcal{U}),$$

where

$$l(z)=1+c_1z+c_2z^2+\dots$$
$$m(w)=1+d_1w+d_2w^2+\dots$$

are in $\mathcal{P}$.
Now, equating the coefficients of (3.5) and (3.6), we get

$$(\mu-\lambda+\lambda[2]_q)a_2 = (1-\beta)c_1 \tag{3.7}$$

$$(\mu-\lambda+\lambda[3]_q)a_3+(\mu-1)\left(\frac{\mu}{2}-\lambda+\lambda[2]_q\right)a_2^2 = (1-\beta)c_2 \tag{3.8}$$

and

$$-(\mu-\lambda+\lambda[2]_q)a_2 = (1-\beta)d_1 \tag{3.9}$$

$$-(\mu-\lambda+\lambda[3]_q)a_3+\{(\mu-1)\left(\frac{\mu}{2}-\lambda+\lambda[2]_q\right) \tag{3.10}$$

$$+2(\mu-\lambda+\lambda[3]_q)\}a_2^2 = (1-\beta)d_2. \tag{3.11}$$

From Equations (3.7) and (3.9), we get

$$c_1 = -d_1 \tag{3.12}$$

and

$$2(\mu-\lambda+\lambda[2]_q)^2a_2^2 = (1-\beta)^2(c_1^2+d_1^2). \tag{3.13}$$

Adding Equations (3.8) and (3.11), we find that

$$\left\{2(\mu-1)\left(\frac{\mu}{2}-\lambda+\lambda[2]_q\right)+2(\mu-\lambda+\lambda[3]_q)\right\}a_2^2 = (1-\beta)(c_2+d_2). \tag{3.14}$$

Now, applying Lemma 1.1 for (3.13) and (3.14), we get

$$|a_2| \le \min\left\{\frac{2(1-\beta)}{\mu-\lambda+\lambda[2]_q},\ \sqrt{\frac{2(1-\beta)}{\mu-\lambda+\lambda[3]_q+(\mu-1)\left(\frac{\mu}{2}-\lambda+\lambda[2]_q\right)}}\right\}.$$

Next, in order to find bound on $|a_3|$, we subtract (3.11) from (3.8), we find that

$$2(\mu-\lambda+\lambda[3]_q)a_3-2(\mu-\lambda+\lambda[3]_q)a_2^2 = (1-\beta)(c_2-d_2). \tag{3.15}$$

Now, using (3.13) in (3.15), which yeilds

$$a_3 = \frac{(1-\beta)^2(c_1^2+d_1^2)}{2(\mu-\lambda+\lambda[2]_q)^2}+\frac{(1-\beta)(c_2-d_2)}{2(\mu-\lambda+\lambda[3]_q)}. \tag{3.16}$$

Applying Lemma 1.1, we get

$$|a_3| \le \frac{4(1-\beta)^2}{(\mu-\lambda+\lambda[2]_q)^2}+\frac{2(1-\beta)}{(\mu-\lambda+\lambda[3]_q)}. \tag{3.17}$$

Using (3.14) in (3.15),we obtain

$$a_3 = \frac{(1-\beta)(c_2+d_2)}{2\left[\mu-\lambda+\lambda[3]_q+(\mu-1)\left(\frac{\mu}{2}-\lambda+\lambda[2]_q\right)\right]}+\frac{(1-\beta)(c_2-d_2)}{2(\mu-\lambda+\lambda[3]_q)}. \tag{3.18}$$

Applying Lemma 1.1, we get

$$|a_3| \leq \frac{2(1-\beta)}{\left[\mu - \lambda + \lambda[3]_q + (\mu-1)\left(\frac{\mu}{2} - \lambda + \lambda[2]_q\right)\right]} + \frac{2(1-\beta)}{(\mu - \lambda + \lambda[3]_q)}. \tag{3.19}$$

Therefore, (3.19) and (3.17) are required estimates on $|a_3|$ as given in (3.4). □

If we choose $q \to 1^-$ in Theorem 3.2, we get following consequence.

Corollary 3.1. *[19] The function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma^\mu(\beta, \lambda)$. Then*

$$|a_2| \leq \min\left\{\frac{2(1-\beta)}{\mu+\lambda}, \sqrt{\frac{4(1-\beta)}{(\mu+2\lambda)(\mu+1)}}\right\}$$

and

$$|a_3| \leq \begin{cases} \min\left\{\frac{4(1-\beta)^2}{(\mu+\lambda)^2} + \frac{2(1-\beta)}{\mu+2\lambda}, \frac{4(1-\beta)}{(\mu+2\lambda)(\mu+1)}\right\}, & 0 \leq \mu < 1 \\ \frac{2(1-\beta)}{\mu+2\lambda}, & \mu \geq 1. \end{cases}$$

If we choose $\mu = 1$ and $q \to 1^-$ in Theorem 3.2, we get following corollary.

Corollary 3.2. *The function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\beta, \lambda)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)}{2\lambda+1}}, \frac{2(1-\beta)}{\lambda+1}\right\} \quad \textit{and} \quad |a_3| \leq \frac{2(1-\beta)}{2\lambda+1}.$$

Above Corollary 3.2 is an improvement for the results obtained by Frasin and Aouf [20].
If we choose $\lambda = 1,\ \mu = 1$ and $q \to 1^-$ in Theorem 3.2, we get following corollary.

Corollary 3.3. *The function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\beta)$. Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & 0 \leq \beta < \frac{1}{3} \\ (1-\beta), & \frac{1}{3} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{3}.$$

Above Corollary 3.3 is an improvement for the results obtained by Srivastava et.al. [21].
If we choose $\lambda = 1,\ \mu = 0$ and $q \to 1^-$ in Theorem 3.2, we get following corollary.

COROLLARY 3.4. *The function $f(z)$ given by (1.1) be in the class $\mathcal{S}^*_\Sigma(\beta)$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)}$$

and

$$|a_3| \leq \begin{cases} 2(1-\beta), & 0 \leq \beta < \frac{3}{4} \\ (1-\beta)(5-4\beta), & \frac{3}{4} \leq \beta < 1. \end{cases}$$

Above Corollary 3.4 is an improvement for the results obtained by Brannan and Taha [9].

REMARK 1. *For $\mu = 1$ and $\lambda = 1$ the results are followed by the results discussed in [23].*

REMARK 2. *For $\mu = 1$ the results are followed by the results discussed in [24].*

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