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A CLASS OF HORIZONTAL SUBMERSION FROM KENMOTSU MANIFOLDS TO RIEMANNIAN MANIFOLDS

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Abstract

We define a type of horizontal submersion from Kenmotsu manifolds to Riemannian manifolds and its geometrical properties are studied. The necessary and sufficient conditions for the distributions to be integrable are derived. We also give the necessary and sufficient condition for the horizontal submersion to be totally geodesic. Finally, we construct a non-trivial example of a Kenmotsu manifold to prove the existence of conformal semi-slant submersion on it.

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1. Introduction

Let M be a Riemannian manifold endowed with a Riemannian metric g. A Kenmotsu manifold is a subclass of almost contact manifold. Since the Riemannian Submersions have many applications in science and technology, especially in the theory of relativity and cosmology, therefore it attracts many researchers to do the research in this area.

In 1966, the theory of Riemannian submersion was initiated by O' Neill [21] and it has been further studied by Gray [10], in 1967. The Riemannian submersions play a vital role not only in the differential geometry but also in science and technology. It is noticed that the theory of Riemannian submersions are capable to handle many issues of the singularity theory, Yang-Mills theory, quantum theory, Kaluza-Klein theory, relativity, superstring theories, etc. (see, [4], [5], [6], [14], [15], [16], [22]). For more details, we cite the books ([8], [26]) and the references therein. The Riemannian submersions motivate the researchers to define the semi-Riemannian submersions and Lorentzian submersion [8], locally conformal Kaehler submersions [23], almost Hermitian submersions ([3], [28]) almost contact submersions [18], semi-slant submersion [7, 13, 24], para-contact submersions [12], para-contact para-complex submersions [19, 20], anti-invariant Riemannian submersions [2]. Recently, Akyol [1] introduced the notion of conformal semi-slant submersion from almost Hermitian manifolds to Riemannian manifolds.

The above studies inspire us to introduce the notion of conformal semi-slant submersion from the Kenmotsu manifolds to the Riemannian manifolds and characterize its geometrical properties. Throughout the paper, we denote the Kenmotsu manifold of dimension $2n+1$ by (M, g_M). We exhibit our work as follows: Section 2 contains some basic results of Kenmotsu manifold, a non-trivial example of Kenmotsu manifold, and well-known lemmas. In Section 3, we give the definition of conformal semi-slant submersions from (M, g_M) to Riemannian manifold and establish the necessary and sufficient conditions for the distributions to be integrable. The necessary and sufficient conditions for the conformal semi-slant submersions to be totally geodesics are given in Section 4. The last section is concerned with a non-trivial example of Kenmotsu manifold with conformal semi-slant submersion.

2. Preliminaries

Kenmotsu Manifolds: [17] Let M be an almost contact metric manifold. So there exist on M, a $(1, 1)$ tensor field ϕ, a vector field ξ, a $1-$form η and a Riemannian metric g_M such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g_M(X, \xi) = \eta(X), \eta(\xi) = 1 \tag{2.2}$$

and

$$g_M(\phi X, \phi Y) = g_M(X, Y) - \eta(X)\eta(Y),$$

$$g_M(\phi X, Y) = -g_M(X, \phi Y), \tag{2.3}$$

for any vector fields X and Y on M and I is the identity tensor field [12]. An almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g_M) is denoted by $(M, \phi, \xi, \eta, g_M)$.

An almost contact metric manifold M is called a Kenmotsu manifold [25] if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection of the Riemannian metric g_M. If $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold, then the following equation holds:

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.5}$$

Example 2.1. *[27] Let (x_i, y_i, z) be cartesian coordinates on $\mathbb{R}^{2n+1}$ for $i = 1, 2, 3, ..., n$. An almost contact metric structure (ϕ, ξ, η, g) is defined as follows:*

$$\phi(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + ... + a_n \frac{\partial}{\partial x_n} + b_1 \frac{\partial}{\partial y_1} + b_2 \frac{\partial}{\partial y_2} + ... + b_n \frac{\partial}{\partial y_n} + c \frac{\partial}{\partial z})$$

$$= (-b_1 \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial y_1} - b_2 \frac{\partial}{\partial x_2} + a_2 \frac{\partial}{\partial y_2} - .... - b_n \frac{\partial}{\partial x_n} + a_n \frac{\partial}{\partial y_n}),$$

where $\xi = \frac{\partial}{\partial z}$ *and* a_i, b_i, c *are* C^∞ *real valued functions in* R^{2n+1}.

Let $\eta = dz$, *g is Euclidean metric and*

$$\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, ..., \frac{\partial}{\partial y_n}, \frac{\partial}{\partial z}\}$$

is orthonormal base field of vectors on $\mathbb{R}^{2n+1}$. *We can easily show that* (ϕ, ξ, η, g) *is Kenmotsu structure on* $\mathbb{R}^{2n+1}$. *Hence, it is Kenmotsu manifold.*

Now, we recall following definition for later use:

Definition 2.2. Let (M, g_M) and (N, g_N) denote the Kenmostu manifold and a Riemannian manifold, respectively, then a differentiable map $f : M \to N$ is said to be horizontally weakly conformal or semi-conformal at $p \in M$, if either $df_p = 0$ or df_p maps the horizontal space $\mathcal{H} = ((\ker f_*)_p)^\perp$ conformally onto $T_{f(p)}$.

It is noticed that df_p is symmetric, which is same as the second condition of Definition 2.2. Thus there exists a number $\chi(p) \neq 0$ such that

$$g_N(f_*X, f_*Y) = \chi(p) g_M(X, Y), \quad \text{for all } X, Y \in ((\ker f_*)_p)^\perp.$$

Here $\chi(p)$ represents the square dilation of f at p and $\lambda(p) = \sqrt{\chi(p)}$ is called the dilation of f at p. The map f is called horizontally weakly conformal or semi-conformal on M if it is horizontally weakly conformal at every point on M. If f has no critical point, then it is said to be a (horizontally) conformal submersion [3].

A horizontally conformal Lorentzian submersion $f : M \to N$ is called horizontally homothetic if the gradient of its dilation λ is vertical, that is,

$$\mathcal{H}(grad\lambda) = 0 \tag{2.6}$$

at point $p \in M$, where $\mathcal{H}$ is the complement orthogonal distribution to $\mathcal{V} = \ker f_*$ in $\Gamma(T_pM)$.

A vector field E on M is called projectable if there exists a vector field $\widehat{E}$ on N such that $f_*(E_p) = \widehat{E}_{f(p)}$ for any $p \in M$, where $f : M \to N$ is a conformal submersion. Note that the vector fields E and $\widehat{E}$ are f-related. Every projectable horizontal vector field Y on M is called basic. It is observed that for a vector field $\widehat{Z}$ of N, there exists a unique basic vector field Z, termed as horizontal lift of $\widehat{Z}$. It is well-known that the fundamental tensors $\mathcal{T}$ and $\mathcal{A}$, defined by O'Neill's [21], satisfy the relations

$$\mathcal{A}_E F = \mathcal{V}\nabla^M_{\mathcal{H}E}\mathcal{H} + \mathcal{H}\nabla^M_{\mathcal{H}E}\mathcal{V}FF, \tag{2.7}$$

$$\mathcal{T}_E F = \mathcal{V}\nabla^M_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla^M_{\mathcal{V}E}\mathcal{V}F \tag{2.8}$$

for any vector fields E and F on M, where $\mathcal{H}$ and $\mathcal{V}$ denote the horizontal and the vertical projections, respectively. In view of equations (2.7) and (2.8), we obtain

$$\nabla_U V = \mathcal{T}_U V + \widehat{\nabla}_U V, \tag{2.9}$$

$$\nabla_U X = \mathcal{H}\nabla_U X + \mathcal{T}_U X, \tag{2.10}$$

$$\nabla_X U = \mathcal{A}_X U + \mathcal{V}\nabla_X U, \tag{2.11}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \tag{2.12}$$

for all $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^\perp$, where $\mathcal{V}\nabla_U V = \widehat{\nabla}_U V$. If X is basic, then $\mathcal{A}_X V = \mathcal{H}\nabla_X V$.

For $p \in M$, $V \in \mathcal{V}_p$ and $X \in \mathcal{H}_p$, the linear operators

$$\mathcal{T}_V, \mathcal{A}_X : T_pM \to T_pM$$

are skew-symmetric, that is,

$$g(\mathcal{A}_X E, F) + g(E, \mathcal{A}_X F) = 0 \text{ and } g(\mathcal{T}_V E, F) + g(E, \mathcal{T}_V F) = 0 \tag{2.13}$$

for all $E, F \in T_pM$. We also see that the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}$ is the second fundamental form of the fibers of f. Since $\mathcal{T}_V$ is skew-symmetric, therefore we observe that f has totally geodesic fibers if and only if $\mathcal{T} = 0$.

Let (M, g_M) be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Let $f : M \to N$ be a smooth map. Then the second fundamental form of f is given by

$$(\nabla f_*)(X, Y) = \nabla^f_X f_* Y - f_*(\nabla_X Y), \text{ for all } X, Y \in \Gamma(T_pM), \tag{2.14}$$

where ∇ denotes the Levi-Civita connection of the metrics g_M and g_N and ∇^f is the pullback connection ([9], [11]). We also know that if f satisfies $(\nabla f_*)(X, Y) = 0$, for all $X, Y \in \Gamma(TM)$, then it is called totally geodesic.

Lemma 2.3. *If $f : M \to N$ denotes a horizontal conformal submersion, then we have*
(i) $(\nabla f_*)(X, Y) + g_M(X, Y) f_*(grad \ln\lambda) = X(\ln\lambda) f_* Y + Y(\ln\lambda) f_* X$,
(ii) $(\nabla f_*)(U, V) + f_*(\mathcal{T}_U V) = 0$,
(iii) $(\nabla f_*)(X, U) = -f_*(\nabla^M_X U) = -f_*(\mathcal{A}_X U)$
for all horizontal vector fields X, Y and vertical vector fields U, V.

3. Conformal semi-slant submersions

This section is concerned with the study of conformal semi-slant submersion from Kenmotsu manifold (M, g_M) to Riemannian manifold (N, g_N).

Definition 3.1. Let (M, g_M) be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Then a horizontal conformal submersion $f : (M, g_M) \to (N, g_N)$ is called a conformal semi-slant submersion if there is a distribution $D_1 \subset (\ker f_*)$ such that

$$\ker f_* = D_1 \oplus_{orth} D_2 \oplus_{orth} < \xi >, \ \ \phi(D_1) = D_1, \tag{3.1}$$

and the angle $\theta = \theta(X)$ between ϕX and the space $(D_2)_p$ is constant for non-zero vector field $X \in (D_2)_p$ and $p \in M$, where D_2, D_1 and $< \xi >$ are mutually orthogonal in $(\ker f_*)$. The angle θ is called the semi-slant angle of the horizontally conformal submersions.

It is known that the distribution $\ker f_*$ is integrable. Hence the Definition 3.1 implies that the integral manifold (fiber) $f^{-1}(q), q \in N$ of $\ker f_*$ is a semi-slant submanifold. Let f be a conformal semi-slant submersion from (M, g_M) onto (N, g_N). For all $V \in \Gamma(\ker f_*)$, we have

$$V = PV + QV + \eta(V)\xi,$$

where $PV \in \Gamma(D_1)$ and $QV \in \Gamma(D_2)$.

For $W \in \Gamma(\ker f_*)$, we have

$$\phi W = \psi W + \omega W, \tag{3.2}$$

where ψW and ωW are vertical and horizontal components of ϕW, respectively. Also, if $Z \in \Gamma(\ker f_*)^\perp$ then we have

$$\phi Z = BZ + CZ, \tag{3.3}$$

where BZ and CZ represent the vertical and horizontal components of ϕZ, respectively.

Next, $\Gamma(\ker f_*)^\perp$ is decomposed as

$$\Gamma(\ker f_*)^\perp = \omega D_2 \oplus \mu, \tag{3.4}$$

where μ denotes the orthogonal complement of ωD_2 in $\Gamma(\ker f_*)^\perp$ and it is invariant with respect to ϕ. In consequence of equations (2.2), (3.2) and (3.3) we infer

$$g_M(\psi X, Y) = -g_M(X, \psi Y), g_M(V, CW) = -g_M(CV, W) \tag{3.5}$$

for all $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$.

Again, the equations (3.2), (3.3), (3.4) and (3.5) together give

$$\psi D_1 = D_1, \omega D_1 = 0, \psi D_2 \subset D_2, B(\Gamma(\ker f_*)^\perp) = D_2.$$

Lemma 3.2. *If $f : (M, g_M) \to (N, g_N)$ is a conformal semi-slant submersion, then*

$$\psi^2 V + B\omega V = -V + \eta(V)\xi, \quad \omega\psi V + C\omega V = 0,$$

$$\psi BZ + BCZ = 0, \quad \omega BZ + C^2 Z = Z$$

for all $V \in \Gamma(\ker f_*)$ and $Z \in \Gamma(\ker f_*)^\perp$.

The covariant derivatives of ψ and ω are defined as:

$$(\nabla_V \psi)W = \widehat{\nabla}_V \psi W - \psi \widehat{\nabla}_V W,$$

$$(\nabla_V \omega)W = \mathcal{H}\nabla_V \omega W - \omega \widehat{\nabla}_V W$$

for all $V, W \in \Gamma(\ker f_*)$, where $\widehat{\nabla}_V W = \mathcal{V}\widehat{\nabla}_V W$.

Lemma 3.3. *Let the map $f : (M, g_M) \to (N, g_N)$ be a conformal semi-slant submersion, then*

1. $\mathcal{V}\nabla_X \psi Y + \mathcal{T}_X \omega Y + g_M(\psi X, Y)\xi + \eta(Y)\psi X = B\mathcal{T}_X Y + \psi\mathcal{V}\nabla_X Y,$
2. $\mathcal{T}_X \psi Y + \mathcal{H}\nabla_X \omega Y + \eta(Y)\omega X = C\mathcal{T}_X Y + \omega\mathcal{V}\nabla_X Y,$

3. $\mathcal{T}_X BV + \mathcal{H}\nabla_X CV = C\mathcal{H}\nabla_X V + \omega\mathcal{T}_X V$,
4. $\mathcal{V}\nabla_X BV + \mathcal{T}_X CV + g_M(\omega X, V)\xi = B\mathcal{H}\nabla_X V + \psi\nabla_X V$,
5. $\mathcal{V}\nabla_V \psi X + \mathcal{A}_V \omega X + g_M(X, BV)\xi + \eta(X)BV = B\mathcal{A}_V X + \psi\mathcal{V}\nabla_V X$,
6. $\mathcal{A}_V \psi X + \mathcal{H}\nabla_X \omega X + \eta(X)CV = C\mathcal{A}_V X + \omega\mathcal{V}\nabla_V X$,
7. $\mathcal{A}_U BV + \mathcal{H}\nabla_U CU = C\mathcal{H}\nabla_U V + \omega\mathcal{A}_U V$,
8. $\mathcal{V}\nabla_U BV + \mathcal{A}_U CV + g_M(CU, V)\xi = B\mathcal{H}\nabla_U V + \psi\mathcal{A}_U V$

for all $X, Y \in \Gamma(\ker f_*)$ *and* $U, V \in \Gamma(\ker f_*)^\perp$.

Proposition 3.4. *Let* $f : (M, g_M) \to (N, g_N)$ *be a conformal semi-slant submersion from* (M, g_M) *onto* (N, g_N). *Then* f *is a proper conformal semi-slant submersion if and only if there exists a constant* $\lambda \in [0, 1]$ *such that*

$$\psi^2 Z = \lambda Z, \text{ for all } Z \in \Gamma(D_2),$$

where $\lambda = -\cos^2\theta$.

Proof. If $Z \in \Gamma(D_2)$ and $Z \neq 0$, then we have

$$\cos\theta = \frac{\|\psi Z\|}{\|\phi Z\|},$$

and

$$\cos\theta = \frac{g_M(\phi Z, \psi Z)}{\|\psi Z\|\|\phi Z\|}, \tag{3.6}$$

where $\theta(Z)$ is the semi-slant angle. Using equations (2.1) and (3.2) in (3.6), we get

$$\cos\theta = \frac{g_M(Z, \psi^2 Z)}{\|\psi Z\|\|\phi Z\|}. \tag{3.7}$$

From equations (3.6) and (3.7), we have

$$\psi^2 Z = -\cos^2\theta Z.$$

If $\lambda = -\cos^2\theta$, then

$$\psi^2 Z = \lambda Z$$

for all $Z \in \Gamma(D_2)$. □

From Proposition (3.4) and equations (2.3), (3.2) and (3.3), we can state the following:

Corollary 3.5. *Let* $f : (M, g_M) \to (N, g_N)$ *be a conformal semi-slant submersion from* (M, g_M) *onto* (N, g_N), *then*

$$g_M(\psi X, \psi Y) = \cos^2\theta g_M(X, Y),$$
$$g_M(\omega X, \omega Y) = \sin^2\theta g_M(X, Y) \tag{3.8}$$

for all $X, Y \in \Gamma(D_2)$.

Proof. Let ω be parallel, then we have $((\nabla_X\omega)Y = \mathcal{H}\nabla_X\omega Y - \omega\mathcal{V}\nabla_X Y = 0)$ and hence from Lemma 3.3(1.), we conclude that

$$\mathcal{H}\nabla_X\omega Y - \omega\mathcal{V}\nabla_X Y + \eta(Y)\omega X = C\mathcal{T}_X Y - \mathcal{T}_X\psi Y,$$

and

$$C\mathcal{T}_X Y = \mathcal{T}_X\psi Y, \text{ for } X, Y \in \Gamma(D_2). \tag{3.9}$$

Interchanging X and Y in the equation (3.9), we have

$$C\mathcal{T}_Y X = \mathcal{T}_Y\psi X, \text{ for } X, Y \in \Gamma(D_2). \tag{3.10}$$

Since $\mathcal{T}$ is symmetric, therefore from equations (3.9) and (3.10), we follow the result. □

Theorem 3.6. *If the map f from (M, g_M) onto (N, g_N) is a conformal semi-slant submersion, then the invariant distribution D_1 is integrable if and only if*

$$\frac{1}{\lambda^2} g_N((\nabla f_*)(X, \phi Y) - (\nabla f_*)(Y, \phi X), f_*(\omega V)) = g_M(\psi(\widehat{\nabla}_X\phi Y - \widehat{\nabla}_Y\phi X), V)$$

for $X, Y \in \Gamma(D_1)$ and $V \in \Gamma(D_2)$.

Proof. We note that the distribution D_1 is integrable if and only if $g_M([X, Y], V) = 0$, $g_M([X, Y], \xi) = 0$ and $g_M([X, Y], W) = 0$, for $X, Y \in \Gamma(D_1), V \in \Gamma(D_2)$ and $W \in (\ker f_*)^\perp$. Since $(\ker f_*)$ is integrable and therefore $g_M([X, Y], W) = 0$. Thus, D_1 is integrable if and only if $g_M([X, Y], \xi) = 0$ and $g_M([X, Y], V) = 0$.

Now, from equations (2.2), (2.3), (2.4) and (3.2) it can be easily proved that

$$\begin{aligned} g_M([X, Y], V) &= g_M(\phi\nabla_X Y, \phi Z) - g_M(\phi\nabla_Y X, \phi Z) \\ &= g_M(\widehat{\nabla}_X\phi Y - \widehat{\nabla}_Y\phi X, \psi V) + g_M(\mathcal{H}\nabla_X\phi Y, \omega V) - g_M(\mathcal{H}\nabla_Y\phi X, \omega V). \end{aligned}$$

Since f is a conformal submersion, therefore from equation (2.14) we have

$$\begin{aligned} g_M([X, Y], V) = &-\frac{1}{\lambda^2} g_N((\nabla f_*)(X, \phi Y) - (\nabla f_*)(Y, \phi X), f_*(\omega V)) \\ &- g_M(\psi(\widehat{\nabla}_X\phi Y - \widehat{\nabla}_Y\phi X), V). \end{aligned}$$

From the above equations, we conclude that the distribution D_1 is integrable if and only if

$$\frac{1}{\lambda^2} g_N((\nabla f_*)(X, \phi Y) - (\nabla f_*)(Y, \phi X), f_*(\omega V)) = g_M(\psi(\widehat{\nabla}_X\phi Y - \widehat{\nabla}_Y\phi X), V),$$

which gives our result. □

Theorem 3.7. *Let f be a conformal semi-slant submersion from (M, g_M) onto (N, g_N). Then the semi-slant distribution D_2 is integrable if and only if*

$$\mathcal{T}_X\omega Y - \mathcal{T}_Y\omega X + \psi(\mathcal{T}_X\omega\psi Y - \mathcal{T}_Y\omega X) \in \Gamma(D_2)$$

for $X, Y \in \Gamma(D_2)$.

Proof. It is observed that the distribution D_2 is integrable if and only if $g_M([X,Y],V) = 0$, $g_M([X,Y],\xi) = 0$ and $g_M([X,Y],W) = 0$, for $X,Y \in \Gamma(D_2), V \in \Gamma(D_1)$ and $W \in (\ker f_*)^\perp$. Since the $\ker f_*$ is integrable and therefore $g_M([X,Y],W) = 0$. Thus, D_2 is integrable if and only if $g_M([X,Y],\xi) = 0$ and $g_M([X,Y],V) = 0$.

From equations (2.2), (2.3) and (3.2), we have

$$\begin{aligned}g_M([X,Y],\phi V) = g_M([X,Y],\phi V) = g_M(\nabla_X\psi Y, V) + g_M(\nabla_X\omega Y, V)\\ -g_M(\nabla_Y\psi X, V) - g_M(\nabla_Y\omega X, V) = g_M(\phi\nabla_X\psi Y, \phi V)\\ +g_M(\phi\nabla_X\omega Y, \phi V) - g_M(\nabla_Y\psi X, V) - g_M(\nabla_Y\omega X, V).\end{aligned}$$

Next, using equation (2.10) and Proposition 3.4, in the above equation, we have

$$\sin^2\theta g_M([X,Y],\phi V) = g_M(\mathcal{T}_X\omega Y - \mathcal{T}_Y\omega X, V) + g_M(\psi(\mathcal{T}_X\omega\psi Y - \mathcal{T}_Y\omega\psi X), V),$$

which follows the result. □

Theorem 3.8. *Let f be a conformal semi-slant submersion from (M, g_M) onto (N, g_N). Then the distribution $(\ker f_*)^\perp$ is integrable if and only if*

$$\begin{aligned}&\frac{1}{\lambda^2}\{g_N(\nabla_Y f_*(CX), f_*(\omega V)) - g_N(\nabla_X f_*(CY), f_*(\omega V))\}\\ &= g_M(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY - \mathcal{V}\nabla_Y BX - \mathcal{A}_Y CX, \psi V)\\ &\quad +g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX - CY(\ln\lambda)X + CX(\ln\lambda)Y\\ &\quad +2g_M(X, CY) grad \ln\lambda, \omega V))\end{aligned}$$

for $X,Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$.

Proof. For $X,Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$, we have

$$\begin{aligned}g_M([X,Y],V) &= g_M(\nabla_X Y, V) - g_M(\nabla_Y X, V)\\ &= g_M(\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - g_M(\nabla_Y BX, \phi V)\\ &\quad -g_M(\nabla_Y CX, \phi V) - \eta(V) g_M([X,Y],\xi)\\ &= g_M(\nabla_X\phi Y, \phi V) - g_M(\nabla_Y\phi X, \phi V)\\ &= g_M(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY - \mathcal{V}\nabla_Y BX - \mathcal{A}_Y CX, \psi V)\\ &\quad +g_M(\mathcal{A}_X BY, \omega V) + g_M(\mathcal{H}\nabla_X CY, \omega V) - g_M(\mathcal{A}_Y BX, \omega V)\\ &\quad -g_M(\mathcal{H}\nabla_Y CX, \omega V),\end{aligned}$$

where equations (2.2), (2.3), (2.12), (2.13), (3.2) and (3.3) have been used. Since f is a conformal semi-slant submersion and therefore from equation (2.14) and Lemma

2.2, we get

$$
\begin{aligned}
g_M([X,Y],V) &= g_M(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY - \mathcal{V}\nabla_Y BX - \mathcal{A}_Y CX, \psi V) \\
&+\frac{1}{\lambda^2} g_N(\nabla_X f_*(CY), f_*(\omega V)) - \frac{1}{\lambda^2} g_N(\nabla_Y f_*(CX), f_*(\omega V)) \\
&+\frac{1}{\lambda^2} g_N(f_*(\mathcal{A}_X BY), f_*(\omega V)) - \frac{1}{\lambda^2} g_N(f_*(\mathcal{A}_Y BX), f_*(\omega V)) \\
&-\frac{1}{\lambda^2} g_N(X(\ln\lambda) f_*(CY) + CY(\ln\lambda) f_*(X) - g_M(X, CY) f_*(grad \ln\lambda), f_*(\omega V)) \\
&+\frac{1}{\lambda^2} g_N(Y(\ln\lambda) f_*(CX) + CX(\ln\lambda) f_*(Y) - g_M(Y, CX) f_*(grad \ln\lambda), f_*(\omega V)) \\
&= g_M(\psi(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY - \mathcal{V}\nabla_Y BX - \mathcal{A}_Y CX), V) \\
&+g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX - CY(\ln\lambda)X + CX(\ln\lambda)Y \\
&+2g_M(X, CY) grad \ln\lambda, \omega V) + \frac{1}{\lambda^2} g_N(\nabla_X f_*(CY), f_*(\omega V)) \\
&-\frac{1}{\lambda^2} g_N(\nabla_Y f_*(CX), f_*(\omega V)),
\end{aligned}
$$

which prove our desired result. □

Theorem 3.9. *Let f be a conformal semi-slant submersion from (M, g_M) onto (N, g_N). Then the distribution $(\ker f_*)$ is a totally geodesic foliation on M if and only if*

$$
\begin{aligned}
& \frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)) \\
= \;& g_M(\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) \\
& -g_M(\mathcal{A}_{\omega V} \phi CX, \psi U) + g_M(\omega U, \omega V) g_M(grad \ln\lambda, \phi CX)
\end{aligned}
$$

for $U, V \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$.

Proof. Let U, $V \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$, then by using equations (2.2), (2.3) and (3.3), we get

$$
\begin{aligned}
g_M(\nabla_U V, X) \;=\; & g_M(\nabla_U \psi V, BX) + g_M(\nabla_U \omega V, BX) + g_M(\nabla_U \psi V, CX) \\
& +g_M(\nabla_U \omega V, CX).
\end{aligned}
$$

Since f is a conformal submersion, therefore equations (2.9), (2.10), (2.14) and Lemma 2.2, together with the above equation give

$$
\begin{aligned}
g_M(\nabla_U V, X) &= g_M(\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) \\
&-g_M(\mathcal{A}_{\omega V} \phi CX, \psi U) - \frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)) \\
&+\frac{1}{\lambda^2} g_N(\omega V(\ln\lambda) f_*(\phi CX) + \phi CX(\ln\lambda) f_*(\omega V) \\
&-g_M(\omega V, \phi CX) f_*(grad \ln\lambda), f_*(\omega U)).
\end{aligned}
$$

Hence, we have

$$\begin{aligned} & g_M(\nabla_U V, X) \\ = \; & g_M(\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) \\ & - g_M(\mathcal{A}_{\omega V} \phi CX, \psi U) - \frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)) \\ & + g_M(\omega U, \omega V) g_M(\mathcal{H} grad \ln \lambda, \phi CX), \end{aligned} \tag{3.11}$$

which gives our require result. □

Theorem 3.10. *Let f be a conformal semi-slant submersion from (M, g_M) onto (N, g_N). Then, any two of the following conditions imply the third one:*
(i) $\ker f_*$ *is a totally geodesic foliation on M,*
(ii) *λ is a constant on $\Gamma(D_1)$,*
(iii) $\frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U))$
$= -g_M(C(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) - g_M(\mathcal{A}_{\omega V} \phi CX, \psi U)$ *for all $U, V \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$.*

Proof. From equation (3.11), we have

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) \\ & - g_M(\mathcal{A}_{\omega V} \phi CX, \psi U) - \frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)) \\ & + g_M(\omega U, \omega V) g_M(\mathcal{H} grad \ln \lambda, \phi CX) \end{aligned}$$

for all $U, V \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$. Now, we suppose that the conditions (i) and (ii) hold, then we obtain

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) \\ & - g_M(\mathcal{A}_{\omega V} \phi CX, \psi U) - \frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)). \end{aligned}$$

From the above equation, we obtain (iii). Similarly, we can easily obtain the other assertions. □

4. Totally geodesic maps

In this section, we examine the conditions for which the conformal semi-slant submersions to be totally geodesics. Throughout the section, we denote (M, g_M) and (N, g_N) for the Kenmotsu manifold and the Riemannian manifold, respectively. Before going to state our results, first we recall the following definition as:

Definition 4.1. A smooth map f from (M, g_M) onto (N, g_N) is said to be totally geodesic if $(\nabla f_*)(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$ ([9], [11]).

Now, we state the following theorem as:

Theorem 4.2. *A conformal semi-slant submersion f from (M, g_M) onto (N, g_N) is totally geodesic if*

$$\nabla_Z f_*(Y) = f_*(C(\mathcal{A}_Z\psi Y_1 + \mathcal{H}\nabla_Z\omega Y_1 + \mathcal{A}_Z BY_2 + \mathcal{H}\nabla_Z CY_2 + \eta(Y_1)Z)$$
$$+\omega(\mathcal{V}\nabla_Z\psi Y_1 + \mathcal{A}_Z\omega Y_1 + \mathcal{V}\nabla_Z BY_2 + \mathcal{A}_Z CY_2))$$

holds for any $Z \in \Gamma(\ker f_*)^\perp$ and $Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(\ker f_*)$ and $Y_2 \in \Gamma(\ker f_*)^\perp$.

Proof. Let $Z \in \Gamma(\ker f_*)^\perp$ and $Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(\ker f_*)$ and $Y_2 \in \Gamma(\ker f_*)^\perp$, then from equations (2.1), (2.4), (2.14), (3.2) and (3.3), we observe that

$$(\nabla f_*)(Z, Y) = \nabla_Z f_*(Y) - f_*((\phi^2 - \eta \oplus \xi)(\nabla_Z Y))$$
$$= \nabla_Z f_*(Y) - f_*(\phi(\nabla_Z\psi Y_1 + \nabla_Z\omega Y_1 + \nabla_Z BY_2 + \nabla_Z CY_2) + \eta(Y_1)Z$$
$$-g_M(Z, Y_2)\xi).$$

Again, using equations (2.10), (2.11), (2.13), (3.2) and (3.3) in the above equation, we obtain

$$\begin{aligned}
&(\nabla f_*)(Z, Y)\\
= \;& \nabla_Z f_*(Y) - f_*((B\mathcal{A}_Z\psi Y_1 + C\mathcal{A}_Z\psi Y_1 + \psi\mathcal{V}\nabla_Z\psi Y_1 + \omega\mathcal{V}\nabla_Z\psi Y_1\\
&+B\mathcal{H}\nabla_Z\omega Y_1 + C\mathcal{H}\nabla_Z\omega Y_1 + \psi\mathcal{A}_Z\omega Y_1 + \omega\mathcal{A}_Z\omega Y_1 + B\mathcal{A}_Z BY_2\\
&+C\mathcal{A}_Z BY_2 + \psi\mathcal{V}\nabla_Z BY_2 + \omega\mathcal{V}\nabla_Z BY_2 + B\mathcal{H}\nabla_Z CY_2 + C\mathcal{H}\nabla_Z CY_2\\
&+\psi\mathcal{A}_Z CY_2 + \omega\mathcal{A}_Z CY_2) + \eta(Y_1)BZ + \eta(Y_1)CZ - g_M(Z, Y_2)\xi).
\end{aligned}$$

Hence the vertical parts infer that

$$\begin{aligned}
&(\nabla f_*)(Z, Y)\\
= \;& \nabla_Z f_*(Y) - f_*(C(\mathcal{A}_Z\psi Y_1 + \mathcal{H}\nabla_Z\omega Y_1 + \mathcal{A}_Z BY_2 + \mathcal{H}\nabla_Z CY_2 + \eta(Y_1)Z)\\
&+\omega(\mathcal{V}\nabla_Z\psi Y_1 + \mathcal{A}_Z\omega Y_1 + \mathcal{V}\nabla_Z BY_2 + \omega\mathcal{A}_Z CY_2)).
\end{aligned}$$

This gives the require result. □

Theorem 4.3. *Let f be a conformal semi-slant submersion from (M, g_M) onto (N, g_N). Then the map f is totally geodesic if and only if*

1. $\frac{1}{\lambda^2}\{g_N((\nabla f_*)(U, \omega\psi V), f_*(Z)) - g_N(\nabla_{\omega V} f_*(\omega U), f_*(\phi CZ))\}$
   $= g_M(\mathcal{T}_U\omega V, BZ) + g_M(\mathcal{A}_{\omega V}\psi U, \phi CZ) + g_M(grad \ln\lambda, \omega V)g_M(\omega U, \phi CZ)$,
2. $g_M(\mathcal{V}\nabla_{U_1}\phi V_1, BZ) = g_M(\mathcal{T}_{U_1}\phi Z, CV_1)$,
3. $\frac{1}{\lambda^2}\{g_N((\nabla f_*)(W, CX), f_*(CY)) + g_N((\nabla f_*)(W, \omega BX), f_*(Y))\}$
   $= g_M(\mathcal{T}_W\psi BX, Y) + g_M(\mathcal{T}_W CX, BY)$,
4. *f is a horizontally homothetic map, for $U_1, V_1 \in \Gamma(D_1), U, V \in \Gamma(D_2), W \in \Gamma(\ker f_*)$ and $X, Y, Z \in \Gamma(\ker f_*)^\perp$.*

Proof. Let $U, V \in \Gamma(D_2)$ and $Z \in \Gamma(\ker f_*)^\perp$, then the equations (2.3), (2.5), (2.6), (2.14) and (3.2) give

$$\begin{aligned}\frac{1}{\lambda^2} g_N((\nabla f_*)(U, V), f_*(Z)) &= -g_M(\nabla_U V, Z)\\ &= -g_M(\nabla_U \psi^2 V, Z) - g_M(\nabla_U \omega\psi V, Z) - g_M(\nabla_U \omega V, \phi Z).\end{aligned}$$

From equations (2.3), (2.5), (2.11), (3.3) and Corollary 3.5, we have

$$\begin{aligned}&\sin^2\theta \frac{1}{\lambda^2} g_N((\nabla f_*)(U, V), f_*(Z))\\ &= -g_M(\nabla_U V, Z) - g_M(\mathcal{H}\nabla_U \omega\psi V, Z) - g_M(\mathcal{T}_U \omega V, BZ) - g_M(\nabla_{\omega V} \phi U, \phi CZ).\end{aligned}$$

Since the smooth map f is a conformal submersion, therefore by using equation (2.14) and Lemma 2.2, we find

$$\begin{aligned}&\sin^2\theta \frac{1}{\lambda^2} g_N((\nabla f_*)(U, V), f_*(Z))\\ &= -g_M(\mathcal{T}_U \omega V, BZ) - \frac{1}{\lambda^2} g_N((\nabla f_*)(U, \omega\psi V), f_*(Z))\\ &\quad -g_M(\mathcal{A}_{\omega V} \psi U, \phi CZ) + \frac{1}{\lambda^2} g_M(\nabla_{\omega V} f_*(\omega U), f_*(\phi CZ))\\ &\quad -g_M(\omega V(\ln\lambda) f_*(\omega U) + \omega U(\ln\lambda) f_*(\omega V)\\ &\quad -g_M(\omega V, \omega U) f_*(grad \ln\lambda), f_*(\phi CZ)),\end{aligned}$$

which implies

$$\begin{aligned}&\sin^2\theta \frac{1}{\lambda^2} g_N((\nabla f_*)(U, V), f_*(Z))\\ &= -g_M(\mathcal{T}_U \omega V, BZ) - \frac{1}{\lambda^2} g_N((\nabla f_*)(U, \omega\psi V), f_*(Z))\\ &\quad -g_M(\mathcal{A}_{\omega V} \psi U, \phi CZ) - \frac{1}{\lambda^2} g_M(\nabla_{\omega V} f_*(\omega U), f_*(\phi CZ))\\ &\quad -g_M(\omega V, \omega U) g_M(grad \ln\lambda, \phi CZ),\end{aligned}$$

which gives the first assertion. To prove the second assertion, we suppose that $U_1, V_1 \in \Gamma(D_1)$ and $Z \in \Gamma(\ker f_*)^\perp$, then making use of equations (2.3), (2.5), (2.6), (2.14) and (3.3), we get

$$\frac{1}{\lambda^2} g_N((\nabla f_*)(U_1, V_1), f_*(Z)) = -g_M(\nabla_{U_1} \phi V_1, BZ) - g_M(\nabla_{U_1} \phi V_1, CZ).$$

Again, using equations (2.10) and (2.11), we get

$$\frac{1}{\lambda^2} g_N((\nabla f_*)(U_1, V_1), f_*(Z)) = -g_M(\mathcal{V}\nabla_{U_1} \phi V_1, BZ) - g_M(\mathcal{T}_{U_1} \phi Z, CV_1).$$

Now, we shall prove the third assertion. For $W \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^\perp$, using equations (2.3), (2.5), (2.6) and (2.14), we have

$$\frac{1}{\lambda^2} g_N((\nabla f_*)(X,Y), f_*(W)) = -g_M(\nabla_X \phi Y, \phi W).$$

Also, in view of equations (2.3), (2.5), (2.10), (2.14), (3.2) and (3.3), we conclude that

$$\begin{aligned} & \frac{1}{\lambda^2} g_N((\nabla f_*)(W,X), f_*(Y)) \\ = & -g_M(\mathcal{T}_W \psi BX, Y) - g_M(\mathcal{T}_W CX, BY) \\ & -g_M(\mathcal{H}\nabla_W \omega BX, Y) - g_M(\mathcal{H}\nabla_W CX, CY). \end{aligned}$$

The equation (2.14) together with our hypothesis that f is a conformal submersion infer

$$\begin{aligned} & \frac{1}{\lambda^2} g_N((\nabla f_*)(W,X), f_*(Y)) \\ = & -g_M(\mathcal{T}_W \psi BX, Y) - g_M(\mathcal{T}_W CX, BY) \\ & +\frac{1}{\lambda^2}\{g_N((\nabla f_*)(W,\omega BX), f_*(Y)) - g_N((\nabla f_*)(W, CX), f_*(CY))\}. \end{aligned}$$

To prove the fourth assertion, we suppose that $X_1, X_2 \in \Gamma(\mu)$ and therefore from Lemma 2.2, we have

$$(\nabla f_*)(X_1, X_2) = X_1(\ln\lambda) f_*(X_2) + X_2(\ln\lambda) f_*(X_1) - g_M(X_1, X_2) f_*(grad \ln\lambda).$$

From the above equation, putting $X_2 = \phi X_1$ for $X_1 \in \Gamma(\mu)$, we get

$$\begin{aligned} & (\nabla f_*)(X_1, \phi X_1) \\ & = X_1(\ln\lambda) f_*(\phi X_1) + \phi X_1(\ln\lambda) f_*(X_1) - g_M(X_1, \phi X_1) f_*(grad \ln\lambda) \\ & = X_1(\ln\lambda) f_*(\phi X_1) + \phi X_1(\ln\lambda) f_*(X_1). \end{aligned}$$

If $(\nabla f_*)(X_1, \phi X_1) = 0$, then we have

$$X_1(\ln\lambda) f_*(\phi X_1) + \phi X_1(\ln\lambda) f_*(X_1) = 0. \tag{4.1}$$

Taking inner product of equation (4.1) with $f_*(\phi X_1)$, we have

$$\begin{aligned} & \frac{1}{\lambda^2}\{g_M(grad \ln\lambda, X_1) g_N(f_*(\phi X_1), f_*(\phi X_1)) \\ & + g_M(grad \ln\lambda, \phi X_1) g_N(f_*(X_1), f_*(\phi X_1))\} = 0, \end{aligned}$$

since f is a conformal submersion. From the above equation, it follows that λ is a constant on $\Gamma(\mu)$. Again, for $U_2, V_2 \in \Gamma(\ker f_*)$, the Lemma 2.2 gives

$$\begin{aligned} & (\nabla f_*)(\omega U_2, \omega V_2) \\ & = \omega U_2(\ln\lambda) f_*(\omega V_2) + \omega V_2(\ln\lambda) f_*(\omega U_2) - g_M(\omega U_2, \omega V_2) f_*(grad \ln\lambda). \end{aligned}$$

Setting $V_2 = U_2$ in the above equation, we have

$$(\nabla f_*)(\omega U_2, \omega U_2) \tag{4.2}$$
$$= 2\omega U_2(\ln\lambda) f_*(\omega U_2) - g_M(\omega U_2, \omega U_2) f_*(grad \ln\lambda).$$

Taking inner product of equation (4.2) with $f_*(\omega U_2)$ and consider that f is a conformal submersion, we get

$$g_M(\omega U_2, \omega U_2) g_M(grad \ln\lambda, \omega U_2) = 0.$$

It follows that λ is constant on $\Gamma(\omega(\ker f_*))$. So, λ is also constant on $\Gamma(\ker f_*)^\perp$. On the other hand, if f is a horizontally homothetic map, then $(\nabla f_*)(X, Y) = 0$. □

Definition 4.4. A conformal semi-slant submersion f from (M, g_M) to (N, g_N) is said to be an $(\omega D_2, \mu)$-totally geodesic map if

$$(\nabla f)(\omega V, X) = 0$$

for $V \in \Gamma(D_2)$ and $X \in \Gamma(\mu)$.

Theorem 4.5. *Let f be a conformal semi-slant submersion from (M, g_M) onto (N, g_N). Then f is an $(\omega D_2, \mu)$-totally geodesic map if and only if it is horizontally homothetic map.*

Proof. For $U \in \Gamma(D_2)$ and $X \in \Gamma(\mu)$, the Lemma 2.2 gives

$$(\nabla f_*)(\omega U, X) = \omega U(\ln\lambda) f_*(X) + X(\ln\lambda) f_*(\omega U) - g_M(\omega U, X) f_*(grad \ln\lambda).$$

If f is a horizontally homothetic map, then $(\nabla f_*)(\omega U, X) = 0$. Conversely, if $(\nabla f_*)(\omega U, X) = 0$ then it is obvious that

$$\omega U(\ln\lambda) f_*(X) + X(\ln\lambda) f_*(\omega U) = 0. \tag{4.3}$$

The inner product of equation (4.3) with $f_*(\omega U)$ infers that

$$g_M(\omega U, \omega U) g_M(grad \ln\lambda, X) = 0,$$

since f is a conformal semi-slant submersion. It is clear from the above equation that λ is a constant on $\Gamma(\mu)$. Again, the inner product of equation (4.3) with $f_*(X)$ gives

$$g_M(X, X) g_M(grad \ln\lambda, \omega U) = 0,$$

which reflects that λ is a constant on $\Gamma(\omega D_2)$ and it is also constant on $\Gamma(\ker f_*)^\perp$. □

Theorem 4.6. *Let f be a conformal semi-slant submersion from (M, g_M) onto (N, g_N). Then f is a totally geodesic map if and only if*

1. $C\mathcal{T}_V \psi Y + \omega \mathcal{V} \nabla_V \psi Y = 0$ for $V, Y \in \Gamma(D_1)$,
2. $C(\mathcal{T}_V \psi W + \mathcal{A}_{\omega V} W) + \omega(\nabla_V \psi W + \mathcal{T}_V \omega W) = 0$ for $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$,
3. $C(\mathcal{T}_U BX + \mathcal{A}_{CX} U) + \omega(\mathcal{V} \nabla_U BX + \mathcal{T}_U CX) = 0$ for $U \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$.

Proof. Let $V, Y \in \Gamma(D_1)$, then from equations (2.1), (2.5), (2.14) and (3.2), we have

$$(\nabla f_*)(V, Y) = -f_*(\phi(\nabla_V \psi Y) - \eta(\nabla_V Y)\xi).$$

Next, from equations (2.10), (3.2) and (3.3) we lead to

$$(\nabla f_*)(V, Y) = -f_*(B\mathcal{T}_V \psi Y + C\mathcal{T}_V \psi Y + \psi \mathcal{V}\nabla_V \psi Y + \omega \mathcal{V}\nabla_V \psi Y$$
$$+B\mathcal{H}\nabla_V \omega Y + C\mathcal{H}\nabla_V \omega Y + C\mathcal{H}\nabla_V \omega Y + \omega \mathcal{T}_V \omega Y - g_M(V, Y)\xi).$$

Since $B\mathcal{T}_V \psi Y + \psi\nabla_V \psi Y - g_M(V, Y)\xi \in \Gamma(\ker f_*)$, therefore we have

$$(\nabla f_*)(V, Y) = -f_*(C\mathcal{T}_V \psi Y + \omega \mathcal{V}\nabla_V \psi Y).$$

Since f is a linear isomorphism between $(\ker f_*)^\perp$ and TM and therefore $(\nabla f_*)(V, Y) = 0 \Leftrightarrow (C\mathcal{T}_V \psi Y + \omega \mathcal{V}\nabla_V \psi Y) = 0$, which prove the first assertion.

Now, we prove the second assertion. For $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$, the equations (2.1), (2.5), (2.14) and (3.2) give

$$(\nabla f_*)(V, W) = -f_*(\phi\nabla_V \psi W + \phi\nabla_V \omega W).$$

Again, in consequence of equations (2.10), (2.11), (3.2) and (3.3), we obtain

$$(\nabla f_*)(V, W) = -f_*(B\mathcal{T}_V \psi W + C\mathcal{T}_V \psi W + \psi \mathcal{V}\nabla_V \psi W + \omega \mathcal{V}\nabla_V \psi W$$
$$+B\mathcal{H}\nabla_V \omega W + C\mathcal{H}\nabla_V \omega W + \psi \mathcal{T}_V \omega W + \omega \mathcal{T}_V \omega W).$$

Since $B\mathcal{T}_V \psi W + \psi \mathcal{V}\nabla_V \psi W + B\mathcal{A}_{\omega V} W + \psi \mathcal{T}_V \omega W \in \Gamma(\ker f_*)$, therefore the above equation assumes the form

$$(\nabla f_*)(V, W) = -f_*(C(\mathcal{T}_V \psi W + \mathcal{A}_{\omega V} W) + \omega(\nabla_V \psi W + \mathcal{T}_V \omega W)).$$

It is noticed that the map f is linear isomorphism between $(\ker f_*)^\perp$ and TN, and thus we have $(\nabla f_*)(V, W) = 0 \Longleftrightarrow C(\mathcal{T}_V \psi W + \mathcal{A}_{\omega V} W) + \omega(\nabla_V \psi W + \mathcal{T}_V \omega W) = 0$.

Next, we shall prove the third assertion. If $U \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$, then from equations (2.1), (2.3), (2.14) and (3.8), we obtain

$$(\nabla f_*)(U, X) = -f_*(\phi(\nabla_U BX + \nabla_U CX)).$$

Using equations (2.10), (3.2) and (3.3) in the above equation, we have

$$(\nabla f_*)(U, X) = -f_*(B\mathcal{T}_U BX + C\mathcal{T}_U BX + \psi \mathcal{V}\nabla_U BX + \omega \mathcal{V}\nabla_U BX$$
$$+\psi \mathcal{T}_U CX + \omega \mathcal{T}_U CX + B\mathcal{H}\nabla_U CX + C\mathcal{H}\nabla_U CX).$$

Since $B\mathcal{T}_U BX + \psi \mathcal{V}\nabla_U BX + \psi \mathcal{T}_U CX + B\mathcal{H}\nabla_U CX \in \Gamma(\ker f_*)^\perp$, therefore the above equation assumes the form

$$(\nabla f_*)(U, X) = -f_*(C(\mathcal{T}_U BX + \mathcal{A}_{CX} U) + \omega(\mathcal{V}\nabla_U BX + \mathcal{T}_U CX)).$$

The map f is linear isomorphism between $(\ker f_*)^\perp$ and TN, and therefore we have $(\nabla f_*)(U, X) = 0 \Leftrightarrow C(\mathcal{T}_U BX + \mathcal{A}_U CX) + \omega(\widehat{\nabla}_U BX + \mathcal{T}_U CX) = 0$. □

5. Example

Let $M = R^7$ be a Kenmotsu manifold as discussed in Example 1 and $N = R^2$ be a Riemannian manifold. The Riemannian metric $g_2 = e^{-2z}(dv_1 \otimes dv_1 + dv_2 \otimes dv_2)$ on R^2. Let $f : R^7 \longrightarrow R^2$ be a submersion defined by

$$f(x_1, x_2, x_3, y_1, y_2, y_3, z) = e^3(\frac{x_2 - y_3}{\sqrt{2}}, y_2).$$

Then by direct calculation, we find the Jacobian matrix of f as

$$\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

After a straightforward computation, we have

$$\ker f_* = span\{V_1 = \frac{\partial}{\partial x_1}, V_2 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_3}), V_3 = \frac{\partial}{\partial x_3}, V_4 = \frac{\partial}{\partial y_1}, V_5 = \frac{\partial}{\partial z}\}$$

and

$$(\ker f_*)^\perp = span\{X_1 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_3}), X_2 = \frac{\partial}{\partial y_3}\}.$$

Thus it follows that distributions $D_1 = span\{V_1, V_4\}$ and $D_2 = span\{V_2, V_3\}$. Therefore, the map f is conformal semi-slant submersion with the semi-slant angle $\theta = \frac{\pi}{4}$ and dilation $\lambda = e^3$.

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