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A STUDY OF η-RICCI SOLITONS ON 3-DIMENSIONAL HYPERBOLIC KENMOTSU MANIFOLDS

JHANTU DAS ✉ and KALYAN HALDER

Abstract

In this paper, we have studied η-Ricci solitons in the context of 3-dimensional hyperbolic Kenmotsu manifolds. Firstly we have established some results on 3-dimensional hyperbolic Kenmotsu manifolds whose metric tensors are η-Ricci solitons. Next, we have studied Ricci symmetric and φ-Ricci symmetric on 3-dimensional hyperbolic Kenmotsu manifold admitting η-Ricci solitons. Also, we consider η-Ricci solitons on 3-dimensional hyperbolic Kenmotsu manifolds having certain special types of Ricci tensor. Finally, we have given an example to prove the existence of η-Ricci solitons on 3-dimensional hyperbolic Kenmotsu manifold and verify the results.

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1. Introduction

In 1982, the concept of *Ricci flow* was first introduced by Richard S. Hamilton [16] to evaluated a canonical metric on a differentiable manifold. A time dependent Riemannian metric $g(t)$ on a differentiable manifold M is said to evolve by the *Ricci flow* if the Riemannian metric g satisfies

$$\frac{\partial}{\partial t}(g(t)) = -2S(g(t))$$

on M, where S is the well-known second order symmetric Ricci tensor field of the metric g. A solution to the *Ricci flow* equation is called a Ricci soliton if it moves only by a one-parameter group of diffeomorphism and scaling [19].

A Ricci soliton is nothing but a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) by

$$\mathcal{L}_{\bar{V}}g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $\bar{V}$ is a vector field on M, λ is a real number and $\mathcal{L}_{\bar{V}}$ is the Lie derivative operator along the potential vector field $\bar{V}$. The potential vector field $\bar{V}$ is Killing if $\mathcal{L}_{\bar{V}}g = 0$,

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whereas conformal Killing if $\mathcal{L}_{\bar{V}}g = \rho g$, where ρ is a real valued smooth function on M. Also, a Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive, respectively [17]. Sharma [23], Bejan and Crasmareanu [3], Baird and Danielo [1], Ghosh [14], and Nagaraja and Premalatha [20], have extensively studied Ricci solitons in contact metric manifolds. Also, Bhattacharyya, et al. [4] studied the torqued vector field and established some applications of torqued vector field on Ricci soliton and conformal Ricci soliton. Ricci solitons have been investigated by several geometers in various contexts ([11], [12], [10], [13], [26], [27], [28]) and many others.

In 2019, the concept of Ricci soliton to η-Ricci soliton is extended by J. T. Cho and M. Kimura [9] on a Riemannian manifold (M, g) by

$$\mathcal{L}_{\bar{V}}g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where λ and μ are real constants on M. In particular, If μ vanishes identically in (1.2), then soliton equation (1.2) becomes a η-Ricci soliton. In [9], J. T. Cho and M. Kimura proved that a real hypersurface admitting an η-Ricci soliton in a non-flat complex space form is a Hopf-hypersurface. In [2], Basu, et al. generated some results on almost conformal Ricci soliton and η-Ricci soliton on 3-dimensional (ϵ, δ)-trans-Sasakian manifold. Recently there has been a lot of work on the η-Ricci solitons on different contact metric manifolds by S. Pahan [21], A. M. Blaga [6], D. G. Prakasha and B. S. Hadimani [22], H. I. Yoldaş et al. [29] and many others.

Motivated by the above studies, we are going to study η-Ricci solitons in the context of 3-dimensional hyperbolic Kenmotsu manifolds. The present paper is constructed in the following way: After introduction, in section 2, we recall some preliminary definitions and results of 3-dimensional hyperbolic Kenmotsu manifolds. Moreover, we have established some results on 3-dimensional hyperbolic Kenmotsu manifolds whose metric tensors are η-Ricci solitons in section 3. Next, we study Ricci symmetric and φ-Ricci symmetric on 3-dimensional hyperbolic Kenmotsu manifold admitting η-Ricci solitons in section 4. Also, we consider η-Ricci solitons on 3-dimensional hyperbolic Kenmotsu manifolds having certain special types of Ricci tensor in section 5 and section 6. Finally in section 7, we give an example to prove the existence of η-Ricci solitons on 3-dimensional hyperbolic Kenmotsu manifold and validate our some results.

2. Preliminaries

An odd dimensional differentiable manifold M^{2m+1} is named to be an almost hyperbolic contact metric manifold [25] if it admits a timelike vector field ξ, a 1-form η, a fundamental tensor field φ of the type $(1, 1)$ and a semi-Riemannian metric g satisfying:

$$\varphi^2(\bar{X}) = \bar{X} + \eta(\bar{X})\xi, \quad \eta(\xi) = -1 \implies \varphi(\xi) = 0, \quad \eta(\varphi\bar{X}) = 0, \tag{2.1}$$

$$g(\varphi\bar{X},\varphi\bar{Y}) = -g(\bar{X},\bar{Y}) - \eta(\bar{X})\eta(\bar{Y}), \qquad g(\bar{X},\xi) = \eta(\bar{X}), \tag{2.2}$$

and

$$g(\varphi\bar{X},\bar{Y}) + g(\bar{X},\varphi\bar{Y}) = 0 \tag{2.3}$$

for all $\bar{X},\bar{Y} \in \chi(M^{2m+1})$. The structure (φ,ξ,η,g) on the manifold M^{2m+1} is named as almost hyperbolic contact metric structure [25]. If an almost hyperbolic contact metric manifold M^{2m+1} is fulfilled the following condition:

$$(\nabla_{\bar{X}}\varphi)\bar{Y} = g(\varphi\bar{X},\bar{Y})\xi - \eta(\bar{Y})\varphi\bar{X} \tag{2.4}$$

for all $\bar{X},\bar{Y} \in \chi(M^{2m+1})$, then the manifold M^{2m+1} is called a hyperbolic Kenmotsu manifold [5]. Here ∇ denotes the Levi-Civita connection of the metric g. From the antecedent equation it is clear that

$$\nabla_{\bar{X}}\xi = -\bar{X} - \eta(\bar{X})\xi \tag{2.5}$$

and

$$(\nabla_{\bar{X}}\eta)\bar{Y} = g(\varphi\bar{X},\varphi\bar{Y}) = -g(\bar{X},\bar{Y}) - \eta(\bar{X})\eta(\bar{Y}). \tag{2.6}$$

In a $(2m+1)$-dimensional hyperbolic Kenmotsu manifold M^{2m+1} [8], we have

$$R(\bar{X},\bar{Y})\xi = \eta(\bar{Y})\bar{X} - \eta(\bar{X})\bar{Y}, \tag{2.7}$$

$$R(X,\xi)\xi = -\bar{X} - \eta(\bar{X})\xi, \tag{2.8}$$

$$R(\xi,\bar{X})\bar{Y} = g(\bar{X},\bar{Y})\xi - \eta(\bar{Y})\bar{X}, \tag{2.9}$$

$$S(\bar{X},\xi) = 2m\eta(\bar{X}), \tag{2.10}$$

$$Q\xi = 2m\xi, \tag{2.11}$$

where R, S and Q are the curvature tensor, the Ricci tensor field and the Ricci operator of M^{2m+1}, respectively.

From [8], we know that for a 3-dimensional hyperbolic Kenmotsu manifold M^3

$$Q\bar{X} = (\frac{r}{2} - 1)\bar{X} + (\frac{r}{2} - 3)\eta(\bar{X})\xi \tag{2.12}$$

and

$$S(\bar{X},\bar{Y}) = (\frac{r}{2} - 1)g(\bar{X},\bar{Y}) + (\frac{r}{2} - 3)\eta(\bar{X})\eta(\bar{Y}). \tag{2.13}$$

Moreover,

$$\begin{aligned} R(\bar{X},\bar{Y})\bar{Z} = \tfrac{r-4}{2}[g(\bar{Y},\bar{Z})\bar{X} - g(\bar{X},\bar{Z})\bar{Y}] + \tfrac{r-6}{2}[g(\bar{Y},\bar{Z})\eta(\bar{X})\xi - g(\bar{X},\bar{Z})\eta(\bar{Y})\xi \\ +\eta(\bar{Y})\eta(\bar{Z})\bar{X} - \eta(\bar{X})\eta(\bar{Z})\bar{Y}]. \end{aligned} \tag{2.14}$$

In addition, the Lie derivative of g along ξ is given by

$$(\mathcal{L}_{\xi}g)(\bar{X},\bar{Y}) = -2[g(\bar{X},\bar{Y}) + \eta(\bar{X})\eta(\bar{Y})]. \tag{2.15}$$

3. η-Ricci solitons on 3-dimensional hyperbolic Kenmotsu manifolds

THEOREM 3.1. *If the metric g of a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits a η-Ricci soliton of the type (g, ξ, λ, μ), then the manifold M^3 becomes a η-Einstein manifold of constant scalar curvature $r = 4 - 2\lambda$ and soliton is shrinking, steady and expanding according as $r > 4$, $r = 4$ and $r < 4$, respectively.*

PROOF. Let us assume that the 3-dimensional hyperbolic Kenmotsu manifold M^3 admits a η-Ricci soliton of the type (g, ξ, λ, μ). Then from (1.2), we have

$$2S(\bar{X}, \bar{Y}) = -(\mathcal{L}_\xi g)(\bar{X}, \bar{Y}) - 2\lambda g(\bar{X}, \bar{Y}) - 2\mu\eta(\bar{X})\eta(\bar{Y}). \tag{3.1}$$

Using equation (2.15) in the equation (3.1), we get

$$S(\bar{X}, \bar{Y}) = (1 - \lambda)g(\bar{X}, \bar{Y}) + (1 - \mu)\eta(\bar{X})\eta(\bar{Y}). \tag{3.2}$$

Hence, we can say that the manifold M^3 is a η-Einstein manifold.

Comparing (3.2) with the equation $S(\bar{X}, \bar{Y}) = (\frac{r}{2} - 1)g(\bar{X}, \bar{Y}) + (\frac{r}{2} - 3)\eta(\bar{X})\eta(\bar{Y})$, we get $\lambda = -\frac{1}{2}(r - 4)$ and $\mu = -\frac{1}{2}(r - 4) + 2$. From which it follows that

$$\mu - \lambda = 2 \tag{3.3}$$

and also

$$r = 4 - 2\lambda. \tag{3.4}$$

This implies that M^3 possesses the constant scalar curvature. This completes the proof. □

In view of the equation (3.3), we can state the following:

COROLLARY 3.2. *A 3-dimensional hyperbolic Kenmotsu manifold M^3 admitting a η-Ricci soliton of the type (g, ξ, λ, μ) is a η-Einstein and the value of the scalar $\lambda = \mu - 2$.*

Next we prove the following theorem:

THEOREM 3.3. *If the metric g of a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits a η-Ricci soliton of the type $(g, \bar{V}, \lambda, \mu)$ such that $\bar{V}$ is pointwise collinear with ξ, then $\bar{V}$ is a constant multiple of ξ and the manifold M^3 becomes a η-Einstein manifold.*

PROOF. Let $\bar{V} = c\xi$ for some smooth function c on M^3. Then from (1.2), we can write

$$c(\mathcal{L}_\xi g)(\bar{X}, \bar{Y}) + \{\bar{X}(c) + \mu\eta(\bar{X})\}\eta(\bar{Y}) + \{\bar{Y}(c) + \mu\eta(\bar{Y})\}\eta(\bar{X}) + 2S(\bar{X}, \bar{Y}) + 2\lambda g(\bar{X}, \bar{Y}) = 0. \tag{3.5}$$

By using the equation (2.15), we get

$$\bar{X}(c)\eta(\bar{Y}) + \bar{Y}(c)\eta(\bar{X}) + 2S(\bar{X}, \bar{Y}) + 2(\lambda - c)g(\bar{X}, \bar{Y}) + 2(\mu - c)\eta(\bar{X})\eta(\bar{Y}) = 0. \tag{3.6}$$

Substituting $\bar{Y}$ with ξ in (3.6) and making use of (2.1) and (2.10), we obtain

$$\bar{X}(c) + \{2\mu - 2\lambda - \xi(c) - 4\}\eta(\bar{X}) = 0. \tag{3.7}$$

Again replacing $\bar{X}$ with ξ in (3.7) and using (2.1) we get

$$\xi(c) = \mu - \lambda - 2. \tag{3.8}$$

Putting the value of $\xi(c)$ in (3.7) we get

$$d(c) = (\lambda - \mu + 2)\eta, \tag{3.9}$$

where d stands for the exterior derivative operator. Taking exterior derivative of (3.9) and using *Poincaré* lemma $d^2 \equiv 0$ yields

$$(\lambda - \mu + 2)d\eta = 0. \tag{3.10}$$

Executing wedge product in the foregoing with η we obtain

$$(\lambda - \mu + 2)\eta \wedge d\eta = 0. \tag{3.11}$$

Since $\eta \wedge \mathbf{d}\eta \neq \mathbf{0}$ in a hyperbolic Kenmotsu manifold, we infer $\mu = \lambda + 2$. Putting the value of μ in (3.9) gives $d(c) = 0$, that is $c = constant$. Consequently, the equation (3.6) reduces to

$$S(\bar{X}, \bar{Y}) = (c - \lambda)g(\bar{X}, \bar{Y}) + (c - \mu)\eta(\bar{X})\eta(\bar{Y}) \tag{3.12}$$

for all $\bar{X}, \bar{Y} \in \chi(M^3)$. From (3.2) and (3.12), it follows that $c = 1$. Thus we can easily see the manifold is a η-Einstein manifold. Also, from (3.8) we have $\lambda = \mu - 2$. This completes the proof. □

In view of (3.8) with the fact that c is a constant, we can easily state the following:

Corollary 3.4. *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits a η-Ricci soliton of the type $(g, \bar{V}, \lambda, \mu)$ such that $\bar{V}$ is pointwise collinear with ξ, then the soliton is shrinking, steady and expanding according as $\mu < 2$, $\mu = 2$ and $\mu > 2$ respectively.*

4. 3-dimensional hyperbolic Kenmotsu manifolds satisfying a η-Ricci solitons

In this section, we obtain some results on 3-dimensional hyperbolic Kenmotsu manifold M^3 admitting a η-Ricci soliton. A hyperbolic Kenmotsu manifold is said to be Ricci symmetric if $\nabla S = 0$, where ∇ denotes the covariant differentiation operator with respect to the metric g.

Theorem 4.1. *If a Ricci symmetric 3-dimensional hyperbolic Kenmotsu manifold M^3 admits a η-Ricci soliton of the type (g, ξ, λ, μ), then the soliton is shrinking with $\lambda = -1$ and $\mu = 1$.*

Proof. From (2.7), we have

$$S(\bar{X}, \bar{Y}) = (1 - \lambda)g(\bar{X}, \bar{Y}) + (1 - \mu)\eta(\bar{X})\eta(\bar{Y}) \tag{4.1}$$

for all $\bar{X}, \bar{Y} \in \chi(M^3)$, where λ and μ are real constants.
Taking covariant derivative of (4.1) along the vector field $\bar{Z} \in \chi(M^3)$, and then using (2.6), we get

$$(\nabla_{\bar{Z}} S)(\bar{X}, \bar{Y}) = (\mu - 1)[g(\bar{X}, \bar{Z})\eta(\bar{Y}) + g(\bar{Y}, \bar{Z})\eta(\bar{X}) + 2\eta(\bar{X})\eta(\bar{Y})\eta(\bar{Z})]. \tag{4.2}$$

We assume that the manifold M^3 is *Ricci symmetric* (i.e., $\nabla S = 0$), the equation (4.2) becomes

$$(\mu - 1)[g(\bar{X}, \bar{Z})\eta(\bar{Y}) + g(\bar{Y}, \bar{Z})\eta(\bar{X}) + 2\eta(\bar{X})\eta(\bar{Y})\eta(\bar{Z})] = 0. \tag{4.3}$$

Replacing $\bar{Y}$ with ξ in the forgoing equation we get

$$(\mu - 1)g(\varphi\bar{X}, \varphi\bar{Z}) = 0 \tag{4.4}$$

for all $\bar{X}, \bar{Z} \in \chi(M^3)$. It follows that $\mu = 1$. From (3.3) we get $\lambda = -1$. Hence the soliton is shrinking. This completes the proof. □

Again setting $\bar{Z} = \xi$ in (4.2), we obtain

$$(\nabla_\xi S)(\bar{X}, \bar{Y}) = 0. \tag{4.5}$$

So, we can state the following:

Corollary 4.2. *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits a η-Ricci soliton of the type (g, ξ, λ, μ), then the second order symmetric Ricci tensor S of M^3 is parallel along ξ.*

A hyperbolic Kenmotsu manifold is said to be *φ-Ricci symmetric* [24] if

$$\varphi^2 \circ \nabla Q = 0.$$

Now we prove the next theorem.

Theorem 4.3. *If a φ-Ricci symmetric 3-dimensional hyperbolic Kenmotsu manifold M^3 admits a η-Ricci soliton (g, ξ, λ, μ), then the soliton is shrinking with $\lambda = -1$ and $\mu = 1$.*

Proof. The Ricci tensor field for a η-Ricci soliton of the type (g, ξ, λ, μ) on 3-dimensional hyperbolic Kenmotsu manifold M^3 is given by

$$S(\bar{X}, \bar{Y}) = (1 - \lambda)g(\bar{X}, \bar{Y}) + (1 - \mu)\eta(\bar{X})\eta(\bar{Y}) \tag{4.6}$$

and hence

$$Q\bar{X} = (1 - \lambda)\bar{X} + (1 - \mu)\eta(\bar{X})\xi. \tag{4.7}$$

Taking covariant derivative of (4.7), and then using (2.5), (2.6) yields

$$(\nabla_{\bar{X}} Q)\bar{Y} = (\mu - 1)[\eta(\bar{Y})\bar{X} + 2\eta(\bar{X})\eta(\bar{Y})\xi + g(\bar{X}, \bar{Y})\xi]. \tag{4.8}$$

Applying φ^2 on both sides of the above equation, we obtain

$$\varphi^2(\nabla_{\bar{X}} Q)\bar{Y} = (\mu - 1)\eta(\bar{Y})\varphi^2\bar{X}. \tag{4.9}$$

Now applying the condition $\varphi^2 \circ \nabla Q = 0$ in the above equation, we have

$$(\mu - 1)\eta(\bar{Y})\varphi^2\bar{X} = 0 \tag{4.10}$$

for all $\bar{X}, \bar{Y} \in \chi(M^3)$. It follows that $\mu = 1$ and from (3.3), we get $\lambda = -1$. Thus, the soliton (g, ξ, λ, μ) is shrinking. This completes the proof. □

Now replacing $\bar{X}$ with ξ in (4.8), we get

$$(\nabla_{\xi} Q)\bar{Y} = 0. \tag{4.11}$$

So, we can state the following:

Corollary 4.4. *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits a η-Ricci soliton of the type (g, ξ, λ, μ), then the Ricci operator Q of M^3 is parallel along ξ.*

5. η-Ricci solitons on 3-dimensional hyperbolic Kenmotsu manifolds with Codazzi type of Ricci tensor

This section deals with η-Ricci solitons on hyperbolic Kenmotsu manifolds whose second order symmetric Ricci tensor is Codazzi type of Ricci tensor. A Gray [15] revealed the idea of cyclic parallel Ricci tensor and Ricci tensor of Codazzi type. Codazzi type of Ricci tensor means that the Levi-Civita connection ∇ of such metric is a Yang-Mills connection while keeping the metric of the manifold fixed. A Riemannian manifold (M, g) is said to have Ricci tensor of Codazzi type if its non-zero second order symmetric Ricci tensor field S satisfies

$$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\nabla_{\bar{Y}} S)(\bar{X}, \bar{Z}) \tag{5.1}$$

for any $\bar{X}, \bar{Y}, \bar{Z} \in \chi(M^3)$. Ki et al. [18] proved that Carten hypersurfaces are manifold with non-parallel Ricci tensor satisfies cyclic parallel Ricci tensor while Bourguigon [7] proved that the interesting result that any metric with Codazzi type of Ricci tensor on a compact orientable 4-manifold with non-vanishing signature is Einstein.

First, we suppose that 3-dimensional hyperbolic Kenmotsu manifolds with η-Ricci solitons satisfy the condition (5.1). Then in view of the equation (4.2), the equation (5.1) becomes

$$(\mu - 1)[g(\bar{X}, \bar{Z})\eta(\bar{Y}) - g(\bar{Y}, \bar{Z})\eta(\bar{X})]. \tag{5.2}$$

Thus by taking $\bar{Y} = \xi$ and using (2.6) yields

$$(\mu - 1)g(\varphi\bar{X}, \varphi\bar{Z}) = 0 \tag{5.3}$$

for any $\bar{X}, \bar{Z} \in \chi(M^3)$. It follows that $\mu = 1$. From (3.3) we obtain $\lambda = -1$. Hence for this values of μ and λ, we have from (3.2) that $S(\bar{X}, \bar{Y}) = 2g(\bar{X}, \bar{Y})$ for any $\bar{X}, \bar{Y} \in \chi(M^3)$. Thus, we can state the following theorem:

Theorem 5.1. *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 with Ricci tensor of Codazzi type admits a η-Ricci soliton of the type (g, ξ, λ, μ), then the soliton is shrinking with $\lambda = -1$ and the manifold M^3 reduces to an Einstein manifold.*

6. η-Ricci solitons on 3-dimensional hyperbolic Kenmotsu manifolds with cyclic parallel Ricci tensor

In this section we study an η-Ricci solitons on hyperbolic Kenmotsu manifolds whose Ricci tensor is cyclic parallel Ricci tensor. Ki et al [18] established that Carten hypersurfaces are manifold, with non-parallel Ricci tensor satisfies cyclic parallel Ricci tensor. A Riemannian manifold (M, g) is said to have cyclic parallel Ricci tensor if its non-zero second order symmetric Ricci tensor S is non-zero and satisfies

$$(\nabla_{\bar{X}}S)(\bar{Y},\bar{Z}) + (\nabla_{\bar{Y}}S)(\bar{Z},\bar{X}) + (\nabla_{\bar{Z}}S)(\bar{X},\bar{Y}) = 0 \tag{6.1}$$

for any $\bar{X},\bar{Y},\bar{Z} \in \chi(M^3)$.
Suppose that 3-dimensional hyperbolic Kenmotsu manifolds with η-Ricci solitons satisfy the condition (6.1). Feeding (4.2) in (6.1), we obtain

$$(\mu - 1)[g(\bar{X},\bar{Y})\eta(\bar{Z}) + g(\bar{Y},\bar{Z})\eta(\bar{X}) + g(\bar{Z},\bar{X})\eta(\bar{Y}) + 3\eta(\bar{X})\eta(\bar{Y})\eta(\bar{Z})] = 0. \tag{6.2}$$

Replacing $\bar{Z} = \xi$ in (6.2) and then using (2.6), we get

$$(1 - \mu)g(\varphi\bar{X}, \varphi\bar{Y}) = 0 \tag{6.3}$$

for any $\bar{X},\bar{Y} \in \chi(M^3)$, which implies that $\mu = 1$. Using this, (3.3) can be reduced to $\lambda = -1$. Therefore from (3.2), we have $S(\bar{X},\bar{Y}) = 2g(\bar{X},\bar{Y})$ for any $\bar{X},\bar{Y} \in \chi(M^3)$. Thus, we can state the following theorem:

Theorem 6.1. *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 with cyclic parallel Ricci tensor admits a η-Ricci soliton of the type (g,ξ,λ,μ), then the soliton is shrinking with $\lambda = -1$ and the manifold M^3 becomes an Einstein manifold.*

7. An example of a 3-dimensional hyperbolic Kenmotsu manifold admitting η-Ricci soliton

We consider the three dimensional manifold $M^3 = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$. Then the vector fields $e_1 = e^z\frac{\partial}{\partial x}, e_2 = e^z\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z} = \zeta$ are linearly independent at each point of M^3 and so they form a basis of the tangent vector space at each point of M^3. Let g be a semi-Riemannian metric defined by:

$$g(e_i, e_j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let η be the 1-form defined by: $g(X, e_3) = \eta(X)$ for all $X \in \chi(M^3)$ and φ is a $(1, 1)$ tensor field defined by $\varphi(e_1) = e_2, \varphi(e_2) = e_1, \varphi(e_3) = 0$. Then using the linearity property of φ and g, we obtain $\eta(e_3) = -1$, $\varphi^2X = X + \eta(X)e_3$, $g(\varphi X, \varphi Y) = -g(X, Y) - \eta(X)\eta(Y)$ for all $X, Y \in \chi(M^3)$. Thus for $e_3 = \xi$, the structure (φ, ξ, η, g) defines an almost hyperbolic contact metric structure on M^3. All possible Lie brackets for the example are as follows:

$$[e_1, e_1] = [e_2, e_2] = [e_3, e_3] = [e_1, e_2] = [e_2, e_1] = 0,$$

$$[e_1, e_3] = -e_1, \qquad [e_3, e_1] = e_1, \qquad [e_3, e_2] = e_2, \qquad [e_2, e_3] = -e_2.$$

Let ∇ be a Riemannian connection with respect to semi-Riemannian metric g. Now using the Koszul's formula $2g(\nabla_X Y, Z) = Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y])$ for all $X, Y, Z \in \chi(M^3)$, we can obtain

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_3} e_1 = \nabla_{e_3} e_2 = \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_3 = -e_1, \nabla_{e_2} e_2 = e_3, \nabla_{e_2} e_3 = -e_2.$$

Then it can be observed that $\nabla_X \xi = -X - \eta(X)\xi$ for all $X \in \chi(M^3)$. Thus $M^3(\varphi, \xi, \eta, g)$ defines a 3-dimensional hyperbolic Kenmotsu manifold. The non-zero components of the curvature tensor R as follows:

$$R(e_1, e_2)e_1 = -e_2, \quad R(e_2, e_1)e_1 = e_2, \quad R(e_1, e_3)e_1 = -e_3, \quad R(e_3, e_1)e_1 = e_3,$$

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_2, e_1)e_2 = e_1, \quad R(e_2, e_3)e_2 = e_3, \quad R(e_3, e_2)e_2 = -e_3,$$

$$R(e_1, e_3)e_3 = -e_1, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_3, e_2)e_3 = e_2.$$

Therefore the non-vanishing components of the Ricci tensor as follows:

$$S(e_1, e_1) = 2, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2. \tag{7.1}$$

Hence we can see that (M^3, g) is a manifold of constant scalar curvature $r = \sum_{r=1}^{3} g(e_r, e_r)S(e_r, , e_r) = 6$. From (3.2), we compute that

$$S(e_1, e_1) = (1 - \lambda), \quad S(e_2, e_2) = -(1 - \lambda), \quad S(e_3, e_3) = -(\mu - \lambda) \tag{7.2}$$

Equating (7.1) with (7.2) we arrive at $\lambda = -1$ and $\mu = 1$. Therefore the structure (g, ξ, λ, μ) is a η-Ricci soliton on 3-dimensional hyperbolic Kenmotsu manifolds $M^3(\varphi, \xi, \eta, g)$, ξ being timelike vector field on M^3. Clearly all the theorems are verified.

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Jhantu Das, Department of Mathematics, Sidho-Kanho-Birsha University, Purulia-723104, West Bengal, India
e-mail: dasjhantu54@gmail.com

Kalyan Halder, Department of Mathematics, Sidho-Kanho-Birsha University, Purulia-723104, West Bengal, India
e-mail: drkalyanhalder@gmail.com