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A FUZZY MATRIX APPROACH TO EXTEND CRISP FUNCTIONS IN MULTI-FUZZY ENVIRONMENT

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Abstract

This paper introduces multi-fuzzy extensions of crisp functions using fuzzy matrices as bridge functions and fuzzy matrices are used for the comparative study of different characterization problems. The study generalizes extensions of crisp functions on neutrosophic sets into multi-fuzzy sets. In this paper $m \times n$ fuzzy matrices act as bridge functions for the multi-fuzzy extensions of crisp functions with the membership functions of the domain and co-domain having dimensions n and m respectively. This extensions map multi-fuzzy sets with zero membership function into zero membership functions and preserve monotonicity . The paper discusses various properties of finite union and intersection of multi-fuzzy sets under multi-fuzzy extensions. Furthermore, we conduct a comparative study of multi-fuzzy extensions based on the order properties and different operations on fuzzy matrices.

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1. Introduction

Zadeh's principle, proposed by Lotfi Zadeh in 1965, is a fundamental concept in fuzzy logic that extends crisp sets and functions to handle uncertainty and vagueness. The extension of crisp functions to fuzzy functions involves the application of Zadeh's extension principle [12]. Wei's generalization of Zadeh's principle [11] further refines and extends this approach to functions. Neutrosophic sets introduce a third component, besides truth and falsehood, called indeterminacy, providing a more sophisticated representation of uncertainty. The works, such as those by S. Sebastian and F. Smarandache in [10] focused on the extension of crisp functions to neutrosophic sets. The introduction of multi-fuzzy sets builds upon the concepts of multi-sets[4], fuzzy multi-sets [5] and multiple sets. Multi-fuzzy sets, as introduced by Sebastian and Ramakrishnan in various works [[6], [7] and [8]]provide a framework for extending crisp functions. These extensions involve the use of bridge functions, such as order homomorphisms, complete lattice homomorphisms, L-fuzzy lattices, and strong L-fuzzy lattices. It offers a powerful framework for handling uncertainty and vagueness in various domains by allowing for the representation of multiple dimensions of fuzziness within a unified structure. In [9] Sebastian and John discussed the relationship between multi-fuzzy sets and the other sets.

In this paper, we focused the multi-fuzzy extensions of crisp functions, introducing a novel approach that employs fuzzy matrices [3] as bridge functions. Here we are using fuzzy matrices to analyse various properties of these extended functions. The key innovation lies in the employment of m × n fuzzy matrices as bridge functions, where the dimensions of the membership functions for the domain and co-domain are n and m respectively

2. Preliminaries

This section discusses some basic concepts related to fuzzy sets, multi-fuzzy sets and fuzzy matrices. Through out this section the positive integers m and n represent dimensions of multi-fuzzy sets with membership values in I^m and I^n (value domains of multi fuzzy sets) where $I = [0, 1]$ and $\mathfrak{F}_{mn}$ be the collection of all $m \times n$ fuzzy matrices over the fuzzy algebra I.

Definition 2.1. [8] Let X be a non-empty set and let N, the set of all natural numbers. A multi-fuzzy set μ in X is a set of ordered sequences
$\mu = \{< x, \mu_1(x), \mu_2(x), ..., \mu_i(x), ... >: x \in X\}$, where $\mu_i \in I$, for all i $\in N$. The set of all multi-fuzzy sets in X of dimension k is denoted by $M^kFS(X)$.

Definition 2.2. [8] Let

$$\mu = \{< x, \mu_1(x), \mu_2(x), ..., \mu_i(x), ... >: x \in X, \mu_i(x) \in \mathrm{I}, \mathrm{i} \in \mathrm{N}\},$$
$$\zeta = \{< x, \zeta_1(x), \zeta_2(x), ..., \zeta_i(x), ... >: x \in X, \zeta_i(x) \in \mathrm{I}, \mathrm{i} \in \mathrm{N}\}$$

be multi-fuzzy sets in a nonempty set set X. We define the following relations and operations:

i) $\mu \subseteq \zeta$ if and only if $\mu_{\mathrm{i}}(x) \leq \zeta_i(x)$, for all $x \in X$ and $i \in N$;

ii) $\mu = \zeta$ if and only if $\mu_{\mathrm{i}}(x) = \zeta_i(x)$, for all $x \in X$ and $i \in N$;

iii) $\mu \cup \zeta = \{< \mathrm{x}, \max(\mu_1(x), \zeta_1(x)), \max(\mu_2(x), \zeta_2(x)), ..., \max(\mu_i(x), \zeta_i(x)), ... >: x \in X\}$;

iv) $\mu \cap \zeta = \{< \mathrm{x}, \min(\mu_1(x), \zeta_1(x)), \min(\mu_2(x), \zeta_2(x)), ..., \min(\mu_i(x), \zeta_i(x)), ... >: x \in X\}$.

Definition 2.3. [2] A fuzzy union or t-conorm s is a binary operation on I that satisfies the following axioms.

i) $s(a, 0) = 0$;

ii) $b \leq c$ implies $s(a, b) \leq s(a, c)$;

iii) $s(a, b) = s(b, a)$;

iv) $s(a, s(b, c)) = s(s(a, b), c)$.

Similarly, fuzzy intersection or $t - norm$ is a non-decreasing, commutative and associative mapping $t : I \times I \to I$ satisfies the boundary condition
$t(a, 1) = a$, for all $a \in I$.

DEFINITION 2.4. [1] Let $\mathfrak{L}$ be a distributive lattice with 0 and 1. Then $\mathfrak{L}_n$ be the set of all $n \times n$ matrices over $\mathfrak{L}$, called $\mathfrak{L}_n$- matrices. Let A_{ij}, the element of $\mathfrak{L}$ which stands in the $(i, j)^{th}$ entry of A. For $A, B \in \mathfrak{L}_n$,

i) $A + B = B$ if and only if $A_{ij} + B_{ij} = C_{ij}$;

ii) $A \leq B$ if and only if $A + B = B$ if and only if $A_{ij} \leq B_{ij}$;

iii) $AB = C$ if and only if $A_{ij}B_{ij} = C_{ij}$;

iv) $A.B$=AB=C if and only if $C_{ij} = \sum\limits_{k=1}^{n} A_{ik}B_{kj}$;

v) For $a \in \mathfrak{L}$,$aA = a.A = C$ if and only if $aA_{ij} = C_{ij}$;

vi) $(0)_{ij} = 0$.

3. Multi-fuzzy Extensions of Crisp Functions using Fuzzy Matrices

This section extends crisp function in multi-fuzzy environment with the help of fuzzy matrix.

DEFINITION 3.1. Let $f : X \longrightarrow Y$ be a crisp function and $\chi : I^n \rightarrow I^m$ be a bridge function. The multi-fuzzy extension $F : M^nFS(X) \rightarrow M^mFS(Y)$ of f with respect to a bridge function χ is
$F(\mu)(y) = \sup\limits_{y=f(x)} \{\chi(\mu(x)) : \mu(x) \in M^nFS(X), x \in X\}$, where χ is
$\chi(\mu(x)) = A\mu(x)$ and A is an $m \times n$ fuzzy matrix , $A = (a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n$.

Notation: The multi fuzzy extension F of a crisp function f with respect to a bridge function $A \in \mathfrak{F}_{mn}$ is denoted by $[F]_A$.

This paper gives some properties of extended functions based on fuzzy matrix as bridge function in two ways. In the first case, the bridge function as the fuzzy matrix in which matrix addition is replaced by fuzzy union or s-norm or t-co-norm and matrix multiplication is replaced by fuzzy intersection or t-norm. Secondly with matrix bridge as the matrix $A = (a_{ij})$ with constraints $0 \leq \sum\limits_{i,j} a_{ij} \leq 1$ and $0 \leq a_{ij} \leq 1$, where the matrix addition and multiplication are usual matrix addition and multiplication.

EXAMPLE 3.2. *Consider* $[F]_A : M^nFS(X) \rightarrow M^mFS(X)$. *If A is a zero matrix defined by* $A = (0_{ij})$, $1 \leq i \leq m, 1 \leq j \leq n$, *then the multi-fuzzy extension of any crisp function is given by* $[F(\mu)]_A(y) = 0$ *, for all* $y \in Y$.

3.1. Properties of multi-fuzzy extensions of crisp functions:

This section introduces some algebraic properties of extended functions described in the first case.

THEOREM 3.3. *Let* $[F]_A : M^nFS(X) \rightarrow M^mFS(Y)$ *be a multi-fuzzy extension of a crisp function* $f : X \longrightarrow Y$ *with respect to a fuzzy matrix bridge function then*
$[F(0_X)]_A = 0_Y$,*where* 0_X *and* 0_Y *are the zero membership function defined by* $< x, 0, 0, ..., 0 >$, *for all* $x \in X$ *and* $< y, 0, 0, ..., 0 >$, *for all* $y \in Y$.

Notation: Here we use the representation $[s(t(a_{i1}, b_1), t(a_{i2}, b_2), ..., t(a_{in}, b_n))]_{i=1}^{i=m}$ for an $m \times 1$ fuzzy matrix

$$\begin{pmatrix} s\left(t\left(a_{11},b_1\right),t\left(a_{12},b_2\right),...,t\left(a_{1n},b_n\right)\right) \\ s\left(t\left(a_{21},b_1\right),t\left(a_{22},b_2\right),...,t\left(a_{2n},b_n\right)\right) \\ ... \\ s\left(t\left(a_{m1},b_1\right),t\left(a_{m2},b_2\right),...,t\left(a_{mn},b_n\right)\right) \end{pmatrix}.$$

Also the membership function $\mu(x)$ is an $n\times 1$ fuzzy matrix $\mu(x) = [\mu_1(x), \mu_2(x), ..., \mu_n(x)]^T$.

In the context of exploring fuzzy matrices as bridge functions, we extend our analysis to consider the property of monotonicity within the framework of extended functions. The monotonicity property holds significant implications for the behavior and relationships of the extended functions under different scenarios.

THEOREM 3.4. *The multi fuzzy extension* $[F]_A : M^nFS(X) \to M^mFS(Y)$ *of a crisp function* $f : X \longrightarrow Y$ *with respect to a fuzzy matrix is monotonically increasing.*

PROOF. Let $A = [a_{ij}]$ be a bridge function, where $1 \le i \le m, 1 \le j \le n$.

$$If \mu \subseteq \zeta,\ \mu_j(x) \leqslant \zeta_j(x), \text{for all } x \in X \text{and } j = 1, 2, ...n. \tag{3.1}$$

Applying the monotonicity property of t-norm in (3.1), we get

$$t(a_{ij}, \mu_j(x)) \le t(a_{ij}, \zeta_j(x)),\ \text{for all}\ \ i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n. \tag{3.2}$$

Now, the i^{th} entry of $A(\mu(x))$ is given by

$$(A\mu(x))_i = [(a_{ij})(\mu_j(x)]_{j=1}^{j=n}. \tag{3.3}$$

Then (3.3) implies

$$A\mu(x) = [s(t(a_{i1}, \mu_1(x)), s(t(a_{i2}, \mu_2(x)), ..., s(t(a_{in}, \mu_n(x)))]_{i=1}^{i=m} \tag{3.4}$$

Applying (3.2) and the monotonicity property of s-norm,

$$A\mu(x) \leqslant [s(t(a_{i1}, \zeta_1(x)), s(t(a_{i2}, \zeta_2(x)), ..., s(t(a_{in}, \zeta_n(x)))]_{i=1}^{i=m} = A\zeta(x). \tag{3.5}$$

$$A\mu(x) \leqslant [s(t(a_{i1}, \zeta_1(x)), s(t(a_{i2}, \zeta_2(x)), ..., s(t(a_{in}, \zeta_n(x)))]_{i=1}^{i=m} = A\zeta(x). \tag{3.6}$$

Therefore, $\chi(\mu(x)) \leqslant \chi(\zeta(x))$, for all $\mu, \zeta \in M^nFS(X)$.

$$\begin{aligned} [F(\mu)]_A(y) &= \sup_{y=f(x)} \{\chi(\mu(x)) : x \in X\} \\ &\le \sup_{y=f(x)} \{\chi(\zeta(x)) : x \in X\} \\ &= [F(\zeta)]_A(y),\ \text{for all } y \in Y. \end{aligned}$$

□

Consider the following s-norm and t-norm in order to look at examples.

For $A, B \in \mathfrak{F}_{mn}$ and let $A = \left(a_{ij}\right) and\ B = \left(b_{ij}\right), 1 \le i \le m, 1 \le j \le n,$

- For the s-norm operations: $a_{ij} + b_{ij}$ denotes $s(a_{ij}, b_{ij})$, where $s(a_{ij}, b_{ij})$is the algebraic sum of a_{ij} and b_{ij}.

- For the t-norm operations: $a_{ij}b_{ij}$ denotes $t(a_{ij}, b_{ij})$, where $t(a_{ij}, b_{ij})$ is the algebraic product of a_{ij} and b_{ij}.

Now, let's consider different situations involving fuzzy matrices as bridge functions:

1×1 **fuzzy matrices as bridge functions:**

Example 3.5. *Consider* $f : \mathbb{R}^2 \to \mathbb{R}$, *where* $f(x, y) = x + y$.
Let $[F]_A : MFS(\mathbb{R}^2) \to MFS(\mathbb{R})$ *,where the bridge function as* 1×1 *fuzzy matrix* $A = (0.2)$ *and let* $\mu = \{< (0,1), 0.2 >, < (1,0), 0.1 >\}$ *and*
$\zeta = \{< (0,1), 0.3 >, < (1,0), 0.7 >, < (2,0), 0.5 >\}$ *be two multi-fuzzy sets. Then,by the definition of extended function,*
$[F(\mu)]_A(1) = \sup\{A\mu(1,0), A\mu(0,1)\} = \sup\{(0.2)(0.1), (0.2)(0.2)\} = (0.04)$,
$[F(\zeta)]_A(1) = \sup\{A\zeta(1,0), A\zeta(0,1)\} = \sup\{(0.2)(0.7), (0.2)(0.3)\} = (0.14)$,
$[F(\mu)]_A(2) = (0)$,
$[F(\zeta)]_A(2) = \sup\{A\zeta(2,0)\} = \sup\{(0.2)(0.5)\} = (0.10)$.
That is, $[F(\mu)]_A(t) \leq [F(\zeta)]_A(t)$, *for all* $t \in \mathbb{R}$, *where* $t = x + y$.
Hence $[F(\mu)]_A \subseteq [F(\zeta)]_A$.

$m \times m$ **fuzzy matrices as bridge functions:**

Example 3.6. *Consider the multi-fuzzy extension of the crisp function* f *discussed in Example by* $[F]_A : M^2FS(\mathbb{R}^2) \to M^2FS(\mathbb{R})$.
Let $\mu = \{< (0,1), 0.2, 0.5 >, < (1,0), 0.3, 0.6 >\}$ *and*
$\zeta = \{< (0,1), 0.3, 0.7 >, < (1,0), 0.4, 0.6 >, < (2,0), 0.2, 0.1 >\}$.
Let the bridge function be $A = \begin{pmatrix} 0.2 & 0.5 \\ 0.1 & 0.4 \end{pmatrix}$.

$$\begin{aligned} [F(\mu)]_A(1) &= \sup_{x+y=1} \{A\mu(x,y) : x, y \in \mathbb{R}\} \\ &= \sup\{A\mu(0,1), A\mu(1,0)\} \\ &= \sup\left\{\begin{pmatrix} 0.2 & 0.5 \\ 0.1 & 0.4 \end{pmatrix}\begin{pmatrix} 0.2 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0.2 & 0.5 \\ 0.1 & 0.4 \end{pmatrix}\begin{pmatrix} 0.3 \\ 0.6 \end{pmatrix}\right\} = \begin{pmatrix} 0.342 \\ 0.263 \end{pmatrix} \end{aligned}$$

$$[F(\zeta)]_A(1) = \sup_{x+y=1} \{A\zeta(x,y) : x, y \in \mathbb{R}\} = \sup\{A\zeta(0,1), A\zeta(1,0)\} = \begin{pmatrix} 0.389 \\ 0.302 \end{pmatrix}$$

$$\begin{aligned} [F(\mu)]_A(2) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ [F(\zeta)]_A(2) &= \sup_{x+y=2} \{A\zeta(x,y) : x, y \in \mathbb{R}\} \\ &= \{A\zeta(2,0)\} \\ &= \sup\left\{\begin{pmatrix} 0.2 & 0.5 \\ 0.1 & 0.4 \end{pmatrix}\begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix}\right\} \\ &= \begin{pmatrix} 0.088 \\ 0.05992 \end{pmatrix} \end{aligned}$$

$$[F(\mu)]_A(t) \leq [F(\zeta)]_A(t), \text{ for all } t \in \mathbb{R}, \text{ where } t = x + y.$$

Hence, $[F(\mu)]_A \subseteq [F(\zeta)]_A$.

$m \times n$ fuzzy matrices as bridge functions, where $m \neq n$:

EXAMPLE 3.7. *Consider* $[F]_A : M^2FS(\mathbb{R}^2) \to M^3FS(\mathbb{R})$ *and the bridge function be* $A = \begin{pmatrix} 0.1 & 0.2 \\ 0.5 & 0.4 \\ 0.6 & 0 \end{pmatrix}$.

Also consider the multi-fuzzy sets μ *and* ζ *defined in Example 2.*

Now

$$\begin{aligned} [F(\mu)]_A(1) &= \sup_{x+y=1} \{A\mu(x,y) : x, y \in \mathbb{R}\} \\ &= \sup \left\{ \begin{pmatrix} 0.1 & 0.2 \\ 0.5 & 0.4 \\ 0.6 & 0 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0.1 & 0.2 \\ 0.5 & 0.4 \\ 0.6 & 0 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.6 \end{pmatrix} \right\} \\ &= \begin{pmatrix} 0.118 \\ 0.354 \\ 0.18 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} [F(\zeta)]_A(1) &= \sup_{x+y=1} \{A\zeta(x,y) : x, y \in \mathbb{R}\} \\ &= \sup \left\{ \begin{pmatrix} 0.1 & 0.2 \\ 0.5 & 0.4 \\ 0.6 & 0 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}, \begin{pmatrix} 0.1 & 0.2 \\ 0.5 & 0.4 \\ 0.6 & 0 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} \right\} \\ &= \begin{pmatrix} 0.166 \\ 0.392 \\ 0.24 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} [F(\mu)]_A(2) &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ [F(\zeta)]_A(2) &= \sup_{x+y=2} \{A\zeta(x,y) : x, y \in \mathbb{R}\} \\ &= \sup \{A\zeta(2,0)\} \\ &= \sup \left\{ \begin{pmatrix} 0.1 & 0.2 \\ 0.5 & 0.4 \\ 0.6 & 0 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix} \right\} \\ &= \begin{pmatrix} 0.039 \\ 0.136 \\ 0.12 \end{pmatrix} \end{aligned}$$

$$[F(\mu)]_A(t) \leq [F(\zeta)]_A(t), \textit{ for all } t \in \mathbb{R}, \textit{ where } t = x + y.$$

Hence $[F(\mu)]_A \subseteq [F(\zeta)]_A$.

THEOREM 3.8. *If* $[F]_A : M^n FS(X) \to M^m FS(Y)$ *is a multi-fuzzy extension of a crisp function* $f : X \to Y$ *and the bridge function* $A \in \mathfrak{F}_{mn}$*, then*

$$[F(\mu)]_A \cup [F(\zeta)]_A \subseteq [F(\mu \cup \zeta)]_A.$$

PROOF.

$$[F(\mu \cup \zeta)]_A(y) = \sup_{y=f(x)} \{(\chi(\mu \cup \zeta)(x)) : x \in X, \mu \cup \zeta \in M^n FS(X)\}$$

and $\chi(\mu \cup \zeta)(x) = A(\mu \cup \zeta)(x), x \in X$.

Consider the i^{th} entry of $A(\mu \cup \zeta)(x)$:

$(A(\mu \cup \zeta)(x))_i = [(a_{ij})(\max(\mu_j(x), \zeta_j(x)))]_{j=1}^{j=n}$

$= [s\,(t(a_{i1}, \max(\mu_1(x), \zeta_1(x))), t(a_{i2}, \max(\mu_2(x), \zeta_2(x))), \ldots, t(a_{in}, \max(\mu_n(x), \zeta_n(x))))]_{i=1}^{i=m}$.

As t is a monotonically increasing function, for $i = 1, 2, \ldots, m$, we have:

$$t(a_{ij}, \max(\mu_j(x), \zeta_j(x))) \geq t(a_{ij}, \mu_j(x)), \text{for all } j = 1, 2, \ldots, n \tag{3.7}$$

$$t(a_{ij}, \max(\mu_j(x), \zeta_j(x))) \geq t(a_{ij}, \zeta_j(x)), \text{for all } j = 1, 2, \ldots, n \tag{3.8}$$

From (3.7) and (3.8), we can write

$$[s(t(a_{i1}, \max(\mu_1(x), \zeta_1(x))), t(a_{i2}, \max(\mu_2(x), \zeta_2(x))), \ldots, t(a_{in}, \max(\mu_n(x), \zeta_n(x))))]_{i=1}^{i=m}$$
$$\geq [s(t(a_{i1}, \mu_1(x)), t(a_{i2}, \mu_2(x)), \ldots, t(a_{in}, \mu_n(x)))]_{i=1}^{i=m} \tag{3.9}$$

$$[s(t(a_{i1}, \max(\mu_1(x), \zeta_1(x))), t(a_{i2}, \max(\mu_2(x), \zeta_2(x))), \ldots, t(a_{in}, \max(\mu_n(x), \zeta_n(x))))]_{i=1}^{i=m}$$
$$\geq [s(t(a_{i1}, \zeta_1(x)), t(a_{i2}, \zeta_2(x)), \ldots, t(a_{in}, \zeta_n(x)))]_{i=1}^{i=m} \tag{3.10}$$

Comparing (3.9) and (3.10), we get

$$[s(t(a_{i1}, \max(\mu_1(x), \zeta_1(x))), t(a_{i2}, \max(\mu_2(x), \zeta_2(x))), \ldots, t(a_{in}, \max(\mu_n(x), \zeta_n(x))))]_{i=1}^{i=m}$$
$$\geq \max\{[s(t(a_{i1}, \mu_1(x)), t(a_{i2}, \mu_2(x)), \ldots, t(a_{in}, \mu_n(x)))]_{i=1}^{i=m},$$
$$[s(t(a_{i1}, \zeta_1(x)), t(a_{i2}, \zeta_2(x)), \ldots, t(a_{in}, \zeta_n(x)))]_{i=1}^{i=m}\}$$

That is,

$$[s(t(a_{i1}, \max(\mu_1(x), \zeta_1(x))), t(a_{i2}, \max(\mu_2(x), \zeta_2(x))), \ldots, t(a_{in}, \max(\mu_n(x), \zeta_n(x))))]_{i=1}^{i=m}$$
$$\geq \max\{A\mu(x), A\zeta(x)\}$$
$$= A\mu(x) \cup A(\zeta(x))$$

Thus, we get,

$$\chi(\mu \cup \zeta)(x) \geq \max\{\chi(\mu(x)), \chi(\zeta(x))\}, x \in X.$$

$$[F(\mu \cup \zeta)]_A(y) \geq \sup_{y=f(x)} \{\max(\chi(\mu(x)), \chi(\zeta(x))) : \mu, \zeta \in M^n FS(X)\}$$
$$= \max\{[F(\mu)]_A(y), [F(\zeta)]_A(y) : y \in Y\}.$$

□

THEOREM 3.9. *Let* $\mu_i \in M^n FS(X), 1 \leq i \leq n$, *then* $\bigcup_{i=1}^{n} [F(\mu_i)]_A \subseteq \left[F\left(\bigcup_{i=1}^{n} \mu_i \right) \right]_A$.

PROOF. Proof by mathematical induction. □

THEOREM 3.10. *Let* $[F]_A : M^n FS(X) \to M^m FS(Y)$ *be a multi-fuzzy extension of a crisp function* $f : X \to Y$. *If* $A \in \mathfrak{F}_{mn}$ *be the bridge function, then*

$$[F(\mu \cap \zeta)]_A \subseteq [F(\mu)]_A \cap [F(\zeta)]_A.$$

PROOF.

$$[F(\mu \cap \zeta)]_A(y) = \sup_{y=f(x)} \{\chi((\mu \cap \zeta)(x)) : x \in X, \mu, \zeta \in M^n FS(X)\}$$

$$\text{and } \chi((\mu \cap \zeta)(x)) = A(\mu \cap \zeta)(x), x \in X.$$

The i^{th} entry of $A(\mu \cap \zeta)(x)$ is given by

$$(A\mu \cap \zeta)(x))_i = [(a_{ij})(\min(\mu_j(x), \zeta_j(x)))]_{j=1}^{j=n}.$$

Now,

$$A(\mu \cap \zeta)(x)$$
$$= [s(t(a_{i1}, \min(\mu_1(x), \zeta_1(x))), t(a_{i2}, \min(\mu_2(x), \zeta_2(x))), \ldots, t(a_{in}, \min(\mu_n(x), \zeta_n(x))))]_{i=1}^{i=m}.$$

Triangular norm is monotonically increasing, for $i = 1, 2, \ldots, m$,

$$t(a_{ij}, \min(\mu_j(x), \zeta_j(x))) \leq t(a_{ij}, \mu_j(x)), \text{ for all } j = 1, 2, \ldots, n, \tag{3.11}$$
$$t(a_{ij}, \min(\mu_j(x), \zeta_j(x))) \leq t(a_{ij}, \zeta_j(x)), \text{ for all } j = 1, 2, \ldots, n \tag{3.12}$$

Then,

From (3.11), $[s(t(a_{i1}, \min(\mu_1(x), \zeta_1(x))), t(a_{i2}, \min(\mu_2(x), \zeta_2(x))), \ldots, t(a_{in}, \min(\mu_n(x), \zeta_n(x))))]_{i=1}^{i=m}$

$$\leq [s(t(a_{i1}, \mu_1(x)), t(a_{i2}, \mu_2(x)), \ldots, t(a_{in}, \mu_n(x)))]_{i=1}^{i=m} \tag{3.13}$$

From (3.12), $[s(t(a_{i1}, min(\mu_1(x), \zeta_1(x))), t(a_{i2}, \min(\mu_2(x), \zeta_2(x))), \ldots, t(a_{in}, \min(\mu_n(x), \zeta_n(x))))]_{i=1}^{i=m}$

$$\leq [s(t(a_{i1}, \zeta_1(x)), t(a_{i2}, \zeta_2(x)), \ldots, t(a_{in}, \zeta_n(x)))]_{i=1}^{i=m} \tag{3.14}$$

Combining (3.13) and (3.14), we can write,

$$[s(t(a_{i1}, \min(\mu_1(x), \zeta_1(x))), t(a_{i2}, \min(\mu_2(x), \zeta_2(x))), \ldots, t(a_{in}, \min(\mu_n(x), \zeta_n(x))))]_{i=1}^{i=m}$$
$$\leq \min\left([s(t(a_{i1}, \mu_1(x)), t(a_{i2}, \mu_2(x)), \ldots, t(a_{in}, \mu_n(x)))]_{i=1}^{i=m},\right.$$
$$\left.[s(t(a_{i1}, \zeta_1(x)), t(a_{i2}, \zeta_2(x)), \ldots, t(a_{in}, \zeta_n(x)))]_{i=1}^{i=m}\right\}$$

$$[s(t(a_{i1}, \min(\mu_1(x), \zeta_1(x))), t(a_{i2}, \min(\mu_2(x), \zeta_2(x))), \ldots, t(a_{in}, \min(\mu_n(x), \zeta_n(x))))]_{i=1}^{i=m}$$
$$\leq A\mu(x) \cap A\zeta(x).$$

Hence,

$$\chi((\mu \cap \zeta)(x)) \leq \min\{\chi(\mu(x)), \chi(\zeta(x))\}, \text{ for all } x \in X.$$

Therefore,

$$\begin{aligned}[F(\mu \cap \zeta)]_A(y) &\leq \sup_{y=f(x)} \{\min(\chi(\mu(x)), \chi(\zeta(x))) : \mu, \zeta \in M^n FS(X)\} \\ &= \min\{[F(\mu)]_A(y), [F(\zeta)]_A(y) : y \in Y\} \\ &= [F(\mu)]_A(y) \cap [F(\zeta)]_A(y).\end{aligned}$$

□

THEOREM 3.11. *If $\mu_i \in M^n FS(X), 1 \leq i \leq n$, then* $\left[F\left(\bigcap_{i=1}^{n} \mu_i\right)\right]_A \subseteq \bigcap_{i=1}^{n} [F(\mu_i)]_A$.

PROOF. Proof by mathematical induction. □

THEOREM 3.12. *Let $[F]_A$ be a multi-fuzzy extension of a crisp function $f : X \to Y$, and let χ_1 and χ_2 be two bridge functions defined by $\chi_1(\mu(x)) = A\mu(x)$ and $\chi_2(\mu(x)) = B\mu(x)$ satisfying $A \leqslant B$. Then $[F(\mu)]_A \leqslant [F(\mu)]_B$.*

PROOF. Let $A = (a_{ij})$, $B = (b_{ij})$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ be bridge functions, and $\mu(x) = [\mu_j(x)]_{j=1}^{j=n}$ be an element of $M^n FS(X)$. Suppose that $A \subseteq B$, that is, $a_{ij} \leq b_{ij}$ for all i and j. Consider $A\mu(x)$ and $B\mu(x)$. By the commutative property of t-norm, $t(a_{ij}, \mu(x)) = t(\mu(x), a_{ij})$ and $t(b_{ij}, \zeta(x)) = t(\zeta(x), b_{ij})$. Since t-norm is monotonic and $a_{ij} \leqslant b_{ij}$, $t(\mu_j(x), a_{ij}) \leq t(\mu_j(x), b_{ij})$, for all i and j.

$$\begin{aligned}A\mu(x) &= [s(t(a_{i1}, \mu_1(x)), t(a_{i2}, \mu_2(x)), ..., (t(a_{in}, \mu_n(x)))]_{i=1}^{i=m} \\ &= [s(t(\mu_1(x), a_{i1}), t(\mu_2(x), a_{i2}), ..., (t(\mu_n(x), a_{in}))]_{i=1}^{i=m} \\ &\leqslant [s(t(\mu_1(x), b_{i1}), t(\mu_2(x), b_{i2}), ..., (t(\mu_n(x), b_{in}))]_{i=1}^{i=m} \\ &= B\mu(x), \text{ for all } x \in X.\end{aligned}$$

Thus,

$$[F(\mu)]_A(y) = \sup_{y=f(x)} \{\chi_1 \mu(x) : \mu(x) \in M^n FS(X)\}$$

$$= \sup_{y=f(x)} \{A\mu(x)\} \leq \sup_{y=f(x)} \{B\mu(x) : x \in X\} = [F(\mu)]_B(y).$$

THEOREM 3.13. *Let $A, B \in \mathfrak{F}_{mn}$. If $A \leqslant B$, then $[F(\mu)]_{A+B} = [F(\mu)]_A$.*

Proof. $$\begin{aligned}[F(\mu)]_{A+B}(y) &= \sup_{y=f(x)} \{(A+B)\mu(x) : \mu(x) \in M^n FS(X)\} \\ &\leq \sup_{y=f(x)} \{BC\mu(x) : \mu(x) \in M^n FS(X)\} \\ &= [F(\mu)]_B. \square\end{aligned}$$

□

THEOREM 3.14. *If $A \leqslant B \in \mathfrak{F}_{mn}$, then for any $C \in \mathfrak{F}_{np}$ and $D \in \mathfrak{F}_{pm}$, $[F(\mu)]_{AC} \leq [F(\mu)]_{BC}$ and $[F(\mu)]_{DA} \leq [F(\mu)]_{DB}$.*

PROOF.
$$\begin{aligned}[F(M)]_{AC} &= \sup_{y=f(x)} \{AC\mu(x) : \mu(x) \in M^n FS(X)\} \\ &\leq \sup_{y=f(x)} \{BC\mu(x) : \mu(x) \in M^n FS(X)\} \\ &= [F(\mu)]_{BC}.\end{aligned}$$
Similarlly we can prove the other result. □

THEOREM 3.15. *Let $A_1, A_2 \in \mathfrak{F}_{mn}$, $B_1, B_2 \in \mathfrak{F}_{np}$. If $A_1 \leq A_2$, $B_1 \leq B_2$, then $[F(\mu)]_{A_1 B1} \leq [F(\mu)]_{A_2 B_2}$.*

PROOF. If $A_1 \leq A_2, B_1 \leq B_2$, then $A_1 B_1 \leq A_2 B_2$ and apply theorem 3.7. □

THEOREM 3.16. *Let $A \in \mathfrak{F}_{mn}$ be a bridge function associated with the multi-fuzzy extension $[F]_A$:$M^n FS(X) \to M^m FS(Y)$ of the crisp function $f : X \to Y$. Let $a \geq b \in \phi_A$, where ϕ_A be the set of all nonzero entries of A, then $[F(\mu)]_{A_a} \leqslant [F(\mu)]_{A_b}$.*

PROOF. Let $A_a = ((A_a)_{ij})$ and $A_b = ((A_b)_{ij})$.
If $((A_a)_{ij}) = 0$, then $((A_a)_{ij}) \leqslant ((A_b)_{ij})$ for all i and j.
If $((A_a)_{ij}) = 1$, then by definition of the cut matrix if $a_{ij} \geq a \geq b$, then $((A_b)_{ij}) = 1$.
That is,$((A_a)_{ij}) \leq ((A_b)_{ij})$, for all i and j,which implies $A_a \leqslant A_b$, for all $a \geq b$.
Hence $[F(\mu)]_{A_a} \leqslant [F(\mu)]_{A_b}$. □

THEOREM 3.17. *If the bridge function $A \in \mathfrak{F}_{mn}$ has a zero row, then the multi fuzzy extension F of the crisp function f with respect to the bridge function $adj(A)A$ is zero.*

PROOF. Suppose A has a zero row. Then, we have $adj(A)A$ is a zero matrix and so $[F(\mu)]_{adj(A)A} = 0$, for all $y \in Y$.

□

Next we look at properties of multi-fuzzy extension of crisp functions described in the second case.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}, \tag{3.15}$$

where the constraints are defined by
$0 \leq \sum_{i,j} a_{ij} \leq 1$ and $0 \leq a_{ij} \leq 1, i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.
For $A, B \in \mathfrak{F}_{mn}$, define matrix addition and multiplication as usual matrix addition $A+B$ and usual matrix multiplication AB. (3.14)

Notation: Here we use the representation $\left[\sum_{j=1}^{n} a_{ij}\mu_j(x)\right]_{i=1}^{i=n}$,

for the $m \times 1$ fuzzy matrix $\begin{pmatrix} a_{11}\mu_1(x) + a_{12}\mu_2(x) + ... + a_{1n}\mu_n(x) \\ a_{21}\mu_1(x) + a_{22}\mu_2(x) + ... + a_{2n}\mu_n(x) \\ ... \\ a_{m1}\mu_1(x) + a_{m2}\mu_2(x) + ... + a_{mn}\mu_n(x) \end{pmatrix}$.

Theorem 3.18. *Let* $[F]_A : M^nFS(X) \to M^mFS(Y)$ *be a multi-fuzzy extension of a crisp function* $f : X \longrightarrow Y$ *and the bridge function is defined as in* (3.14). *Then*

a) If $\mu \subseteq \zeta$ *,then* $[F(\mu]_A \subseteq [F(\zeta)_A$*, for all* $\mu, \zeta \in M^nFS(X)$*;*

b) $\bigcup_{i=1}^{n} [F(\mu_i)]_A \subseteq \left[F\left(\bigcup_{i=1}^{n} \mu_i\right)\right]_A$ *,* $\mu_i \in M^nFS(X)$*,* $i = 1, 2, ..., m$*;*

c) $\left[F\left(\bigcap_{i=1}^{n} \mu_i\right)\right]_A \subseteq \bigcap_{i=1}^{n} [F(\mu_i)]_A, \mu_i \in M^nFS(X)$*,* $i = 1, 2, ..., m$*.*

Proof. For part a), take $\mu(x) = \left[\mu(x_j)\right]_{j=1}^{j=n}$ and $\zeta(x) = \left[\zeta(x_j)\right]_{j=1}^{j=n}$. Suppose that $\mu \subseteq \zeta$. Then $\mu(x) \leqslant \zeta(x)$, for all $x \in X$. Consider the bridge function $\chi : I^n \to I^m$ be defined by $\chi(\mu(x)) = A\mu(x)$, where $A \in \mathfrak{F}_{\mathrm{mn}}$.

$$A\mu(x) = \left[\sum_{j=1}^{n} a_{ij}\mu_j(x)\right]_{i=1}^{i=m} \leq \left[\sum_{j=1}^{n} a_{ij}\zeta_j(x)\right]_{i=1}^{i=m} = A\zeta(x)$$

That is, $\chi(\mu(x)) \leqslant \chi(\zeta(x))$, for all $\mu, \zeta \in M^nFS(X)$.

$$F(\mu)]_A(y) = \sup_{y=f(x)} \{\chi(\mu(x)); x \in X\} \leq \sup_{y=f(x)} \{\chi(\zeta(x)); x \in X\} = [F(\zeta)]_A(y),$$

for all $y \in Y$. Hence $[F(\mu)]_A \subseteq [F(\zeta)]_A$.

For part b)

$$[F(\mu \cup \zeta)]_A(y) = \underset{y=f(x)}{sup} \{\chi((\mu \cup \zeta)(x)) : x \in X, \mu \cup \zeta \in M^nFS(X)\}$$

and $\chi((\mu \cup \zeta)(x)) = A(\mu \cup \zeta)(x)$, $x \in X$.

$$(A(\mu \cup \zeta)(x))_i = A\left[\max_{1\leq j\leq n}\left(\mu_j(x), \zeta_j(x)\right)\right]$$

Then

$$\begin{aligned} A((\mu \cup \zeta)(x)) &= \left[\sum_{j=1}^{n} a_{ij}\left(\max_j\left(\mu_j(x), \zeta_j(x)\right)\right)\right]_{i=1}^{i=m} \\ &\geq \left[\max\left(\sum_{j=1}^{n} a_{ij}\mu_j(x), \sum_{j=1}^{n} a_{ij}\zeta_j(x)\right)\right]_{i=1}^{i=m} \\ &= \max\{A\mu(x), A\zeta(x) : x \in X\} \end{aligned}$$

That is, $\chi(\mu \cup \zeta)(x) \geq \max(\chi(\mu(x)), \chi(\zeta(x))$, for all $x \in X$.

$$[F(\mu \cup \zeta)]_A(y) \geq \sup_{y=f(x)} \{\max(\chi(\mu(x)), \chi(\zeta(x)) : \mu, \zeta \in M^n FS(X)\}$$
$$= \sup\{([F(\mu)]_A(y), [F(\zeta)]_A(y)) : y \in Y\}.$$

Therefore, $\bigcup_{i=1}^{n} [F(\mu_i)]_A \subseteq \left[F\left(\bigcup_{i=1}^{n} \mu_i\right)\right]_A$.

For part c)

$$[F(\mu \cap \zeta)]_A(y) = \sup_{y=f(x)} \{\chi(\mu \cap \zeta)(x) : x \in X, \mu \cap \zeta \in M^n FS(X)\}$$

and $\chi(\mu \cap \zeta)(x) = A(\mu \cap \zeta)(x)$, $x \in X$.

$$(A(\mu \cap \zeta)(x))_i = A \min(\mu_j(x), \zeta_j(x)), j = 1, 2, ...n.$$

$$Therefore, A(\mu \cap \zeta)(x) = \left[\sum_{j=1}^{n} a_{ij} \min\left(\mu_j(x), \zeta_j(x)\right)\right]_{i=1}^{i=m}$$
$$= \left[\sum_{j=1}^{n} \min\left(a_{ij}\mu_j(x), a_{ij}\zeta_j(x)\right)\right]_{i=1}^{i=m}$$
$$\leq \left[\min\left(\sum_{j=1}^{n} a_{ij}\mu_j(x), \sum_{j=1}^{n} a_{ij}\zeta_j(x)\right)\right]_{i=1}^{i=m}$$
$$= \min\{A\mu(x), A\zeta(x)) : x \in X\}.$$

□

Remark 3.19. *Theorems 3.4 and 3.6 hold in this case also.*

4. Discussions

This paper significantly contributes to the literature by introducing and exploring multi-fuzzy extensions of crisp functions using fuzzy matrices as bridge functions. The comparative study of different characterization problems and the generalization of crisp function extensions to neutrosophic sets highlight the novelty of this work. The use of m × n fuzzy matrices as bridge functions for multi-fuzzy extensions, mapping zero membership functions and preserving monotonicity, adds to the methodological toolbox. The examination of properties related to finite union and intersection of multi-fuzzy sets under these extensions further enriches the understanding of their behavior. With regard to future developments, the formulation and analysis of algebraic properties lay the ground work for structured approach to multi-fuzzy modeling, ranking and descion making.

Conflicts of Interest: The authors declare no conflict of interest.

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