GANITA, Vol.74(1), 2024, 269-278

ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH SALAGEAN DERIVATIVE OPERATOR

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Abstract

A subclass of analytic functions with negative coefficients using Salagean derivative operator [7] is introduced. Further properties like coefficient estimates, distortion theorem, closure properties and radii of convexity has been obtained for the classes.

2010 *Mathematics subject classification:* 30C45.
Keywords and phrases: Analytic functions, Salagean derivative, Coefficient estimate, Growth and distortion theorem, Integral operator, Closure Theorem.

1. INTRODUCTION

Let A denote the subclass of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in $U = \{z : |z| < 1\}$ and let S represent the subclass of A containing univalent functions in U satisfying normalisation conditions $f(0) = 0$ and $f'(0) = 1$.

A function $f(z) \in S$ is said to be starlike of order α where $0 \le \alpha < 1$ in U if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in U.$$

Also, a function $f(z) \in S$ is called convex of order α $(0 \le \alpha < 1)$ in U if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad z \in U.$$

We denote $S^*(\alpha)$ and $K(\alpha)$ to the classes of functions in S which are starlike and convex of order α respectively in U. It is well known that

$$K(\alpha) \subset S^*(\alpha) \subset S \qquad (0 \le \alpha < 1).$$

The work of third author (S. S. Joshi) is supported by NBHM research project (2/48(7)/2016/NBHM/RP/R and D II/14977). Authors would like to thank refrees for their valuable suggestions.

Let T denote the subclass of S consisting functions of the type

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \geqslant 0, \quad z \in U). \tag{1.2}$$

Ruscheweyh [6] studied and investigated the family of order α where the function

$$S_\alpha(z) = z(1-z)^{-2(1-\alpha)}, \qquad 0 \leq \alpha \leq 1,$$

is the well known extremal function for $S^*(\alpha)$. Let

$$C(\alpha, k) = \frac{\prod_{i=2}^{k}(i-2\alpha)}{(k-1)!}, \qquad k \geq 2$$

then

$$S_\alpha(z) = z + \sum_{k=2}^{\infty} C(\alpha, k) z^k.$$

Clearly, $C(\alpha, k)$ is a decreasing function in α and that

$$\lim_{k\to\infty} C(\alpha, k) = \begin{cases} \infty, & \alpha < 1/2 \\ 1, & \alpha = 1/2 \\ 0, & \alpha > 1/2. \end{cases}$$

The Hadamard product of two functions (1.1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

is denoted by $(f * g)(z)$, and is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

For a function $f(z) \in A$, we define

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= Df(z) = zf'(z) \\ &\vdots \\ D^n f(z) &= D(D^{n-1} f(z)). \end{aligned}$$

The differential operator D^n was introduced by Salagean [7]. It can be easily seen that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

Definition 1.1. A function $f(z)$ of the form (1.1) is said to be in the class $S_{n,\alpha}(\beta, \gamma, \mu)$ if

$$\left| \frac{(\phi_{n,\alpha}(z))' - 1}{2\gamma\left[(\phi_{n,\alpha}(z))' - \mu\right] - \left[(\phi_{n,\alpha}(z))' - 1\right]} \right| < \beta \tag{1.3}$$

where $0 \le \alpha < 1, 0 \le \mu < 1, 0 < \beta \le 1, \frac{1}{2} < \gamma \le 1$ and $\phi_{n,\alpha}(z) = D^n(f * s_\alpha)(z)$.

Thus, we can deduce that

$$\phi_{n,\alpha}(z) = z + \sum_{k=2}^{\infty} k^n C(\alpha, k) a_k z^k, \qquad a_k \ge 0, \quad z \in U.$$

Further we define $T_{n,\alpha}(\beta, \gamma, \mu) = S_{n,\alpha}(\beta, \gamma, \mu) \cap T$.

We observe that, by specializing the parameters β, γ, μ, n *and* α the class $T_{n,\alpha}(\beta, \gamma, \mu)$ generalizes the subclasses studied by various authors viz. Aouf and Cho [1], Silverman [9], Aouf *et. al.* [2, 3] and others {[5],[7],[8]}.

Among ample of definitions studied by many researchers from fractional calculus, we restrict ourselves to the definitions given by Srivastava *et. al.* [10] of fractional integral operator.

Definition 1.2. [9] For real numbers $\lambda > 0, \delta,$ *and* η the fractional integral operator $I_{0,z}^{\lambda,\delta,\eta}$ is defined by

$$I_{0,z}^{\lambda,\delta,\eta} f(z) = \frac{z^{\lambda-\delta}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} {}_2F_1(\lambda+\delta, -\eta, 1-\frac{t}{z}) f(t) dt$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin with order

$$f(z) = o(|z|^{\varepsilon}), \qquad z \to 0$$

where $\varepsilon > \max(0, \delta - \eta) - 1$, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

We easily note that,

$$I_{0,z}^{\lambda,\delta,\eta} f(z) = D_z^{-\lambda} f(z), \qquad (\lambda > 0)$$

where $D_z^{-\lambda} f(z)$ is the fractional integral operator considered by Owa [4].

Lemma 1.3. [9] *If* $\lambda > 0$ *, and* $k > \delta - \eta - 1$*, then*

$$I_{0,z}^{\lambda,\delta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\delta+\eta+1)}{\Gamma(k-\delta+1)\Gamma(k+\lambda+\eta+1)} z^{k-\delta}.$$

2. COEFFICIENT ESTIMATE

Theorem 2.1. *A function* $f(z)$ *defined by* (1.1) *belongs to the class* $S_{n,\alpha}(\beta,\gamma,\mu)$ *if*

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k)\{1+\beta(2\gamma-1)\} a_k \le 2\gamma\beta(1-\mu). \tag{2.1}$$

The result (2.1) *is sharp.*

Proof. Assume that, the inequality (2.1) holds true and let $|z| = 1$ then from (1.3) we have

$$\begin{aligned}
&\left|(\phi_{n,\alpha}(z))' - 1\right| - \beta\left|2\gamma\{(\phi_{n,\alpha}(z))' - \mu\} - \{(\phi_{n,\alpha}(z))' - 1\}\right| \\
&= \left|-\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k) a_k z^{k-1}\right| - \left|2\gamma\beta(1-\mu) - \beta(2\gamma-1)\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k) a_k z^{k-1}\right| \\
&\le \sum_{k=2}^{\infty} k^{n+1} C(\alpha,k) a_k \{1+\beta(2\gamma-1)\} - 2\gamma\beta(1-\mu) \le 0.
\end{aligned}$$

Hence, by the maximum modulus principle, $f(z) \in S_{n,\alpha}(\beta,\gamma,\mu)$. □

Theorem 2.2. *A function* $f(z)$ *defined by* (1.2) *belongs to the class* $T_{n,\alpha}(\beta,\gamma,\mu)$ *if and only if*

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k)\{1+\beta(2\gamma-1)\} a_k \le 2\gamma\beta(1-\mu). \tag{2.2}$$

The result (2.2) *is sharp.*

Proof. In the view of Theorem (2.1), we need only to prove the necessity. Assume that $f(z)$ defined by (1.2) is in the class $T_{n,\alpha}(\beta,\gamma,\mu)$ then we have

$$\left|\frac{(\phi_{n,\alpha}(z))' - 1}{2\gamma\{(\phi_{n,\alpha}(z))' - \mu\} - \{(\phi_{n,\alpha}(z))' - 1\}}\right| = \left|\frac{-\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k) a_k z^{k-1}}{2\gamma(1-\mu) - (2\gamma-1)\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k) a_k z^{k-1}}\right| < \beta.$$

Since $|Re(z)| \le |z|$ for all z, we have

$$Re\left\{\frac{-\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k) a_k z^{k-1}}{2\gamma(1-\mu) - (2\gamma-1)\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k) a_k z^{k-1}}\right\} < \beta. \tag{2.3}$$

Now choosing the values of z on real axis such that $[\phi_{n,\alpha}(z)]'$ takes real values. Further on clearing the denominator in (2.3) and taking limit as $z \to 1^-$, through real values, we get

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_k \leq 2\gamma\beta(1-\mu) - \beta(2\gamma - 1) \sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_k$$

which proves the inequality given by (2.2). The result (2.2) is sharp for the function

$$f(z) = z - \frac{2\gamma\beta(1-\mu)}{k^{n+1} C(\alpha, k)\{1 + \beta(2\gamma - 1)\}} z^k, \qquad k \geq 2.$$

□

Now using Lemma (1.3), we state and prove following Theorem.

3. DISTORTION AND GROWTH THEOREM

THEOREM 3.1. *Let $\lambda > 0, \delta < 2, \lambda + \eta > -2, \delta(\lambda + \eta) \leq 3\lambda$. If $f(z)$ defined by (1.2) is in the class $T_{n,\alpha}(\beta, \gamma, \mu)$, then*

$$\left| I_{0,z}^{\lambda,\delta,\eta} f(z) \right| \geq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)} \left(1 - \frac{\gamma\beta(2-\delta+\eta)}{(2-\delta)(2+\lambda+\eta)2^{n-1}\{1+\beta(2\gamma-1)\}} \right) \tag{3.1}$$

and

$$\left| I_{0,z}^{\lambda,\delta,\eta} f(z) \right| \leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)} \left(1 + \frac{\gamma\beta(2-\delta+\eta)}{(2-\delta)(2+\lambda+\eta)2^{n-1}\{1+\beta(2\gamma-1)\}} \right) \tag{3.2}$$

for $z \in U$. The result is sharp and is given by

$$f(z) = z - \frac{\gamma\beta}{2^{n-1}} z^2. \tag{3.3}$$

PROOF. Since the function $f(z)$ defined by (1.2) belongs to the class $T_{n,\alpha}(\beta, \gamma, \mu)$. Then, from Theorem (2.2) we have,

$$\sum_{k=2}^{\infty} a_k \leq \frac{\gamma\beta}{2^n\{1+\beta(2\gamma-1)\}}. \tag{3.4}$$

Now, from Lemma (1.3) we conclude that

$$\begin{aligned} I_{0,z}^{\lambda,\delta,\eta} f(z) &= \frac{\Gamma(2-\delta+\eta) z^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)} - \frac{\sum_{k=2}^{\infty} \Gamma(k+1)\Gamma(k-\delta+\eta+1)}{\Gamma(k-\delta+1)\Gamma(k+\lambda+\eta+1)} a_k z^{k-\delta} \\ &= \frac{\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)} z^{-\delta} \left(z - \frac{\sum_{k=2}^{\infty} (1)_k (2-\delta+\eta)_{k-1}}{(2-\delta)_{k-1}(2+\lambda+\eta)_{k-1}} a_k z^k \right) \\ &= \frac{\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)} z^{-\delta} \left(z - \sum_{k=2}^{\infty} h(k) a_k z^k \right) \end{aligned}$$

where

$$h(k) = \frac{(1)_k(2-\delta+\eta)_{k-1}}{(2-\delta)_{k-1}(2+\lambda+\eta)_{k-1}}, \qquad k \geq 2.$$

Setting

$$H(z) = \frac{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)}{\Gamma(2-\delta+\eta)} z^{\delta} I_{0,z}^{\lambda,\delta,\eta} f(z) = z - \sum_{k=2}^{\infty} h(k) a_k z^k \tag{3.5}$$

it can be easily verified that h(k) is non-increasing for $k \geq 2$ and thus we have

$$0 \leq h(k) \leq h(2) = \frac{(2-\delta+\eta)2}{(2-\delta)(2+\lambda+\eta)}. \tag{3.6}$$

Now, using (3.6) and (3.4) in (3.5) we have

$$|H(z)| \geq |z| - \frac{(2-\delta+\eta)2\gamma\beta}{(2-\delta)(2+\lambda+\eta)2^n\{1+\beta(2\gamma-1)\}}|z|^2. \tag{3.7}$$

Hence, from (3.5) we get

$$\frac{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)}{\Gamma(2-\delta+\eta)}|z|^{\delta}\left|I_{0,z}^{\lambda,\delta,\eta} f(z)\right| \geq |z| - \frac{(2-\delta+\eta)2\gamma\beta}{(2-\delta)(2+\lambda+\eta)2^n\{1+\beta(2\gamma-1)\}}|z|^2.$$

This proves (3.1), and the remaining part (3.2) can be proved in similar manner,details are omitted. □

Now, replacing δ by -λ in Theorem (3.1),we obtain following corollary:

Corollary 3.2. *Let the function $f(z)$ defined by (1.2) be in the class $T_{n,\alpha}(\beta,\gamma,\mu)$ then*

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left(1 - \frac{\gamma\beta}{(2+\lambda)2^{n-1}\{1+\beta(2\gamma-1)\}}\right)$$

and

$$\left|D_z^{-\lambda} f(z)\right| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left(1 + \frac{\gamma\beta}{(2+\lambda)2^{n-1}\{1+\beta(2\gamma-1)\}}\right)$$

for $\lambda > 0, z \in U$. The result is sharp for the function

$$D_z^{-\lambda} f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)}\left(1 - \frac{\gamma\beta}{(2+\lambda)2^{n-1}\{1+\beta(2\gamma-1)\}}\right)$$

4. INTEGRAL OPERATOR

Theorem 4.1. *Let c be a real number such that $c > -1$. If $f(z) \in T_{n,\alpha}(\beta,\gamma,\mu)$, then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \tag{4.1}$$

also belongs to $T_{n,\alpha}(\beta,\gamma,\mu)$.

Proof. Let $f(z)$ given by (1.2) belongs to the class $T_{n,\alpha}(\beta,\gamma,\mu)$ then using (4.1) we can write $F(z)$ as

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k$$

where

$$b_k = \frac{c+1}{c+k} a_k \qquad (k \geq 2).$$

Therefore

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k)\{1+\beta(2\gamma-1)\} b_k < \sum_{k=2}^{\infty} k^{n+1} C(\alpha, k)\{1+\beta(2\gamma-1)\} a_k \leq 2\gamma\beta(1-\mu).$$

Since $f(z) \in T_{n,\alpha}(\beta, \gamma, \mu)$. Hence from Theorem (2.2), $F(z) \in T_{n,\alpha}(\beta, \gamma, \mu)$. □

Theorem 4.2. *Let the function*

$$F(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \geq 0)$$

belongs to the class $T_{n,\alpha}(\beta, \gamma, \mu)$, *and c be any real number such that* $c > -1$. *Then* $F(z)$ *given by* (4.1) *is univalent in* $|z| < R^*$, *where*

$$R^* = \inf_{*} \left\{ \frac{(c+1) C(\alpha, k)\{1+\beta(2\gamma-1)\} k^n}{2\gamma\beta(1-\mu)(c+k)} \right\}^{\frac{1}{k-1}}, \qquad (k \geq 2). \tag{4.2}$$

The result (4.2) *is sharp.*

Proof. From (4.1) we have

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} = z - \sum_{k=2}^{\infty} \frac{c+k}{c+1} a_k z^k.$$

To prove the required result it is sufficient to show that $|f'(z) - 1| < 1$ in $|z| < R^*$. Now,

$$|f'(z) - 1| = \left| - \sum_{k=2}^{\infty} k \frac{c+k}{c+1} a_k z^{k-1} \right| \leq \sum_{k=2}^{\infty} k \frac{c+k}{c+1} a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{k=2}^{\infty} k \frac{c+k}{c+1} a_k |z|^{k-1} \leq 1 \tag{4.3}$$

but Theorem (2.1) assures that

$$\sum_{k=2}^{\infty} \frac{k^{n+1} C(\alpha, k)\{1+\beta(2\gamma-1)\} a_k}{2\gamma\beta(1-\mu)} \leq 1.$$

Now, (4.3) will be satisfied if

$$k\frac{c+k}{c+1}a_k|z|^{k-1} \leq \frac{k^{n+1}C(\alpha,k)\{1+\beta(2\gamma-1)\}a_k}{2\gamma\beta(1-\mu)}, \qquad (k \geq 2)$$

or if

$$|z| \leq \left\{\frac{k^n(c+1)C(\alpha,k)\{1+\beta(2\gamma-1)\}}{2\gamma\beta(1-\mu)(c+k)}\right\}^{\frac{1}{k-1}}, \qquad (k \geq 2). \tag{4.4}$$

The required result holds from (4.4). The result is sharp for the function

$$f(z) = z - \frac{2\gamma\beta(1-\mu)(c+k)}{k^nC(\alpha,k)(c+1)\{1+\beta(2\gamma-1)\}}z^k, \qquad (k \geq 2). \tag{4.5}$$

□

5. RADIUS OF CONVEXITY

Theorem 5.1. *If $f(z)$ given by (1.2) belongs to the class $T_{n,\alpha}(\beta,\gamma,\mu)$ then $f(z)$ is convex in $|z| < R^*$ where*

$$R^* = \inf_{*}\left\{\frac{k^{n+1}C(\alpha,k)\{1+\beta(2\gamma-1)\}}{2\gamma\beta(1-\mu)}\right\}^{\frac{1}{k-1}}, \qquad (k \geq 2). \tag{5.1}$$

The result (5.1) is sharp.

Proof. To establish the required result, we need to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1, \qquad \text{in} \quad |z| < R^*.$$

Now, using (1.2) we get

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum_{k=2}^{\infty} k(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty} ka_k|z|^{k-1}}.$$

Thus we have

$$\sum_{k=2}^{\infty} k^2a_k|z|^{k-1} \leq 1. \tag{5.2}$$

In view of Theorem (2.2), (5.2) is satisfied if

$$k^2|z|^{k-1} \leq \frac{k^{n+1}C(\alpha,k)\{1+\beta(2\gamma-1)\}}{2\gamma\beta(1-\mu)}, \qquad (k \geq 2).$$

or if

$$|z| \leq \left\{\frac{k^{n+1}C(\alpha,k)\{1+\beta(2\gamma-1)\}}{2\gamma\beta(1-\mu)}\right\}^{\frac{1}{k-1}}, \qquad (k \geq 2).$$

Hence, $f(z)$ is convex in $|z| < R^*$. The result is sharp and is given by (5.1). □

6. CLOSURE THEOREMS

Now, we want to prove the following result corresponding to the closure of function in the class $T_{n,\alpha}(\beta,\gamma,\mu)$.

Theorem 6.1. *Let the function* $f_j(z) \quad (j = 1, 2, ..., m)$ *defined by*

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{kj} z^k, \qquad (a_{kj} \geq 0)$$

for $z \in U$ *belongs to the class* $T_{n,\alpha}(\beta,\gamma,\mu)$, *then the function* $h(z)$ *defined by*

$$h(z) = z - \sum_{k=2}^{\infty} b_k z^k$$

also belongs to the class $T_{n,\alpha}(\beta,\gamma,\mu)$, *where*

$$b_k = \frac{1}{m} \sum_{j=1}^{m} a_{kj}.$$

Proof. Since $f_j(z) \in T_{n,\alpha}(\beta,\gamma,\mu)$, invoking Theorem (2.2), we have

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k)\{1+\beta(2\gamma-1)\} a_{kj} \leq 2\gamma\beta(1-\mu), \qquad (\forall j = 1, 2, ..., m).$$

Thus,

$$\begin{aligned}\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k)\{1+\beta(2\gamma-1)\} b_k &= \sum_{k=2}^{\infty} k^{n+1} C(\alpha,k)\{1+\beta(2\gamma-1)\}\left(\frac{1}{m}\sum_{j=1}^{m} a_{kj}\right) \\ &= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k)\{1+\beta(2\gamma-1)\} a_{kj}\right) \\ &\leq 2\gamma\beta(1-\mu).\end{aligned}$$

Therefore, by Theorem (2.1), $h(z) \in T_{n,\alpha}(\beta,\gamma,\mu)$. □

Applying similar method used by Silverman [9] , and taking help of Theorem (2.2), we can deduce the following. Proofs are obvious and hence omitted.

Theorem 6.2. *The class* $T_{n,\alpha}(\beta,\gamma,\mu)$ *is closed under linear combination.*

Theorem 6.3. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{2\gamma\beta(1-\mu)}{k^{n+1} C(\alpha,k)\{1+\beta(2\gamma-1)\}} z^k, \qquad (k \geq 2) \tag{6.1}$$

for $\quad 0 \leq \alpha < 1, 0 < \beta \leq 1, 0 < \gamma \leq 1.$

Then, $f(z)$ belongs to the class $T_{n,\alpha}(\beta,\gamma,\mu)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \Psi_k f_k(z), \qquad \Psi_k \geq 0, \quad k \geq 1$$

and

$$\sum_{k=1}^{\infty} \Psi_k = 1.$$

COROLLARY 6.4. *The extreme points of the class $T_{n,\alpha}(\beta,\gamma,\mu)$ are the functions $f_k(z), k \geq 1$, given by Theorem (6.3).*

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