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SOLUTIONS IN INTEGERS FOR THE QUADRATIC DIOPHANTINE EQUATION $w^2 - 5z^2 + 12w - 30z - 45 = 0$

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Abstract

The research throws light on the polynomial solutions of Diophantine equation $D : w^2 - 5z^2 + 12w - 30z - 45 = 0$ is considered in $\mathbb{Z}(x)$. In the process, unearthed a few recurrence relations and an identified formulae among the solutions (w_n, z_n) is also discussed in detail.

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1. Introduction

The science of numbers or shall we say "The mystery of numbers". That is what aptly describes the field of Number Theory and its applications. As we immerse into this vast expanse of number theory, the applications which apply universally across the universe, the galaxy astounds us and holds us spellbound. One such stream of this science is that of Diophantine equations and its analysis. With further probing we understand it as a discussion of the rational or integer solutions as regards polynomial equation $f(x_1, x_2, ..., x_n) = 0$ which has integer coeffcients. Most research insights are being made into Quadratic Diophantine equations and form a primary and major contributor in the world of research. As we progress, we will find that there exist a few Diophantine equations in which the complete solution is known. One such well-known example is that of the pythagorean equation $x^2 + y^2 = z^2$ and the Pell's equation $x^2 = dy^2 + 1$. Also, we will realize that no universal method exists to solve these type of equations.

In this paper, we investigate positive integral solutions of the Quadratic Diophantine equation $w^2 - 5z^2 + 12w - 30z - 45 = 0$. The method and the approach taken involves the reduction to Pell's equation with the application of the concepts of continued fraction.

2. The Diophantine Equation $w^2 - 5z^2 + 12w - 30z - 45 = 0$

Consider the Diophantine equation

$$\boxed{D : w^2 - 5^2 + 12w - 30z - 45 = 0} \tag{2.1}$$

which is to be solved over $\mathbb{Z}$. The equation (2.1) given above does not follow the usual and tested method to find its solutions. We identified a linear transformation T approach to solve the equation (2.1) and subsequently transformed it to a simpler form for which we can determine the integral solutions. Let T : be indicated as

$$\boxed{T = \begin{cases} \text{w=M+h} \\ \text{z=N+k} \end{cases}} \tag{2.2}$$

be the transformation where $h, k \in \mathbb{Z}$. As we progress with the solution identification, and applying T to D; we get

$$\boxed{T(D) : (M + h)^2 - 5(N + k)^2 + 12(M + h) - 30(N + k) = 45} \tag{2.3}$$

Taking it a step ahead, and equating the co-efficients of M and N to 0, we get $h = -6$ and $k = -3$. Hence for $w = (M - 6)$ and $z = (N - 3)$, we have the Diophantine equation as

$$\boxed{\bar{D}{:}M^2 - 5N^2 = 36} \tag{2.4}$$

which is a Pell's equation. This reduction to the Pell's equation enables us to find all the integer solutions (M_n, N_n) of $\bar{D}$ and then re-transfer all the results from $\bar{D}$ to D by using the inverse of T. Let's consider the most general Pell's equation

$$\boxed{M^2 - 5N^2 = 1} \tag{2.5}$$

Theorem 2.1. *We consider $\bar{D}$ be the Diophantine equation in (2.4), then*
1. The continued fraction expansion of $\sqrt{5} = [2 : \bar{4}]$
2. The fundamental solution of $M^2 -$ [U+3016]5N[U+3017]$^2 = 1$ is $(u_1, v_1) = (9, 4)$
3. For $n \geq 4$,

$$M_n = 19(M_{n-1} - M_{n-2}) + M_{n-3}$$
$$N_n = 19(N_{n-1} - N_{n-2}) + N_{n-3}$$

Proof. 1. The continued fraction expansion of

$$\begin{aligned}\sqrt{5} &= 2 + (\sqrt{5} - 2)\\ &= 2 + \frac{1}{(\sqrt{5}+2)}\\ 2 + \sqrt{5} &= 4 + \frac{1}{(\sqrt{5}+2)}\\ &= 4 + \cfrac{1}{4 + \cfrac{1}{(\sqrt{5}+2)}}\\ &= 4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{(\sqrt{5}+2)}}}}\end{aligned}$$

Hence

$$\sqrt{5} = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{(\sqrt{5}+2)}}}}}$$

Therefore, the continued fraction expansion of $\sqrt{5}$ is $[2 : \bar{4}]$
2. It is easily identified and seen that $(u_1, v_1) = (9, 4)$ is a solution of $M^2 - 5N^2 = 1$. Since $M^2 - 5N^2 = 81 - 5 \times 16 = 81 - 80 = 1$
3. If $(u_1, v_1) = (9, 4)$ is considered as the fundamental solution of $M^2 - 5N^2 = 1$, then the other solutions (u_n, v_n) of $M^2 - 5N^2 = 1$ can be derived by using the equation $(u_n + v_n\sqrt{5}) = (u_1 + v_1\sqrt{5})^n$ for $n \geq 2$, as

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & 5v_1 \\ v_1 & u_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore, it can be shown by mathematical induction on ’n’ that, the solution set is

n	1	2	3	4	5	6
u_n	9	161	2889	51841	930249	16692641
v_n	4	72	1292	23184	416020	7465176

The solution set satisfies the recurrence relations
$u_n = 19(u_{n-1} - u_{n-2}) + u_{n-3}$
$v_n = 19(v_{n-1} - v_{n-2}) + v_{n-3}$

It should be noted that we denote the integer solutions of $M^2 - 5N^2 = 36$ by (M_n, N_n) and the integer solution of $M^2 - 5N^2 = 1$ by (u_n, v_n). Then we have the following theorem.

THEOREM 2.2. *Let's define the sequence* (M_n, N_n) *of positive integers by* $(M_1, N_1) = (9, 3)$ *and*

$$\boxed{M_n = 20u_{n-1} + 48v_{n-1}} \tag{2.6}$$

$$\boxed{N_n = 8u_{n-1} + 20v_{n-1}} \tag{2.7}$$

where (u_n, v_n) *is considered to be a sequence of positive solutions of* $M^2 - 5N^2 = 1$, *then*
1. (M_n, N_n) *is a solution of* $M^2 - 5N^2 = 36$ *for any integer* $n \geq 1$.
2. For $n \geq 2$

$$M_{n+1} = 9M_n + 20N_n\,,\; N_{n+1} = 4M_n + 9N_n$$

3. For $n \geq 4$

$$M_n = 19(M_{n-1} - M_{n-2}) + M_{n-3}\,, N_n = 19(N_{n-1} - N_{n-2}) + N_{n-3}$$

PROOF. 1. It is seen that $(M_1, N_1) = (9, 3)$ is a solution of $M^2 - 5N^2 = 36$.
Since $M_1^2 - 5N_1^2 = (9)^2 - 5(3)^2 = 81 - 45 = 36$

2. Let (M_n, N_n) be the integer solution of $M^2 - 5N^2 = 36$ and (u_n, v_n) be the integer solution of $M^2 - 5N^2 = 1$. We have

$$\boxed{(M_{n+1} + N_{n+1}\sqrt{d}) = (u_1 + v_1\sqrt{d})^n (M_1 + N_1\sqrt{d})} \tag{2.8}$$

$$\begin{pmatrix} M_{n+1} \\ N_{n+1} \end{pmatrix} = \begin{pmatrix} u_1 & 5v_1 \\ v_1 & u_1 \end{pmatrix}^n \begin{pmatrix} 9 \\ 3 \end{pmatrix}$$

The value $n = 1$ in (2.8) culminates into $M_2 +$ [U+3016] $\sqrt{5}N$[U+3017]$_2 = 141 + 63\sqrt{5}$ which gives (141,63) as the second solution of equation (2.4). Expanding equation (2.8) and equating the rational and irrational coefficients, we obtain the relation connecting (2.4) and (2.5) as

$$\boxed{M_n = \text{[U+3016]}9u_{n-1} + \text{[U+3016]}15v_{n-1}}$$

$$\boxed{N_n = 3u_{n-1} + 9v_{n-1}} \tag{2.9}$$

We see that,

$$\begin{aligned} M_n^2 - 5N_n^2 &= (9u_{n-1} + 15v_{n-1})^2 - 5(3u_{n-1} + 9v_{n-1})^2 \\ &= 36u_{n-1}^2 - 180v_{n-1}^2 \\ &= 36(u_{n-1}^2 - 5v_{n-1}^2) \\ &= 36 \end{aligned}$$

Therefore, (M_n, N_n) given by (2.9) is a solution of $M^2 - 5N^2 = 36$. To achieve the generalization, we apply Brahmagupta's Lemma between the solutions of (2.4) and (2.5), we get $M_{n+1} = u_1 M_n + v_1 N_n d$ and $N_{n+1} = v_1 M_n + u_1 N_n$
Since $u_1 = 9$ and $v_1 = 4$.

$$\boxed{M_{n+1} = 9M_n + 20N_n}$$

$$\boxed{N_{n+1} = 4M_n + 9N_n} \tag{2.10}$$

3. For different values of n, we get the solution set of equation (2.4) which is expressed as a table given below

n	1	2	3	4	5
M_n	9	141	2529	45361	814329
N_n	3	63	1131	20295	364179

For $n \geq 4$, it can be shown that the set of solutions of equation (2.4) satisfies the recurrence relations.

$$\boxed{M_n = 19(M_{n-1} - M_{n-2}) + M_{n-3}}$$

$$\boxed{N_n = 19(N_{n-1} - N_{n-2}) + N_{n-3}} \tag{2.11}$$

Until now, we have proved that $(M_1, N_1) = (9, 3)$ is the fundamental solution of $\bar{D}$. Also, we showed that $h = -6$ and $k = -3$, so the base of T is $T[h, k] = \{-6, -3\}$ as we had claimed. The Diophantine equation D could be transformed into the Diophantine equation $\bar{D}$ by the transformation $T : w = M - 6$ and $z = N - 3$.

THEOREM 2.3. *Let's consider D be the Diophantine equation in (2.1) then*
1. The fundamental solution of D is $(w_1, z_1) = (3, 0)$.
2. We also define the sequence $\{(w_n, z_n)\}n \geq 1 = \{(M_n - 6, N_n - 3)\}$ *where* $\{(M_n, N_n)\}$ *defined in (2.10). Then* (w_n, z_n) *is a solution of D. So it has infinitely many solutions* $(w_n, z_n) \in \mathbb{Z}X\mathbb{Z}$

3. The solution (w_n, z_n) *satisfy the recurrence relation* $w_n = 9w_{n-1} + 20z_{n-1} + 108$*,* $z_n = 4w_{n-1} + 9z_{n-1} + 33$*.*
4.The solution (w_n, z_n) *satisfy the recurrence relations* $w_n = 19(w_{n-1} - w_{n-2}) + w_{n-3}$ *,* $z_n = 19(z_{n-1} - z_{n-2}) + z_{n-3}$ *.*

Proof. 1. It is easily seen that $(w_1, z_1) = (3, 0)$ is the fundamental solution of D.

2. We prove by mathematical induction
Let $n = 1$, then $(w_1, z_1) = (M_1 - 6, N_1 - 3) = (3, 0)$ which is the fundamental solution and so is a solution of D. Let us assume that the Diophantine equation in (2.1) is satisfied for $(n - 1)$, that is

$$(M_{n-1} - 6)^2 - 5(N_{n-1} - 3)^2 + 12(M_{n-1} - 6) - 30(N_{n-1} - 3) - 45 = 0$$

We want to show that this equation is also satisfied for n.
$w_n^2 - 5z_n^2 + 12w_n - 30z_n - 45$
$= (M_n - 6)^2 - 5(N_n - 3)^2 + 12(M_n - 6) - 30(N_n - 3) - 45$
$= M_n^2 - 5N_n^2 - 36$
$= 0$
Since (M_n, N_n) is a solution of $\bar{D}$; from (2.2), we get $(w_n, z_n) = (M_n - 6, N_n - 3)$ is a solution of D.
3. Using (2.10)
$M_{n+1} = 9M_n + 20N_n$
$N_{n+1} = 4M_n + 9N_n$

Adding -6 on both sides, we get

$$M_n - 6 = 9M_{n-1} + 20N_{n-1} - 6$$

Adding -3 on both sides, we get

$$N_n - 3 = 4M_{n-1} + 9N_{n-1} - 3$$

We know that $w_n = M_n - 6$ and $z_n = N_n - 3$
Therefore, $M_n = w_n + 6$ and $N_n = z_n + 3$
$M_n - 6 = 9M_{n-1} + 20N_{n-1} - 6$
$(w_n + 6) - 6 = 9(w_{n-1} + 6) + 20(z_{n-1} + 3) - 6$
Therefore, we get

$$\boxed{w_n = 9w_{n-1} + 20z_{n-1} + 108} \quad (2.12)$$

Similarly, we get

$$\boxed{z_n = 4w_{n-1} + 9z_{n-1} + 48} \quad (2.13)$$

4. The following table represents the solution set of (2.12) and (2.13)

n	1	2	3	4	5
w_n	3	135	2523	45375	814323
z_n	0	60	1128	20292	364176

We can see that for $n = 4$

$$\begin{aligned} w_4 &= 19(w_3 - w_2) + w_1 \\ &= 19(45375 - 2523) + 135 \\ &= 814323 \end{aligned}$$

Let's us assume that this relation is satisfied for $(n-1)$

$$\boxed{w_{n-1} = 19(w_{n-2} - w_{n-3}) + w_{n-4}} \tag{2.14}$$

Then applying the previous assertion, equation (2.12) and equation (2.14) we can conclude that $w_n = 19(w_{n-1} - w_{n-2}) + w_{n-3}$ for $n \geq 4$.
Now we can prove that z_n also satisfies the recurrence relation. For $n = 4$ we get
$z_4 = 19(z_3 - z_2) + z_1 = 19(20292 - 1128) + 60 = 364176$
Let's also assume that this relation is satisfied for $(n-1)$

$$\boxed{z_{n-1} = 19(z_{n-2} - z_{n-3}) + z_{n-4}} \tag{2.15}$$

Then applying the previous assertion by equation (2.13) and equation (2.15) , we can conclude that $z_n = 19(z_{n-1} - z_{n-2}) + z_{n-3}$ for $n \geq 4$.

3. Conclusion

With each research effort and the subsequent observations derived; it is pretty evident that the area of Diophantine equations are sure to spring surprises in the near future. Mathematicians from the world over are striving to deep dive and come up with possible solutions to hitherto unknown features of the number systems; thereby assimilate this knowledge to solve complex problems.

As illustrated in the beginning and explained earlier on in this document, we will acknowledge that Diophantine Equations are one of many mind boggling and mysterious wonders in the world of Mathematics. The mystery and their hold on the intellect draws many aclaimed mathematicians from all over the world to unravel the hidden knowledge, and to illuminate the seeking with the richness of the computational world. As mathematicians rake their brains and unravel these mysteries, these equations have time and again proved beyond doubt that there is no unique and defined method to find the solutions. The quest definitely looks easy but with time engulfs the man-brain into an untangled heap of equations, theorems, numbers, and concepts.

For the given quadratic Diophantine equation, we have seen the derived solutions and conforming to a particular pattern.

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